Calibration of Wrist-Mounted Robotic Sensors by Solving Homogeneous Transform Equations of the Form $AX = XB$

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***Abstract*—In order to use a wrist-mounted sensor (such as a camera) for a robot task, the position and orientation of the sensor with respect to the robot wrist frame must be known. We can find the sensor mounting position by moving the robot and observing the resulting motion of the sensor. This yields a homogeneous transform equation of the form $AX = XB$, where A is the change in the robot wrist position, B is the resulting sensor displacement, and X is the sensor position relative to the robot wrist. The solution to an equation of this form has one degree of rotational freedom and one degree of translation freedom if the angle of rotation of A is neither 0 nor π radians. To solve for X uniquely, it is necessary to make two arm movements and form a system of two equations of the form: $A_1X = XB_1$ and $A_2X = XB_2$. A closed-form solution to this system of equations is developed and the necessary conditions for uniqueness are stated.**

I. Introduction

The investigation into the solution of the homogeneous transform equation of the form $AX = XB$, where A are B are known and X is unknown, is motivated by a need to solve for the position between a wrist-mounted sensor and the manipulator wrist center (T_6). Throughout this paper, the homogeneous transform T_6 is used in the same manner as in Paul's text [28]; it is used to represent the position and orientation of the robot wrist frame with respect to the robot base frame. In some literature, ${}^0T_6$ is used instead of T_6.

We want to find the sensor position relative to the robot wrist instead of to other robot links, because of the following reasons: 1) The sensor is usually mounted to the wrist (last link of the robot), to allow itself all 6 degrees of freedom. If, for example, the sensor is mounted on the fifth link of the robot, its motion will be limited to 5 degrees of freedom. 2) Robot motions are conventionally specified in terms of the position of the last robot link (the wrist); it is therefore natural to find the sensor position relative to this link. 3) Once the sensor position relative to the last link is found, it is straightforward to find the sensor position relative to other links, using encoder readings and link specifications.

Much research has been done on using a sensor to locate an object. The three-dimensional position and orientation of an object can be found by monocular vision, stereo vision, dense/sparse range sensing, or tactile sensing. Monocular vision locates an object using a single view, and the object dimensions are assumed to be known *a priori* [2], [6], [8], [10], [13], [22], [29], [31], [32]. Stereo vision uses two views instead of one so that the range information of feature points can be found [1], [6], [12], [14], [20], [24], [32]. A dense range sensor scans a region of the world and there are as many sensed points as its resolution allows [3], [7], [17], [25]. A sparse range sensor scans only a few points, and if the sensed points are not sufficient to locate the object, additional points will be sensed [5], [15], [16]. Tactile sensing is similar to sparse range sensing in that it obtains the same information: range and surface normal of the sensed points [4], [15], [16].

A sensing system refers to object positions with respect to a coordinate frame attached to the sensor, but robot motions are specified by the wrist positions (T_6). In order to use the sensor information for a robot task, the relative position between the sensor and the wrist must be known.

Direct measurements are difficult because there may be obstacles to obstruct the measurement path, the points of interests may be inside a solid and be unreachable, and the coordinate frames may differ in their orientations. The measurement path can be obstructed by the geometry of the sensor or the robot, the sensor mount, wires, etc. The unreachable coordinate frames include T_6 and the camera frame: T_6 is unreachable because it is the intersection of various link axes, the camera frame is unreachable because its origin is at the focal point, inside the camera. Instead of direct measurement, we can compute the camera position by displacing the robot and observing the changes in the sensor frame. This method works for any sensors capable of finding the three-dimensional position and orientation of an object. Figs. 1 and 2 show the cases of a monocular vision system and a robot hand with tactile sensors.

In order to formulate a homogeneous transform equation, Fig. 1 is re-drawn in Fig. 3. If the robot is moved from position T_{6_1} to T_{6_2}, and the position of the fixed object relative to the camera frame is found to be OBJ_1 and OBJ_2, respectively, then the following equation is obtained:

$$T_{6_1} X \,\mathrm{OBJ}_1 = T_{6_2} X \,\mathrm{OBJ}_2 \tag{1}$$

where X is the unknown transform representing the camera

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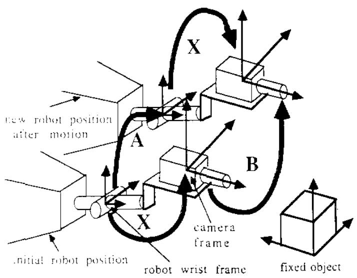

Fig. 1. Finding the mounting position of a camera by solving a homogeneous transform equation of the form $AX = XB$, where A is the robot motion, B is the resulting camera motion, and X is the camera mounting position.

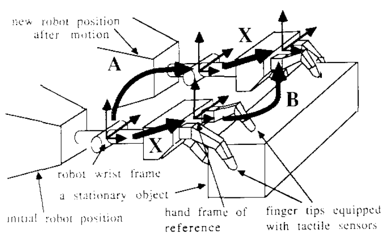

Fig. 2. Finding the mounting position of a robot hand equipped with tactile sensors, by solving a homogeneous transform equation of the form $AX = XB$, where A is the robot motion, B is the resulting motion of the hand coordinate frame, and X is the mounting position of the hand.

