Hand to Sensor Calibration: A Geometrical Interpretation of the Matrix Equation $AX = XB$

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In this paper, the matrix equation $AX = XB$ used for *hand to sensor* calibration of robot-mounted sensors is analyzed using a geometrical approach. The analysis leads to an original way to describe the properties of the equation and to find all of its solutions. It will also be highlighted why, when multiple instances $A_i X = XB_i$ $(i = 1, 2, \ldots)$ of the equation are to be solved simultaneously, the system is overconstrained. Finally, singular cases are also discussed.

1. INTRODUCTION

A sensor mounted on a robot gripper is sometimes used to measure the position of one body.[1–8] The sensor should be able to measure the pose of the body (position and orientation) with respect to an intrinsic frame defined on the sensor. Practical applications include (but are not restricted to) robot calibration, localization of a mechanical part, or self-localization of mobile robots.

To obtain high precision, "*hand to sensor*" calibration is required. This expression is used to indicate the operation performed to identify, with high precision, the pose of the measuring device with respect to the end-effector of the manipulator. The end-effector is also called the *hand*, and when the sensor is a vision system, the operation is called *hand-eye* calibration. The hand-to-sensor calibration is generally performed as follows:

The measuring device is mounted on the gripper in an arbitrary and a priori unknown position. Then, the robot is maneuvered to an object located in an arbitrary unknown position, and the pose of the object with respect to the sensor is measured by the sensor itself.

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The measure is repeated several times using different robot configurations, and the measures are coupled in pairs.

Using 4*4 transformation matrices, an equation of the form $A_iX=XB_i$ is written for each pair of collected data ($i=1,2,\ldots$); matrices A and B depend on the robot kinematics and on the data collected by the sensor. Matrix X, which represents the unknown pose of the robot sensor, is determined by simultaneously solving two or more matrix equations of the form $AX=XB$.

Different mathematical procedures are available to solve one equation of this form or a system of them. This equation has been studied in linear algebra,[9] but the discussion is insufficient for this application. In some cases, the problem is addressed both by an analytical and a numerical approach.[6,7] In other papers, numerical approaches are privileged;[3,4] but in ref. 4 it is not guaranteed that the solution is a 4*4 transformation matrix. Reference 1 is quite detailed, and it is based on eigenvalues and eigenvectors, refs. 5 and 7 are based on quaternions, ref. 8 on dual quaternions, ref. 6 is based on Euclidean groups, and ref. 2 is devoted to special cases of robots with limited degrees of freedom kinematics.

In this paper, a geometrical interpretation of these equations is presented, making use of rototranslation and screws. This interpretation gives an alternative way to find the solutions of the equation and presents new ways to interpret its properties.

This paper is organized as follows. After presenting the general properties of the equation (Section 2), we initially consider one instance of the equation $AX=XB$ presenting a geometrical interpretation (Section 3) which permits us to find easily all its possible solutions. As a second step in Section 4, we will analyze the cases where one, two, or more instances of the equation are available, and the unique solution is found by the intersection of the solutions of the individual equations. Singular cases are analyzed in Section 4.4. Conclusions are drawn in Section 5. For the reader's convenience, some known useful properties of rototranslation, points, lines, and planes are summarized in the Appendix. The Appendix also clarifies the notation adopted in the paper.

2. GENERAL PROPERTIES OF THE EQUATION *AX=XB*

Denoting X as the relative position of the sensor with respect to the robot gripper, Y as the unknown absolute pose of the object, T_j as the jth known end-effector pose, and U_j as the measured pose of the object with respect to the sensor when the robot is in the jth pose, it is possible to write

$$Y = T_jXU_j,$$

where X and Y are unknowns, and X is to be determined. Once X is found, Y is also known. Combining the just mentioned relation for two different gripper poses 1 and 2, we get

$$T_1XU_1 = T_2XU_2.$$

This equation is generally simplified by multiplying by the inverse of U_1 on the right, and by the inverse of T_2 on the left, yielding

$$AX = XB \tag{1}$$

with

$$A = T_2^{-1}T_1, \qquad B = U_2U_1^{-1}.$$

This simplification is always possible, since matrices U and T are nonsingular, representing rototranslation.