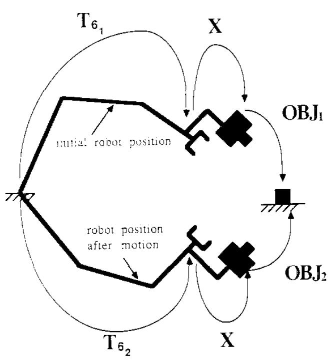

Fig. 3. If the robot is moved from position T_{6_1} to T_{6_2} and the position of the fixed object relative to the camera frame is found to be OBJ_1 and OBJ_2, respectively, then the following equation is obtained: $T_{6_1}X\,OBJ_1 = T_{6_2}X\,OBJ_2$, where X is the unknown transform representing the camera mounting position relative to the robot wrist frame.

mounting position relative to the robot wrist frame. Premultiplying both sides of the equation by $T_{6_2}^{-1}$ and postmultiplying them by OBJ_1^{-1}, we have

$$T_{6_2}^{-1}T_{6_1}X = X\,OBJ_2\,OBJ_1^{-1}. \tag{2}$$

$T_{6_2}^{-1}\,T_{6_1}$ can be interpreted as the relative motion made by the robot and we denote it by A; thus

$$A = T_{6_2}^{-1}T_{6_1}. \tag{3}$$

Similarly, we denote $OBJ_2OBJ_1^{-1}$ by B and it can be interpreted as the relative motion of the camera frame.

$$B = OBJ_2\,OBJ_1^{-1}. \tag{4}$$

The transform matrices A and B are known since T_{6_1} and T_{6_2} can be calculated by the robot controller from the joint measurements, and OBJ_1 and OBJ_2 can be found by the vision system. The case of the tactile sensor shown in Fig. 2 is similar to that of the vision system, where a homogeneous transform equation of the form $AX = XB$ results.

Matrix equations of the form $AX = XB$ have been discussed in linear algebra [11]; however, the results are not specific enough to be useful for our application. In order to solve for a unique solution, we must have a geometric understanding of the equation and use properties specific to homogeneous transforms. Using Gantmacher's results [11], the solution to the 3×3 rotational part of X (R_X) is any linear combination of n linearly independent matrices: $R_X = k_1M_1 + \cdots + k_nM_n$, where n is determined by properties of eigenvalues of R_A and R_B (rotational parts of A and B), $k_1, \cdots, k_n$ are arbitrary constants, and $M_1, \cdots, M_n$ are linearly independent matrices. Gantmacher's solution is for general matrices; the given solution may not be a homogeneous transform. To restrict the solution to homogeneous transforms, we must impose the conditions that the 3×3 rotational part of the solution be orthonormal and that the right-handed screw rule is satisfied. These restrictions will result in nonlinear equations in terms of $k_1, \cdots, k_n$. Formulating the problem in the above manner does not solve the problem because of the following reasons: 1) There is an infinite number of solutions to an equation of the form $AX = XB$. In order to find a way to solve for a unique answer, we must have a geometric understanding of the equation; however, the above formulation does not enable us to do so. 2) Only iterative solutions are possible, since nonlinear equations are involved. 3) The solution cannot be expressed symbolically and in closed form.

The approach in this paper is based on the geometric interpretations of the eigenvalues and eigenvectors of a rotational matrix. The solution is discussed in the context of finding the sensor position with respect to T_6; however, the results are general and can possibly be useful for other applications which require the solutions to homogeneous transform equations of the form $AX = XB$.

Since this paper investigates the solution to the homogeneous transform equation of the form $AX = XB$ in the context of finding a sensor's mounting position, we will relate the mathematics to this problem throughout the paper. Section II is a review on expressing a homogeneous transform in terms of rotation about an axis of rotation and translations in the x, y, and z directions. Some properties of the eigenvalues and eigenvectors of rotational matrices are also explored. Section III discusses the general solution to the equation and its geometric interpretation. Section IV deals with the solution to a system of two such equations and the conditions for uniqueness. Section V contains an example showing how we can solve for a sensor position using the proposed method. Section VI addresses the issues of noise sensitivity.

II. Homogeneous Transforms and Rotation About an Arbitrary Axis

Homogeneous transforms [28] can be viewed as the relative position and orientation of a coordinate frame with respect to another coordinate frame. The elements of a homogeneous transform T is usually denoted as follows:

$$T=\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

We also denote $[n_x, n_y, n_z]^T$ as $\boldsymbol{a}$, $[o_x, o_y, o_z]^T$ as $\boldsymbol{o}$, and $[a_x, a_y, a_z]^T$ as $\boldsymbol{a}$. $\boldsymbol{n}$, $\boldsymbol{o}$, and $\boldsymbol{a}$ can be interpretated as unit vectors which indicate the $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ directions of coordinate frame T; p can be viewed as the origin of T. The vectors $\boldsymbol{n}$, $\boldsymbol{o}$, $\boldsymbol{a}$, and $\boldsymbol{p}$ are referenced with respect to a frame represented by a transform to which T is post-multiplied. If there is no transform to the left of T, then $\boldsymbol{n}$, $\boldsymbol{o}$, $\boldsymbol{a}$, and $\boldsymbol{p}$ will be vectors relative to the world or absolute frame.

We will refer to the upper-left 3×3 submatrix of T as the rotational submatrix since it contains information about the orientation of the coordinate frame. A rotational submatrix can be expressed as a rotation around an arbitrary axis. From [28], the matrix representing a right-hand-rule rotation of θ around an axis $[k_x, k_y, k_z]^T$ is

$$\mathrm{Rot}\,(\mathbf{k}, \theta)=\begin{bmatrix} k_xk_x \text{ vers } \theta+\cos\theta & k_yk_x \text{ vers } \theta-k_z\sin\theta & k_zk_x \text{ vers } \theta+k_y\sin\theta \\ k_xk_y \text{ vers } \theta+k_z\sin\theta & k_yk_y \text{ vers } \theta+\cos\theta & k_zk_y \text{ vers } \theta-k_x\sin\theta \\ k_xk_z \text{ vers } \theta-k_y\sin\theta & k_yk_z \text{ vers } \theta+k_x\sin\theta & k_zk_z \text{ vers } \theta+\cos\theta \end{bmatrix} \tag{6}$$

where vers θ = $(1 - \cos\theta)$.

Given the rotational part of a homogeneous transform in the form of (5), the angle of rotation and the axis of rotation can be solved for symbolically, provided the rotational submatrix is not an identity matrix. If we are given an identity matrix (which is equivalent to zero rotation), it will not be possible to determine $\boldsymbol{k}$, since zero rotation about any vector will yield an identity matrix. In this paper, we will follow the convention that $0 \leqq \theta \leqq \pi$. From Paul's text [28], we have the following two equations:

$$\cos\theta=\frac{1}{2}\,(n_x+o_y+a_z-1) \tag{7}$$

and

$$\sin\theta=\pm\frac{1}{2}\,\sqrt{((o_z-a_y^2)+(a_x-n_z^2)+(n_y-o_x^2)}. \tag{8}$$

Since $0 \leqq \theta \leqq \pi$, we only take the positive sign of (8). Thus we have only one solution for θ

$$\theta=\text{atan } 2(\sqrt{(o_z-a_y^2)+(a_x-n_z^2)+(n_y-o_x^2)},\ n_x+o_y+a_z-1). \tag{9}$$

We can now find $\boldsymbol{k}$ using θ computed by (9). The set of equations used depends on whether n_x, o_y, or a_z is most positive. From Paul's text, if n_x is most positive

$$k_x=\text{sgn }(o_z-a_y)\sqrt{\frac{n_x-\cos\theta}{\text{vers }\theta}} \tag{10a}$$

$$k_y=\frac{n_y+o_x}{2k_x \text{ vers }\theta} \tag{10b}$$

$$k_z=\frac{a_x+n_z}{2k_x \text{ vers }\theta} \tag{10c}$$

where sgn (e) = $+1$ if $e \geqq 0$ and sgn (e) = -1 if $e < 0$. (Note that our definition of sgn (e) is different from that in Paul's text. We will discuss this later on.) If o_y is the most positive

$$k_y=\text{sgn }(a_x-n_z)\sqrt{\frac{o_y-\cos\theta}{\text{vers }\theta}} \tag{11a}$$

$$k_x=\frac{n_y+o_x}{2k_y \text{ vers }\theta} \tag{11b}$$

$$k_z=\frac{o_z+a_y}{2k_y \text{ vers }\theta}. \tag{11c}$$

Finally, if a_z is the most positive

$$k_z=\text{sgn }(n_y-o_x)\sqrt{\frac{a_z-\cos\theta}{\text{vers }\theta}} \tag{12a}$$

$$k_x=\frac{a_x+n_z}{2k_z \text{ vers }\theta} \tag{12b}$$

$$k_y=\frac{o_z+a_y}{2k_z \text{ vers }\theta}. \tag{12c}$$

From a geometric point of view, when $\theta = \pi$, there are two solutions to $\boldsymbol{k}$, one opposite to the other. Also, when $\theta = \pi$, we can see from (6) that $o_z - a_z = 0$, $a_x - n_z = 0$, and $n_y - o_x = 0$. In this case, we can use either sgn (0) = $+1$ or sgn (0) = -1 for (10a), (11a), and (12a); we have two solutions for $\boldsymbol{k}$. However, it is desirable to use some convention so that we can solve for $\boldsymbol{k}$ uniquely even when $\theta = \pi$. To do this, we define sgn (0) = $+1$, so that we have unique θ and $\boldsymbol{k}$ for each rotational matrix.

In order to provide some background for later proofs, we will present the exponential representation of a general rotational matrix which was discussed in [23], [26]. Furthermore, we will express $\boldsymbol{k}$ and θ in terms of the eigenvectors and eigenvalues of a rotational matrix. A general rotational matrix can be represented as the exponent of a skew-symmetric matrix [26]

$$\mathrm{Rot}\,(\boldsymbol{k}, \theta)=e^{K\theta} \tag{13}$$

where

$$K=\begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}.$$

Lemma 1: The eigenvalues of a general rotation matrix not equal to identity are 1, $e^{j\theta}$, and $e^{-j\theta}$. Let $e^{j\theta}$ and $e^{-j\theta}$ be denoted by λ and $\bar{\lambda}$. Then θ can be calculated by

$$\theta=\text{atan }2(|\text{Re }(\lambda-\bar{\lambda})|,\ \lambda+\bar{\lambda}). \tag{14}$$

Proof: Fisher [9] has shown that the eigenvalues of K are $0, j$, and $-j$. Since these eigenvalues are distinct, K from (13) can be diagonalized [26]. Let E be the diagonalizing matrix whose columns contain linearly independent eigenvectors, we have

$$K=E\begin{bmatrix} 0 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & -j \end{bmatrix}E^{-1}. \tag{15}$$

By definition

$$e^{K\theta}=\sum_{i=0}^{\infty}\frac{(K\theta)^i}{i!}.$$

Using this definition and after simplification, we obtain

$$e^{K\theta}=E\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{j\theta} & 0 \\ 0 & 0 & e^{-j\theta} \end{bmatrix}E^{-1}. \tag{16}$$

This diagonalized form shows that the eigenvalues of $e^{K\theta}$ or Rot $(\boldsymbol{k}, \theta)$ are 1, $e^{j\theta}$, and $e^{-j\theta}$. Since $\lambda = e^{j\theta}$ and $\bar{\lambda} = e^{-j\theta}$, or $\lambda = \cos\theta + j\sin\theta$ and $\bar{\lambda} = \cos\theta - j\sin\theta$,

$$\cos\theta=\frac{1}{2}(\lambda+\bar{\lambda})$$

and

$$\sin\theta=-\frac{1}{2}j(\lambda-\bar{\lambda}).$$

Since we cannot distinguish between λ and $\bar{\lambda}$ from the eigenvalues of a rotational matrix, we should rewrite the equation for $\sin\theta$ in a way that we do not need to distinguish between λ and $\bar{\lambda}$. Knowing that $0 \leqq \theta \leqq \pi$, we have

$$\sin\theta=\left|\text{Re}\left(\frac{1}{2}(\lambda-\bar{\lambda})\right)\right|.$$

Thus we have Lemma 1.

□

Lemma 2: For a general rotation matrix not equal to identity, the eigenvector corresponding to the eigenvalue 1 can be expressed as a vector with real components and is either parallel or antiparallel to the axis of rotation. Furthermore, if the angle of rotation of the matrix is not equal to π, the remaining two eigenvectors cannot be expressed as real vectors.