Equation (1) is linear in the unknown X. It is equivalent to a linear system whose right-hand side is null, and whose coefficient matrix is singular; thus, it has an infinite number of solutions. In fact, $X=0$ is a solution of the equation, and we also know that a solution for $X\neq 0$ exists (the actual sensor position). Thus, the system has at least two solutions. Since the number of solutions of a linear system is 0, 1, or infinite, having found two of them, we conclude that Eq. (1) has infinite solutions, and so the matrix of the coefficients is singular.

The number of solutions is infinite even if they are searched under the restriction that A, B, and X are matrices representing rototranslation, thus having special properties. This is illustrated by the following theorem.

Theorem: If X is a solution of Eq. (1), then $X'=Q_aXQ_b$ is also a solution if Q_a and Q_b are arbitrary rototranslations around the same screw axes of A and B.

Proof: If two rototranslations have a common screw axis, their product is commutative, and so it yields $AQ_a=Q_aA$ and $BQ_b=Q_bB$. Thus, inserting X' in Eq. (1), we get $AQ_aXQ_b=Q_aXQ_bB$, which can be converted to $Q_aAXQ_b=Q_aXBQ_b$, which is equivalent to Eq. (1), since matrices Q_a and Q_b are nonsingular, and it is also possible to multiply the just found equation on the left by Q_a^{-1} and on the right by Q_b^{-1}.

It is known that the sought unique solution X can be found if the pose of the object can be measured using additional robot configurations, or if the pose of another object is measured.[1–7] In both cases, it is possible to write a second equation

$$A_2 X = X B_2, \tag{2}$$

to be solved simultaneously with Eq. (1).

However, as will be proved in a following section, the system of two or more matrix equations of the form $AX=XB$ is overconstrained, and in the presence of measuring noise or roundoff errors, the set of simultaneous equations does not have an exact solution. An estimation of X can be obtained with a least-squares criterion.

When high precision is required, more than two equations can be considered simultaneously. Each of these equations is denoted as $(i=1,2,\dots,n)$

$$A_i X = X B_i. \tag{3}$$

3. GEOMETRICAL INTERPRETATION OF THE EQUATION *AX=XB*

Rewriting Eq. (1) as

$$A = X B X^{-1}, \tag{4}$$

A and B can be interpreted as the representation of the same rototranslation "watched" by two different reference frames whose relative pose is represented by X (see Section 6.5). Alternatively, A and B can be interpreted as two rototranslations of the same amplitude (same amount of rotation and translation) around two different screw axes $\$_a$ and $\$_b$. In this case, X is the rototranslation transforming $\$_b$ into $\$_a$ (Figures 1 and 2).

We will denote $\$_u$ as the screw axis around which the rototranslation X takes place.

The determination of X (and likewise $\$_u$) can be performed by the steps described in the following sections.

3.1. Rotation and Translation Parts of *X*

To analyze Eq. (1) more easily, it can be split into its rotation and translation parts as

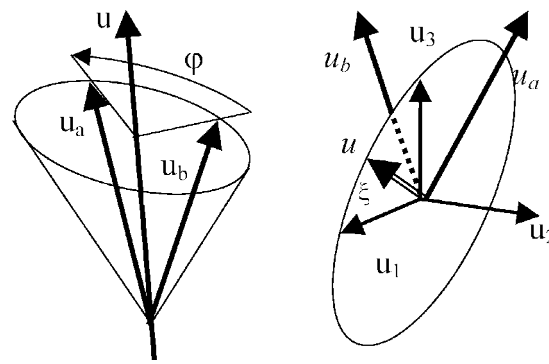

Figure 1. Geometrical representation of the equation $u_a = R_u u_b$ and the "intrinsic" frame defined by u_a and u_b. A rotation φ around u transforms u_b in u_a.

$$\begin{cases} R_a R_u = R_u R_b, \\ R_a T_u + T_a = R_u T_b + T_u, \end{cases} \qquad X = \left[\begin{array}{ccc|c} & R_u & & T_u \\ \hline 0 & 0 & 0 & 1 \end{array}\right],$$

$$A = \left[\begin{array}{ccc|c} & R_a & & T_a \\ \hline 0 & 0 & 0 & 1 \end{array}\right],$$

$$B = \left[\begin{array}{ccc|c} & R_b & & T_b \\ \hline 0 & 0 & 0 & 1 \end{array}\right],$$

moreover (see Sections 6.1 and 6.6, considering the matrix representation L_a and L_b of the screw axes of