Proof: Fisher [9] has shown that the eigenvectors of K are as follows: $c_1[k_x, k_y, k_z]^T$ corresponding to an eigenvalue of 0,

$$c_2\ [\sin\beta-jk_z\cos\beta,\ -\cos\beta-jk_z\sin\beta,\ j\sqrt{1-k_z^2}]^T$$

corresponding to an eigenvalue of $j\theta$, and

$$c_3\ [\sin\beta+jk_z\cos\beta,\ -\cos\beta+jk_z\sin\beta,\ -j\sqrt{1-k_z^2}]^T$$

corresponding to an eigenvalue of $-j\theta$, where c_1, c_2, and c_3 are arbitrary complex constants and $\beta = \tan^{-1}(k_y/k_x)$. From the proof of Lemma 1, we have

$$e^{K\theta}=E\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{j\theta} & 0 \\ 0 & 0 & e^{-j\theta} \end{bmatrix}E^{-1} \tag{17}$$

where E is the eigenvector matrix of K. Thus the eigenvectors of $e^{K\theta}$ corresponding to eigenvalues of 1, $e^{j\theta}$, and $e^{-j\theta}$ will be the same as the eigenvectors of K corresponding to 0, $j\theta$, and $-j\theta$, except that they may differ by a constant multiplier. We can see that the eigenvector of a rotation matrix can be expressed as a real vector (when c_1 is real), and that it is either parallel or antiparallel to the axis of rotation $\boldsymbol{k}$.

If the angle of rotation is not equal to 0 or π, the three eigenvalues are distinct and the eigenvectors associated with each eigenvalues are unique (ignoring the scaling factors) and can be written symbolically as shown earlier in this proof. The eigenvectors associated with $e^{j\theta}$ and $e^{-j\theta}$ cannot be expressed in terms of real vectors because this will require that both $\sin\beta$ and $\cos\beta$ be zero simultaneously, contradicting the identity $\sin^2\beta + \cos^2\beta = 1$. (Notice that this lemma does not hold when $\theta = \pi$. In this case, we will have -1 as an eigenvalue with multiplicity 2, and the eigenvectors associated with $e^{j\theta}$, and $e^{-j\theta}$ will no longer be unique.)

□

Lemma 3: If R is a rotation matrix and R Rot $(\boldsymbol{k}, \theta)$ = Rot $(\boldsymbol{k}, \theta)R$ and $\theta \neq 0$ or π, then R = Rot $(\boldsymbol{k}, \beta)$, where β is arbitrary.

Proof: We will first prove that R and Rot $(\boldsymbol{k}, \theta)$ have the same set of eigenvectors (up to a scaling factor). Since Rot $(\boldsymbol{k}, \theta)$ is a rotation matrix, it can be diagonalized and Rot $(\boldsymbol{k}, \theta) = E\Lambda E^{-1}$. Substituting this into R Rot $(\boldsymbol{k}, \theta)$ = Rot $(\boldsymbol{k}, \theta)R$ and rearranging, we have $\Lambda E^{-1}RE = E^{-1}RE\Lambda$. Denoting $E^{-1}RE$ by R', we have $\Lambda R' = R'\Lambda$. From Lemma 1, the eigenvectors of Rot $(\boldsymbol{k}, \theta)$ are 1, $e^{j\theta}$, and $e^{-j\theta}$. Rewritting R' in terms of its 9 elements (r_1 to r_9), we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{j\theta} & 0 \\ 0 & 0 & e^{-j\theta} \end{bmatrix}\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{j\theta} & 0 \\ 0 & 0 & e^{-j\theta} \end{bmatrix}. \tag{18}$$

Expanding the above, we have

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 e^{j\theta} & r_5 e^{j\theta} & r_6 e^{j\theta} \\ r_7 e^{-j\theta} & r_8 e^{-j\theta} & r_9 e^{-j\theta} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 e^{j\theta} & r_3 e^{-j\theta} \\ r_4 & r_5 e^{j\theta} & r_6 e^{-j\theta} \\ r_7 & r_8 e^{j\theta} & r_9 e^{-j\theta} \end{bmatrix}. \quad (19)$$

Equating elements of both sides and knowing $\theta \neq 0$ or π, we can conclude that all but the diagonal elements of R' are zero. Recalling that $R = ER'E^{-1}$, we now have

$$R = E \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_5 & 0 \\ 0 & 0 & r_9 \end{bmatrix} E^{-1}. \quad (20)$$

Thus R must have the same set of eigenvectors as Rot $(\boldsymbol{k}, \theta)$, except the scaling constants.

If β is the angle of rotation of R, then the eigenvalues r_1, r_5, and r_9 must be a certain permutation of 1, $e^{j\beta}$, and $e^{-j\beta}$. In fact, $r_1 = 1$, otherwise a contradiction will result when $\beta \neq 0$ or π. From Lemma 2, Rot $(\boldsymbol{k}, \theta)$ has one eigenvector (first column of E) corresponding to an eigenvalue of 1 and the remaining two eigenvectors (second and third columns of E) are complex. If r_1 in (20) is not one, then either r_5 or r_9 equals one and its associated eigenvectors (second or third column of E) must be real. This contradicts that both the second and third columns of E are complex.

From Lemma 2, the real eigenvector corresponding to an eigenvalue of one is either parallel or antiparallel to the axis of rotation. Since Rot $(\boldsymbol{k}, \theta)$ and R have the same eigenvector associated with an eigenvalue of one, they must have their axes of rotation parallel or antiparallel to one another and R can be expressed as Rot $(\boldsymbol{k}, \beta)$, where β is arbitrary.

□

III. Solution to the Equation $AX = XB$

We will solve for the rotational and translational components of X separately in order to make the geometric interpretation easier. Dividing a homogeneous transform into its rotational and translational components, $AX = XB$ becomes

$$\begin{bmatrix} \boldsymbol{R}_A & \boldsymbol{P}_A \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_X & \boldsymbol{P}_X \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_X & \boldsymbol{P}_X \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_B & \boldsymbol{P}_B \\ \boldsymbol{0} & 1 \end{bmatrix} \quad (21)$$

where $\boldsymbol{R}$ is a 3×3 rotational matrix, $\boldsymbol{P}$ is a 3×1 translation vector, and $\boldsymbol{0}$ is a row of 3 zeros. Multiplying out and equating the first row of (21), we have

$$\boldsymbol{R}_A \boldsymbol{R}_X = \boldsymbol{R}_X \boldsymbol{R}_B \quad (22)$$

and

$$\boldsymbol{R}_A \boldsymbol{P}_X + \boldsymbol{P}_A = \boldsymbol{R}_X \boldsymbol{P}_B + \boldsymbol{P}_X. \quad (23)$$

We will show that R_A and R_B have the same angle of rotation and that the rotational matrix R_X has one degree of freedom. Also, if R_X is fixed, P_X has one degree of freedom.

Lemma 4: If R_A and R_B are rotation matrices such that $R_A R = R R_B$ for any rotation matrix R, then R_A and R_B must have the same angle of rotation.

Proof: From Lemma 1, the product of the eigenvalues of a rotational matrix is 1. Thus a rotational matrix has a determinant of 1 and is always invertible. R_A and R_B are similar, since $R_A = RR_BR^{-1}$. R_A and R_B must have the same eigenvalues since similar matrices have the same eigenvalues [26]. From Lemma 1, R_A and R_B must have the same angle of rotation.

□

Before we formally state and prove the solution to $R_A R_X = R_X R_B$ in Theorem 1, we first examine the geometry of the problem. Let us rewrite R_A and R_B as Rot $(\boldsymbol{k}_A, \theta)$ and Rot $(\boldsymbol{k}_B, \theta)$, respectively. We will show that $\boldsymbol{k}_A$ referenced to the base frame (${}^{\text{base}}\boldsymbol{k}_A$) and $\boldsymbol{k}_B$ references to the frame R_X (${}^{R_X}\boldsymbol{k}_B$) both point in the same direction if a common frame of reference is used. Notice that, from Lemma 4, R_A and R_B have the same angle of rotation. We can now rewrite (22) as

$$\text{Rot}(\boldsymbol{k}_A, \theta) R_X = R_X \text{Rot}(\boldsymbol{k}_B, \theta). \quad (24)$$

For the following discussion, we will think of R_X as a coordinate frame relative to the base frame. Using the geometrical interpretation of post-multiplication of homogeneous transforms [28], the left side of the equation can be interpreted as rotation of R_X frame with respect to ${}^{\text{base}}\boldsymbol{k}_A$ by an angle θ. Similarly, the right hand side of the equation is the rotation of R_X frame with respect to ${}^{R_X}\boldsymbol{k}_B$ by θ. As a result, (24) can be interpreted as follows: R_X is a coordinate frame such that rotating R_X about a vector ${}^{\text{base}}\boldsymbol{k}_A$ by any angle β is equivalent to rotating R_X about ${}^{R_X}\boldsymbol{k}_B$ by the same amount, where ${}^{\text{base}}\boldsymbol{k}_A$ is referenced with respect to the base frame (the world frame), and ${}^{R_X}\boldsymbol{k}_B$ is referenced with respect to R_X. This is shown in Fig. 4. In order that rotating R_X about ${}^{\text{base}}\boldsymbol{k}_A$ being the same as rotating it about ${}^{R_X}\boldsymbol{k}_B$, ${}^{\text{base}}\boldsymbol{k}_A$ and ${}^{R_X}\boldsymbol{k}_B$ must be the same physical vector in three-dimensional space.

We will now show that the solution to (24) has one degree of rotational freedom. A formal proof will be given in Theorem 1. If R_X is a solution to (24) and it is rotated about the axis of rotation (${}^{R_X}\boldsymbol{k}_B$ or ${}^{\text{base}}\boldsymbol{k}_A$) by an angle, it will still satisfy (24). Thus the solution to (24) has one degree of freedom. To show this mathematically, rotation of a particular solution R_{XP} about the axis by any angle β can be written as R_{XP} Rot $(\boldsymbol{k}_B, \beta)$ or Rot $(\boldsymbol{k}_A, \beta) R_{XP}$. We will use the later form for the rest of the paper. Since R_{XP} is a particular solution

$$\text{Rot}(\boldsymbol{k}_A, \theta) R_{XP} = R_{XP} \text{Rot}(\boldsymbol{k}_B, \theta).$$

Also, since

$$\text{Rot}(\boldsymbol{k}_A, -\beta)\, \text{Rot}(\boldsymbol{k}_A, \beta) = I$$

$$\text{Rot}(\boldsymbol{k}_A, \theta)\, \text{Rot}(\boldsymbol{k}_A, -\beta)\, \text{Rot}(\boldsymbol{k}_A, \beta) R_{XP} = R_{XP}\, \text{Rot}(\boldsymbol{k}_B, \theta).$$

Using the commutative properties of rotational matrices with a common axis of rotation and that

$$\text{Rot}(\boldsymbol{k}_A, -\beta)^{-1} = \text{Rot}(\boldsymbol{k}_A, \beta)$$

we have

$$\text{Rot}(\boldsymbol{k}_A, \theta)\, \text{Rot}(\boldsymbol{k}_A, \beta) R_{XP} = \text{Rot}(\boldsymbol{k}_A, \beta) R_{XP}\, \text{Rot}(\boldsymbol{k}_B, \theta)$$