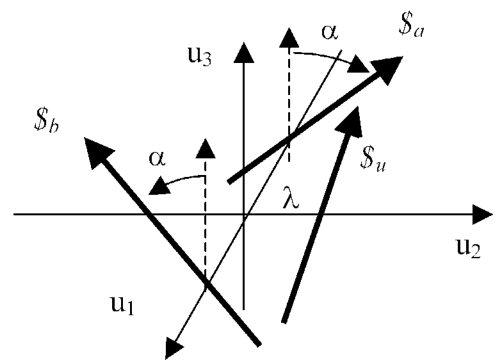

Figure 2. Intrinsic frame constructed on $\$_a$ and $\$_b$.

A and B from Eq. (4), we can also deduce that $L_a = XL_bX^{-1}$, and so

$$\begin{cases} \underline{u}_a R_u = R_u \underline{u}_b, \\ \underline{u}_a T_u + t_a = R_u t_b, \end{cases} \tag{5}$$

with

$$L_a = \left[\begin{array}{ccc|c} & \underline{u}_a & & t_a \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \quad L_b = \left[\begin{array}{ccc|c} & \underline{u}_b & & t_b \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

where $\underline{u}$ is a matrix representation of the unit vector of the screw axis, while t is its momentum (see Sections 6.1, 6.5, and 6.6).

3.2. Determination of the Rotation Part of *X*

This section is devoted to the determination of the rotation part of X. It will be shown that the rotation axis can be freely chosen with the restriction that it must lie in a given plane and, after choosing the axis, the rotation angle is uniquely determined.

From Eq. (5), we can deduce $\underline{u}_a = R_u \underline{u}_b R_u^t$, and so matrix R_u is the rotation which transforms the unit vector u_b in u_a ($u_a = R_u u_b$).

There is an infinite number of possible rotations achieving this result (Figure 1). In fact, considering an arbitrary unit vector u which lies on the plane orthogonal to the difference between u_a and u_b, a rotation of an appropriate angle φ around u transforms u_b into u_a. To determine φ, it is convenient to represent vectors in an "intrinsic" frame whose axes u_1, u_2, and u_3 are parallel to vectors v_1, v_2, and v_3

$$v_1 = u_a \times u_b, \qquad u_a = [0\ B\ C]^t,$$

$$v_2 = u_a - u_b, \qquad u_b = [0\ -B\ C]^t,$$

$$v_3 = u_a + u_b, \qquad u = [\cos(\xi)\ 0\ \sin(\xi)]^t,$$

with $B^2 + C^2 = 1$. The trivial case is $u = u_3$, for which we obviously get $\varphi = \pi$. As a second simple choice, we can select $u = u_1$ getting $\cos(\varphi) = u_a \cdot u_b$.

The value of φ for the general case is found by expanding $u_b = R_u u_a$ into its scalar terms [see Eq. (A1) in the Appendix]; we get

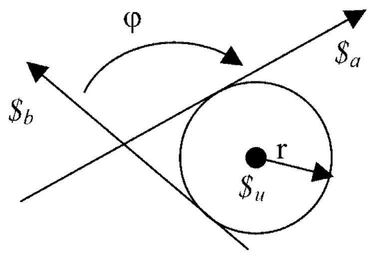

Figure 3. Projection of $\$_a$, $\$_b$ onto a plane orthogonal to $\$_u$.

$$\varphi = \operatorname{atan2}\left(-2\frac{\cos(\xi)CB}{\cos(\xi)^2C^2 + B^2}, \frac{\cos(\xi)^2C^2 - B^2}{\cos(\xi)^2C^2 + B^2}\right), \tag{6}$$

where atan2(y,x) is the four-quadrant extension of $\arctan(y/x)$. This term evaluates the angle of a vector whose components are y and x.

The rotation matrix R_u can be evaluated by u and φ by means of Eq. (A1). The unit vector, u, represents the direction of the screw axis $\$_u$.