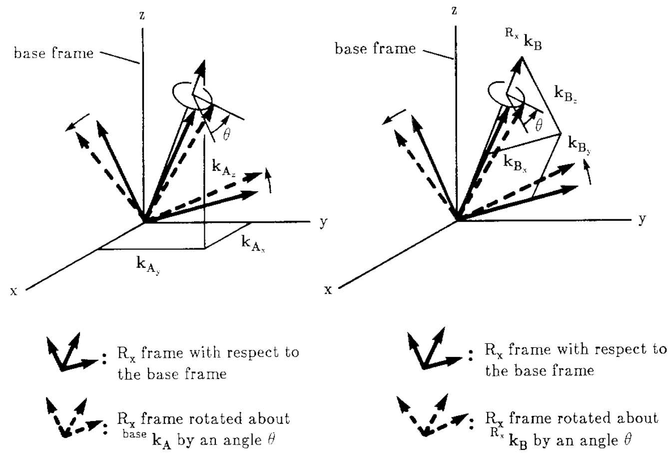

Fig. 4. Rotating R_X about ${}^{\text{base}}\boldsymbol{k}_A$ by θ is equivalent to rotating R_X about ${}^{R_X}\boldsymbol{k}_B$ by the same angle. $\boldsymbol{k}_A$ is the axis of rotation of A and $\boldsymbol{k}_B$ is the axis of rotation of B in the homogeneous transform equation $AX = XB$.

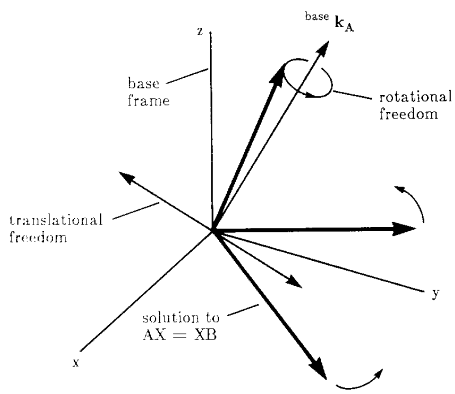

Fig. 5. The rotational and translational degrees of freedom of the solution to $AX = XB$. The frame in the figure can rotate about ${}^{\text{base}}\boldsymbol{k}_A$ and slide along the axis as shown.

from which we can see that Rot $(\boldsymbol{k}_A, \beta)R_{XP}$ is a solution. In Fig. 5, it is shown that a general solution has one degree of rotational freedom; any particular solution rotated about ${}^{\text{base}}\boldsymbol{k}_A$ by any angle is also a solution.

Definition: A homogeneous transform equation of the form $AX = XB$ is *solvable* if there exists a homogeneous transform U such that $B = U^{-1}AU$.

Theorem 1: The general solution to the rotational part of a solvable homogeneous transform equation of the form $R_A R_X = R_X R_B$, the angle of rotation of A being neither 0 nor π, is

$$R_X = \text{Rot}\,(\boldsymbol{k}_A, \beta) R_{XP} \tag{25}$$

where $\boldsymbol{k}_A$ is the axis of rotation of R_A, R_{XP} is a particular solution to the equation, and β is any arbitrary angle.

Proof: Assume Rot $(\boldsymbol{k}_A, \beta)R_{XP}$ is not a general solution. Then, there must exist some rotation matrix R' such that

$$R_A R' = R' R_B \tag{26}$$

and $R' \neq$ Rot $(\boldsymbol{k}_A, \beta)R_{XP}$ for any β. Since R_{XP} is a particular solution to (22), $R_A R_{XP} = R_{XP} R_B$, or $R_B = R_{XP}^{-1} R_A R_{XP}$. Substituting this into (26) we have

$$R'^{-1} R_A R' = R_{XP}^{-1} R_A R_{XP}. \tag{27}$$

Rewriting R_A as Rot $(\boldsymbol{k}_A, \theta)$ and rearranging, we have

$$\text{Rot}\,(\boldsymbol{k}_A, \theta) R' R_{XP}^{-1} = R' R_{XP}^{-1} \text{Rot}\,(\boldsymbol{k}_A, \theta). \tag{28}$$

Thus Rot $(\boldsymbol{k}_A, \theta)$ and $R' R_{XP}^{-1}$ are commutative. Moreover, we know that $\theta \neq 0$ or π. If $R' R_{XP}^{-1} \neq I$, from Lemma 3, the axis of rotation of $R' R_{XP}^{-1}$ must be parallel or antiparallel to $\boldsymbol{k}_A$. Thus there must exist a γ such that $R' R_{XP}^{-1} =$ Rot $(\boldsymbol{k}_A, \gamma)$. We have $R' =$ Rot $(\boldsymbol{k}_A, \gamma) R_{XP}$, which is a contradiction. If $R' R_{XP}^{-1} = I$, $R' = R_{XP}$ Rot $(\boldsymbol{k}_A, 0)$, which is also a contradiction.

□

Next we will look at the translation part of the equation $AX = XB$. It has one degree of freedom, as shown in Fig. 5. From (23), we have

$$(R_A - I) P_X = R_X P_B - P_A. \tag{29}$$

If R_X is already solved for, the only unknown in this equation will be P_X. We thus have a system of 3 linear equations having the x, y, and z components of P_X as unknown. P_X has one degree of freedom because $(R_A - I)$ has a rank of two, as will be shown next in Theorem 2.

Theorem 2: The translational part (P_X) of the solution to a solvable homogeneous transform equation $AX = XB$, where $R_A \neq I$ and $R_B \neq I$, has one degree of freedom.

Proof: We can see that $R_A - I$ is similar to a matrix of rank two if $R_A \neq I$, since

$$R_A - I = E\Lambda_A E^{-1} - EIE^{-1} = E \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \bar{\lambda} - 1 \end{bmatrix} E^{-1}. \tag{30}$$

Thus $R_A - I$ must have a rank of two. Thus from (29), there may be no solution or there are infinite number of solutions to P_X. The first case is ruled out since the physical system guarantees the existence of a solution. The solution must exist and consist of all the vectors in the null space of $R_A - I$ translated by a particular solution to (29) [30]. The null space of $R_A - I$ has a dimension of 3 − rank $(R_A - I)$, thus the solution to (29) has one degree of freedom.

□

Finally, we need to find a particular solution to the rotational part of $AX = XB$. From the geometric interpretation of the general solution, we will show that any transformation that rotates $\boldsymbol{k}_B$ into $\boldsymbol{k}_A$ is a solution.