3.3. Determination of the Location of the Screw Axis of *X*

To determine the location of the screw axis $\$_u$, it is useful to represent the screw axes $\$_a$ and $\$_b$ in a reference frame (the intrinsic frame) with the orientation described in Section 3.2, and an origin in the middle of the common normal between $\$_a$ and $\$_b$ (Figures 2–5).

Since the rototranslation X should transform $\$_b$

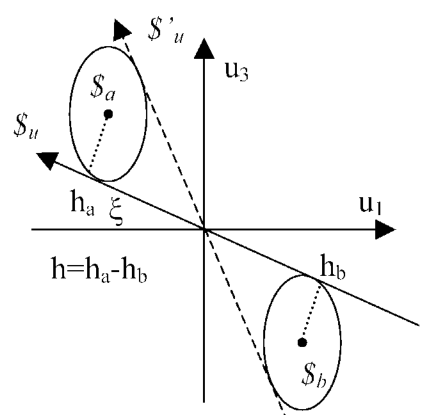

Figure 4. Intersection between axes $\$_a$ and $\$_b$ with a plane parallel to u_1u_3 at a distance λ from the origin.

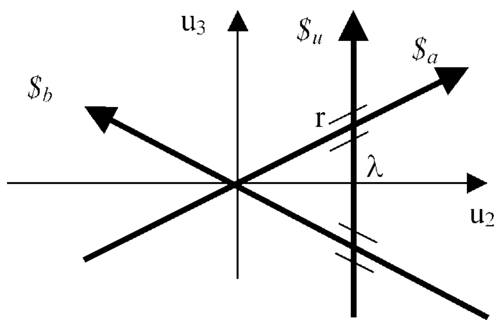

Figure 5. Projection of $\$_a$, $\$_b$, and $\$_u$ onto plane u_2u_3.

into $\$_a$, then the screw axis $\$_u$ of X should be at the same distance from both $\$_a$ and $\$_b$, and it should form the same angle with both of them. From Section 3.2, we learned that $\$_u$ lies in a plane orthogonal to u_a-u_b, and so it is parallel to the plane u_1u_3. To respect all of these restrictions, it follows that $\$_u$ should intersect axis u_2 of the intrinsic frame. This result is shown in Figures 2–5. Figure 2 is a 3D representation, while the others are, respectively, projections into a plane orthogonal to $\$_u$, into plane u_3u_1, and into plane u_3u_2; λ is the distance of $\$_u$ from the origin of the intrinsic frame, and r is the distance from $\$_u$ to $\$_a$ and $\$_b$. In Figure 4, the two ellipses represent the section of cylinders with a radius equal to r whose axes are $\$_a$ and $\$_b$. Two solutions for $\$_u$ are possible for any value of r ($\$_u$ and $\$'_u$ in Figure 4); $h=h_a-h_b$ is the amount of translation of X, while h_a and h_b are the distances from the origin to the common normal between $\$_u$ and $\$_a$ and $\$_b$ ($|h_a|=|h_b|$).

The value of λ can be freely chosen.

3.4. Determination of the Translational Part of *X*

After choosing the value of λ, the corresponding value of the translation h of X can be evaluated, observing that any point P_b of $\$_b$ must transform into one point P_a of $\$_a$ ($P_a=XP_b$):

$$P_a = P'_a + \rho u_a = R_u P_b + T_u, \qquad T_u = t + hu, \tag{7}$$

where R_u is the rotation part of X, $t=-\underline{u}P_u$, P_u is one point of $\$_u$, and ρ is an unknown scalar (point P'_a is one point of $\$_a$). Point P_u can be evaluated, for example, by considering the point P_0 (the origin of the intrinsic frame) which lies in the middle of the common normal between $\$_a$ and $\$_b$ and adding to it a translation λ along axis u_2, such that $P_u=P_0+\lambda u_2$ (see also Section 6.8). With the adopted reference frame, it yields $P_0=[0\ 0\ 0\ 0]^T$.

The value of h together with ρ is obtained after solving a linear system of two equations obtained by projecting Eq. (7) in the directions of u and u_a, and performing some simplifications that are possible because of the particular structure of R_u [Eq. (A1) of the Appendix]:

$$\begin{cases} u_a^t P_a \\ u^t P_a \end{cases} \Rightarrow \begin{cases} \rho - h u_a^t u = u_a^t(-P'_a + RP_b + t), \\ \rho u^t u_a - h = u^t(P_b - P'_a), \end{cases} \tag{8}$$

in fact $u^t\underline{u}=0$; $u^t u=1$.

4. GENERAL SOLUTION OF *AX=XB*

4.1. The Infinite Solutions of *AX=XB*

Summarizing the results of the previous sections, the general solution of one equation $AX=XB$ can be found performing the following steps:

1. Extract u_a, P_a, u_b, and P_b from A and B;

2. Choose an arbitrary unit vector u in the plane normal to u_a-u_b;

3. Determine the rotation angle φ using Eq. (6);

4. Choose an arbitrary distance λ and determine the translation h by Eq. (8).

There are an infinite number of solutions because of the arbitrary choices made in steps 2 and 4. Therefore, to find a unique solution, a system of two or more simultaneous equations needs to be solved.

4.2. Determination of the Solution of a System of Two Equations

For the case in which two equations $A_1X=XB_1$ and $A_2X=XB_2$ are to be solved simultaneously, two intrinsic frames $u_{11}u_{21}u_{31}$ and $u_{12}u_{22}u_{32}$ can be constructed, one for each matrix equation. Two planes π_1 and π_2 can be defined by vectors u_{a1}-u_{b1} and u_{a2}-u_{b2}, where u_{a1}, u_{b1}, u_{a2}, and u_{b2} are the unit vectors of the rototranslations A_1, B_1, A_2, and B_2. The direction of u (and so that of $\$_u$) can be determined observing that it must be parallel to both planes π_1 and π_2. The screw axis $\$_u$ must also intersect both axes u_{21} and u_{22}.

Therefore, to determine $\$_u$ and X, the following steps must be performed:

1. Extract u_{a1}, P_{a1}, u_{b1}, and P_{b1} from A_1 and B_1.

2. Extract u_{a2}, P_{a2}, u_{b2}, and P_{b2} from A_2 and B_2.

3. Determine the direction of the planes π_1 and

π_2 in which unit vector u must lay: $w_1 = u_{a1} - u_{b1}$ and $w_2 = u_{a2} - u_{b2}$.

4. Determine unit vector u (intersection of π_1 and π_2) as $u = w_1 \times w_2 / \|w_1 \times w_2\|$.

5. Determine angle φ either from $A_1X = XB_1$ or from $A_2X = XB_2$.

6. Determine one point P_u of $\$_u$ by the intersection of the axes u_{21} and u_{22} of the intrinsic frames.

7. Determine the value of h either from $A_1X = XB_1$ or from $A_2X = XB_2$.

In steps 5 and 7 of the procedure, it is possible to evaluate φ and h from one or the other matrix equation. Moreover, in step 6 the intersection between two lines in the space is evaluated. If the lines do not intersect, or if the value of φ or the value of h is different if it is evaluated from the first or the second equation, a valid solution X does not exist. In our case, we know that a solution must exist because it represents the actual pose of the sensor. However, in the case of numerical or measuring errors, the values of the matrices A_1, A_2, B_1, and B_2 are not known exactly, and so the two lines will not intersect. Moreover, it is impossible to find values for φ and h that satisfy both $A_1X = XB_1$ and $A_2X = XB_2$. In these cases, no solution exists.

We conclude that the system of two matrix equations is overconstrained (see also Section 4.3).

4.3. Solution of a System of More Equations of the Form *AX*=*XB*

When two or more matrix equations of the form $AX = XB$ are to be solved simultaneously, the system is overconstrained and the "*best approximate solution*" can be determined by evaluating the values of u, φ, P_u, and h that minimize the following index based on the Euclidean norm of AX-XB:

$$I = \sum_i \|A_i X - XB_i\|, \qquad X = X(u, \varphi, P_u, h).$$

This minimization can be performed, for instance, using some general iterative optimization technique after initializing u, φ, P_u, h with the values obtained as described in Section 4.2 for a couple of equations. Alternative procedures are available.[3,4,6,7]

4.4. Singular Cases

One singular case for the equation $AX = XB$ is when $u_a = u_b$. Two subcases must be considered:

(a) if $\$_a = \$_b$, then X can be an arbitrary rototranslation around $\$_u = \$_a = \$_b$;

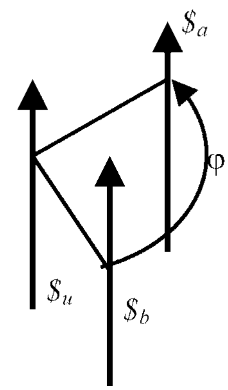

Figure 6. Determination of angle φ in the singular cases.