Lemma 5:

$$\text{Rot}\,(R\boldsymbol{k}, \theta) = R\,\text{Rot}\,(\boldsymbol{k}, \theta)R^{-1} \tag{31}$$

for any axis of rotation $\boldsymbol{k}$, any $\theta \subset [0, \pi]$, and any 3×3 rotation matrix R.

Proof: For the purpose of this proof, we will represent a rotation matrix in a form used by [23]. Let $[\boldsymbol{n}\ \boldsymbol{o}\ \boldsymbol{a}]$ be a homogeneous transform and $[\boldsymbol{n}'\ \boldsymbol{o}'\ \boldsymbol{a}']$ be the former transform rotated by Rot $(\boldsymbol{k}, \theta)$. Thus

$$\text{Rot}\,(\boldsymbol{k}, \theta) = [\boldsymbol{n}'\ \boldsymbol{o}'\ \boldsymbol{a}'][\boldsymbol{n}\ \boldsymbol{o}\ \boldsymbol{a}]^{-1}. \tag{32}$$

If we premultiply $\boldsymbol{n}, \boldsymbol{o}, \boldsymbol{a}, \boldsymbol{n}', \boldsymbol{o}', \boldsymbol{a}'$, and $\boldsymbol{k}$ by R, the angular relationship between $R\boldsymbol{n}, R\boldsymbol{o}, R\boldsymbol{a}, R\boldsymbol{n}', R\boldsymbol{o}', R\boldsymbol{a}'$, and $R\boldsymbol{k}$ will be the same as before the premultiplication, because of the angular preservation property of R as a rotational matrix. Since $\boldsymbol{n}' = \text{Rot}\,(\boldsymbol{k}, \theta)\boldsymbol{n}$ before the premultiplication, $R\boldsymbol{n}' = \text{Rot}\,(R\boldsymbol{k}, \theta)R\boldsymbol{n}$. Similar relationships hold for other vectors as well; therefore

$$[R\boldsymbol{n}'\ R\boldsymbol{o}'\ R\boldsymbol{a}'] = \text{Rot}\,(R\boldsymbol{k}, \theta)[R\boldsymbol{n}\ R\boldsymbol{o}\ R\boldsymbol{a}]$$

and

$$\text{Rot}\,(R\boldsymbol{k}, \theta) = [R\boldsymbol{n}'\ R\boldsymbol{o}'\ R\boldsymbol{a}'][R\boldsymbol{n}\ R\boldsymbol{o}\ R\boldsymbol{a}]^{-1}. \tag{33}$$

From (33)

$$\begin{aligned}\text{Rot}\,(R\boldsymbol{k}, \theta) &= R[\boldsymbol{n}'\ \boldsymbol{o}'\ \boldsymbol{a}'][\boldsymbol{n}\ \boldsymbol{o}\ \boldsymbol{a}]^{-1}R^{-1}\\ &= R\,\text{Rot}\,(\boldsymbol{k}, \theta)R^{-1}.\end{aligned}$$

□

Theorem 3: Any rotation matrix R that satisfies

$$\boldsymbol{k}_A = R\boldsymbol{k}_B \tag{34}$$

is a solution to

$$R_A R_X = R_X R_B \tag{35}$$

where $\boldsymbol{k}_A$ is the axis of rotation of R_A and $\boldsymbol{k}_B$ is the axis of rotation of R_B.

Proof: Let us rewrite (35) as

$$\text{Rot}\,(\boldsymbol{k}_A, \theta)R_X = R_X\,\text{Rot}\,(\boldsymbol{k}_B, \theta). \tag{36}$$

Substituting R into R_X and $R\boldsymbol{k}_B$ into $\boldsymbol{k}_A$, the left-hand side becomes Rot $(R\boldsymbol{k}_B, \theta)R$. By Lemma 5, this becomes $R\,\text{Rot}\,(\boldsymbol{k}_B, \theta)R^{-1}R = R\,\text{Rot}\,(\boldsymbol{k}_B, \theta)$, which is the same as the right-hand side when R_X is replaced by R.

□

Since any rotational matrix R satisfying (34) is a particular solution, one method to find a particular solution is a rotation about an axis perpendicular to both $\boldsymbol{k}_B$ and $\boldsymbol{k}_A$. Thus

$$R_{XP} = \text{Rot}\,(\boldsymbol{v}, \omega) \tag{37}$$

where

$$\boldsymbol{v} = \boldsymbol{k}_B \times \boldsymbol{k}_A \tag{38}$$

and

$$\omega = \text{atan}\,2(|\boldsymbol{k}_B \times \boldsymbol{k}_A|, \boldsymbol{k}_B \cdot \boldsymbol{k}_A). \tag{39}$$

The above method will not work when $\boldsymbol{k}_A$ and $\boldsymbol{k}_B$ are parallel or antiparallel to one another since it will produce a zero vector. However, particular solutions for these two special cases can be found easily by other methods. In the first case, the identity matrix will be a valid particular solution. In the second, case, any rotation matrix with its rotation axis perpendicular to $\boldsymbol{k}_A$ and its angle of rotation equal to π will be a particular solution.

IV. Solving for a Unique Solution Using Two Simultaneous Equations

We have seen that the solution to a homogeneous transform equation of the form $AX = XB$ has two degrees of freedom. However, in our application, we need to find a unique solution for ${}^{T_6}T_{\text{CAM}}$. We can find a unique solution to this equation if we have two equations of the form

$$A_1 X = X B_1 \tag{40}$$

and

$$A_2 X = X B_2. \tag{41}$$

In order to obtain two such equations, we need to move the robot twice and use the vision system to find the corresponding changes in the camera frame. It is also desirable to know when this method will not yield a unique solution and the physical interpretation of this situation.