(b) if $\$_a$ is parallel to $\$_b$, $\$_u$ is an arbitrary axis equispaced and parallel to them, the translation h is free, and the rotation φ can be evaluated as Figure 6 suggests.

Singular cases for the system of two equations happen when w_1 and w_2 are parallel to each other, so the planes π_1 and π_2 are also parallel, and vector u cannot be unequivocally identified. Other singular situations are:

(a) when unit vectors u_{21} and u_{22} are parallel to each other and do not intersect,

(b) when they coincide, and thus have an infinite numbers of intersections.

In these cases, the system cannot be unequivocally solved, and other data must be collected to construct further matrix equations of the form $A_iX = XB_i$.

5. CONCLUSION

A discussion of the equation $AX = XB$ has been performed on the bases of geometrical considerations. The discussion pointed out that the equation has ∞^2 solutions, because the direction and the location of the axis of the rototranslation contained in X can be chosen arbitrary, resulting in establishing the values of two parameters. However, the simultaneous adoption of two or more matrix equations will result in an overconstrained system. This could be a problem if numerical or measuring errors are contained in the matrices A_i and B_i. In these cases, a least-squares solution can be performed, resulting in a minimization of the errors. Finally, it was shown that for an incorrect choice of A_i or B_i, the system is singular and cannot be unequivocally solved.

6. APPENDIX A

This appendix summarizes, without proof, some properties of rototranslations, points, lines, planes, and their matrix representations. The description is adapted from refs. 10–15.

6.1. Matrix Representation of Vectors

A vector v can be represented in matrix form using its components along the axes of the adopted reference frame. Two representations can be adopted[10,11,15]

$$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \qquad \underline{v} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}.$$

The "*underlined*" 3*3 form can be used to represent the cross product. Considering three vectors a, b, and c, the cross product $c = a \times b$ can be written in matrix form as $c = \underline{a}b = -\underline{b}a$. The dot product $\mu = a \cdot b$, can be represented in matrix form as $\mu = a^t b = b^t a$.

If (1) and (2) are two different reference frames, and $v_{(1)}$ and $v_{(2)}$ are the representation of the same vector v into the two reference frames, it yields

$$v_{(1)} = R_{12} v_{(2)}, \qquad \underline{v}_{(1)} = R_{12} \underline{v}_{(2)} R_{21},$$

where R_{12} is a 3*3 matrix representing the rotation which superimposes frame (1) onto frame (2). Matrix R_{12} is orthogonal, and so it yields $R_{21} = R_{12}^{-1} = R_{12}^T$.

For any unit vector u and any integer $k \geqslant 0$, it yields

$$\underline{u}^{k+2} = -\underline{u}^k, \qquad \underline{u}^{2k} = (\underline{u}^{2k})^T, \qquad \underline{u}^{2k+1} = -(\underline{u}^{2k+1})^T.$$

6.2. Rotation Matrix

The matrix describing a rotation of an angle φ around an axis u can be evaluated as[1,10,11,13,15]

$$R = 1 + \underline{u}\sin(\varphi) + \underline{u}^2[1 - \cos(\varphi)]. \tag{A1}$$

Rotation matrices are orthogonal, and so their transpose and inverse coincide: $R^T = R^{-1}$. The vector a obtained by rotating vector b with angle φ around the axis u is $a = Rb$.

If (1) and (2) are two different reference frames, one rotation R is described in the two difference frames by two matrices $R_{(1)}$ and $R_{(2)}$ obtained by expressing vector u of Eq. (A1) in the two different references. It yields[10,11]

$$R_{(1)} = R_{12} R_{(2)} R_{21}. \tag{A2}$$

The eigenvalues of $R_{(1)}$ and $R_{(2)}$ are identical for both matrices $\lambda_{1,2,3} = \{1, e^{i\varphi}, e^{-i\varphi}\}$, while the eigenvector associated with $\lambda_1 = 1$ is the axis about which the rotation take places, it is different for $R_{(1)}$ and $R_{(2)}$, and it has the form $u_{(i)} = [u_x\ u_y\ u_z]$ with $u_{(2)} = R_{21} u_{(1)}$ and $\underline{u}_{(2)} = R_{21} \underline{u}_{(1)} R_{12}$. The axis of rotation u and the amplitude φ of the rotation angle can be extracted from the matrix R as

$$\varphi = \operatorname{atan2}(s, c),$$

where $\operatorname{atan2}(s,c)$ is the four-quadrant extension of $\arctan(s/c)$, and c and s are evaluated from the elements r_{ij} of R as

$$s = \sin(\varphi) = \pm\sqrt{(r_{32} - r_{23})^2 + (r_{13} - r_{31})^2 + (r_{12} - r_{21})^2}$$

and

$$c = \cos(\varphi) = (r_{11} + r_{22} + r_{33} - 1)/2.$$

If $s \neq 0$, the axis of rotation is extracted as

$$\underline{u} = \frac{1}{2s}(R_u - R_u^T).$$

If $s = 0$ and $c = 1$, then $\varphi = 0$ and the axis u is undefined. For $s = 0$ and $c = -1$, $\varphi = \pi$, and it is possible to adopt the alternative formulation

$$u_x = \pm\sqrt{\frac{r_{11} - 1}{1 - c} + 1}, \qquad r_{32} + r_{23} = 2u_y u_z,$$

$$u_y = \pm\sqrt{\frac{r_{22} - 1}{1 - c} + 1}, \qquad r_{31} + r_{13} = 2u_x u_z,$$

$$u_z = \pm\sqrt{\frac{r_{33}-1}{1-c}+1}, \qquad r_{12}+r_{21}=2u_xu_y.$$

6.3. Matrix Representations of Points

Points can be represented by a column vector embedding their homogeneous coordinates in the adopted reference frames[10,11,13,14]

$$P=[x \quad y \quad z \quad 1]^T.$$

If (1) and (2) are two different reference frames, the point is represented in the two frames by two column vectors $P_{(1)}$ and $P_{(2)}$ such that

$$P_{(1)}=M_{12}P_{(2)},$$

where M_{12} is the 4^*4 matrix which represents the rototranslation which superimposes frame (1) onto frame (2). Matrix M_{12} has the following form:

$$M_{12}=\left[\begin{array}{ccc|c} & R_{12} & & T_{12} \\ \hline 0 & 0 & 0 & 1\end{array}\right],$$

where R_{12} is the already mentioned rotation matrix, and T_{12} is a vector representing the origin of frame (2) with respect to frame (1). It yields $M_{21}=M_{12}^{-1}$.

6.4. Matrix Representation of Planes

A plane π can be represented by a row vector containing the unit vector u normal to the plane and the distance μ of the origin of the reference system from the plane[14]

$$\pi=[u_x \quad u_y \quad u_z \quad \mu].$$

The distance (with sign) d of one point P from the plane π can be evaluated as

$$d=\pi P,$$

and the planes transform from one reference to another as

$$\pi_{(2)}=\pi_{(1)}M_{12}.$$

6.5. Rototranslations

If a point, whose initial position is P_i, undergoes a rototranslation Q, its final position P_f is[10,11]

$$P_f=QP_i, \tag{A3}$$

where matrix Q can be evaluated knowing the rototranslation parameters, which are the unit vector of the rototranslation axis u, the amplitude of the rotation angle φ, the translation h along the axis, and the position of a point P^* on the axis

$$Q=\left[\begin{array}{ccc|c} & R & & T \\ \hline 0 & 0 & 0 & 1\end{array}\right],$$

where R is the already mentioned rotation matrix [Eq. (A1)] and $T=hu+(1-R)P^*$ with $P^*=[x\ y\ z]^T$.

If (1) and (2) are two different reference frames, the same rototranslation Q is described in the two different frames by two matrices $Q_{(1)}$ and $Q_{(2)}$ obtained by expressing vector u and point P^* in the two different references. It yields[10,11]

$$Q_{(1)}=M_{12}Q_{(2)}M_{21}, \quad M_{12}=M_{21}^{-1},$$

and the eigenvalues of both $Q_{(1)}$ and $Q_{(2)}$ are $\lambda_{1,2,3,4}=\{1,e^{i\varphi},e^{-i\varphi},1\}$. The eigenvector associated with $\lambda_1=1$ has the form $u_1=[u_x\ u_y\ u_z\ 0]^T$ (see Section 6.2).

When the rototranslation Q is known, the rotation angle φ and the unit vector u (and so $\underline{u}$) can be extracted from R (as shown in Section 6.2), while the translation h and the position of one point P on the axis can be extracted as

$$h=u^tT, \qquad P=\frac{(R^T-1)T}{2[1-\cos(\varphi)]}+\lambda u=P^*+\lambda u,$$

where λ is an arbitrary scalar. The pitch, p, is defined as $p=h/\varphi$.

6.6. Matrix Representation of a Line or of a Screw Axis

One screw axis $\$$ can be represented in matrix form as

$$L = \left[\begin{array}{ccc|c} & \underline{u} & & -\underline{u}P^* + pu \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} & \underline{u} & & t \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

where $p = h/\varphi$ is the pitch of the screw, u is its unit vector, P^* is one point on the axis, and $t = -\underline{u}P^* + pu$ is the momentum of the axis. A line is a screw with null pitch $p = 0$.

Screw axes are often represented by *dual* vectors $\$ = u + \varepsilon t$, where ε is a special *dual* operator for which $\varepsilon^2 = 0$. If a screw axis undergoes a rototranslation Q from an initial position, L_i, to a final position L_f, it yields[10,11]

$$L_f = QL_iQ^{-1}.$$

A line or a screw axis $\$ = (L'\ M'\ N';\ P'\ Q'\ R')$ can also be represented by its Plucker coordinates L', M', N', P', Q', and R' as

$$L = \left[\begin{array}{ccc|c} 0 & -N' & M' & P' \\ N' & 0 & -L' & Q' \\ -M' & L' & 0 & R' \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

6.7. Exponential Formula for Rototranslations

The matrix Q describing a rototranslation of angle φ around a screw axis \$ can be evaluated as[10–12]

$$\mathrm{Q} = \exp(\mathrm{L}\varphi) = 1 + \mathrm{L}\varphi + \mathrm{L}^2\frac{\varphi^2}{2} + \cdots + \mathrm{L}^n\frac{\varphi^n}{n!},$$

$$n = 1 \cdots \infty,$$

where $\exp(A)$ is the exponential of the matrix A, and L is the matrix representation of the screw axis \$.

In a similar way for pure rotations, we can write

$$R = \exp(\underline{u}\varphi).$$

6.8. Distance and Intersection Between Lines

One line can be represented by its unit vector u and by a point P^* laying on it. A generic point P of the line is

$$P = P^* + du,$$

where d is the abscissa of P. Two lines a and b intersect if it is possible to find two scalars α and β for which $P_a^* + \alpha u_a = P_b^* + \beta u_b$. The distance δ between two lines can be evaluated by using the equation

$$P_a^* + \alpha u_a - P_b^* - \beta u_b = \delta u_d, \qquad u_d = \frac{u_a \times u_b}{\|u_a \times u_b\|}, \tag{A4}$$

where u_d is the common normal to the two lines, and δ is their distance ($\delta = 0$ if they intersect). The parameters α and β can be determined by solving the following linear system obtained by projecting Eq. (A4) on u_a and u_b:

$$\begin{cases} \alpha - \beta c = u_a^t(P_b^* - P_a^*), \\ \alpha c - \beta = u_b^t(P_b^* - P_a^*), \end{cases} \qquad c = u_a^t u_b = u_b^t u_a.$$

The value of δ is thus

$$\delta = u_d^t(P_a^* + \alpha u_a - P_b^* - \beta u_b) = u_d^t(P_a^* - P_b^*).$$

In fact, u_a and u_b are orthogonal to u_d.

The point P_0 in the middle of the common normal between the two lines is

$$P_0 = \frac{1}{2}(P_a^* + \alpha u_a + P_b^* + \beta u_b).$$

If the two lines intersect, P_0 is the intersection.

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