A unique solution to R_X (the rotational part of X) can be found by associating the general solutions of the two equations $R_{A_1}R_X = R_X R_{B_1}$ and $R_{A_2}R_X = R_X R_{B_2}$. Let R_{XP_1} Rot $(\boldsymbol{k}_{A_1}, \beta_1)$ and R_{XP_2} Rot $(\boldsymbol{k}_{A_2}, \beta_2)$ be the general solutions to the above two equations, we then have

$$\text{Rot}\,(\boldsymbol{k}_{A_1}, \beta_1)R_{XP_1} = \text{Rot}\,(\boldsymbol{k}_{A_2}, \beta_2)R_{XP_2}. \tag{42}$$

Let the particular solutions be written as follows:

$$R_{XP_i} = \begin{bmatrix} n_{x_i} & o_{x_i} & a_{x_i} & p_{x_i} \\ n_{y_i} & o_{y_i} & a_{y_i} & p_{y_i} \\ n_{z_i} & o_{z_i} & a_{z_i} & p_{z_i} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad i = 1, 2. \tag{43}$$

Rearranging and writing it in more condensed form, we have

$$\begin{bmatrix} -n_{x_1}+k_{x_1}\boldsymbol{n}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{n}_1\times\boldsymbol{k}_{A_1})_x & n_{x_2}-k_{x_2}\boldsymbol{n}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{n}_2\times\boldsymbol{k}_{A_2})_x \\ -o_{x_1}+k_{x_1}\boldsymbol{o}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{o}_1\times\boldsymbol{k}_{A_1})_x & o_{x_2}-k_{x_2}\boldsymbol{o}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{o}_2\times\boldsymbol{k}_{A_2})_x \\ -a_{x_1}+k_{x_1}\boldsymbol{a}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{a}_1\times\boldsymbol{k}_{A_1})_x & a_{x_2}-k_{x_2}\boldsymbol{a}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{a}_2\times\boldsymbol{k}_{A_2})_x \\ -n_{y_1}+k_{y_1}\boldsymbol{n}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{n}_1\times\boldsymbol{k}_{A_1})_y & n_{y_2}-k_{y_2}\boldsymbol{n}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{n}_2\times\boldsymbol{k}_{A_2})_y \\ -o_{y_1}+k_{y_1}\boldsymbol{o}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{o}_1\times\boldsymbol{k}_{A_1})_y & o_{y_2}-k_{y_2}\boldsymbol{o}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{o}_2\times\boldsymbol{k}_{A_2})_y \\ -a_{y_1}+k_{y_1}\boldsymbol{a}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{a}_1\times\boldsymbol{k}_{A_1})_y & a_{y_2}-k_{y_2}\boldsymbol{a}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{a}_2\times\boldsymbol{k}_{A_2})_y \\ -n_{z_1}+k_{z_1}\boldsymbol{n}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{n}_1\times\boldsymbol{k}_{A_1})_z & n_{z_2}-k_{z_2}\boldsymbol{n}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{n}_2\times\boldsymbol{k}_{A_2})_z \\ -o_{z_1}+k_{z_1}\boldsymbol{o}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{o}_1\times\boldsymbol{k}_{A_1})_z & o_{z_2}-k_{z_2}\boldsymbol{o}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{o}_2\times\boldsymbol{k}_{A_2})_z \\ -a_{z_1}+k_{z_1}\boldsymbol{a}_1\cdot\boldsymbol{k}_{A_1} & (\boldsymbol{a}_1\times\boldsymbol{k}_{A_1})_z & a_{z_2}-k_{z_2}\boldsymbol{a}_2\cdot\boldsymbol{k}_{A_2} & (-\boldsymbol{a}_2\times\boldsymbol{k}_{A_2})_z \end{bmatrix} \begin{bmatrix} \cos\beta_1 \\ \sin\beta_1 \\ \cos\beta_2 \\ \sin\beta_2 \end{bmatrix} = \begin{bmatrix} -k_{x_2}\boldsymbol{n}_2\cdot\boldsymbol{k}_{A_2}+k_{x_1}\boldsymbol{n}_1\cdot\boldsymbol{k}_{A_1} \\ -k_{x_2}\boldsymbol{o}_2\cdot\boldsymbol{k}_{A_2}+k_{x_1}\boldsymbol{o}_1\cdot\boldsymbol{k}_{A_1} \\ -k_{x_2}\boldsymbol{a}_2\cdot\boldsymbol{k}_{A_2}+k_{x_1}\boldsymbol{a}_1\cdot\boldsymbol{k}_{A_1} \\ -k_{y_2}\boldsymbol{n}_2\cdot\boldsymbol{k}_{A_2}+k_{y_1}\boldsymbol{n}_1\cdot\boldsymbol{k}_{A_1} \\ -k_{y_2}\boldsymbol{o}_2\cdot\boldsymbol{k}_{A_2}+k_{y_1}\boldsymbol{o}_1\cdot\boldsymbol{k}_{A_1} \\ -k_{y_2}\boldsymbol{a}_2\cdot\boldsymbol{k}_{A_2}+k_{y_1}\boldsymbol{a}_1\cdot\boldsymbol{k}_{A_1} \\ -k_{z_2}\boldsymbol{n}_2\cdot\boldsymbol{k}_{A_2}+k_{z_1}\boldsymbol{n}_1\cdot\boldsymbol{k}_{A_1} \\ -k_{z_2}\boldsymbol{o}_2\cdot\boldsymbol{k}_{A_2}+k_{z_1}\boldsymbol{o}_1\cdot\boldsymbol{k}_{A_1} \\ -k_{z_2}\boldsymbol{a}_2\cdot\boldsymbol{k}_{A_2}+k_{z_1}\boldsymbol{a}_1\cdot\boldsymbol{k}_{A_1} \end{bmatrix} \tag{44}$$

where the notation $(\boldsymbol{u} \times \boldsymbol{v})_w$ denotes the w component of the cross product $\boldsymbol{u} \times \boldsymbol{v}$. Equation (44) is a system of linear equations involving $\cos\beta_1$, $\sin\beta_1$, $\cos\beta_2$, and $\sin\beta_2$. Once these values are solved for, we can find β_1 and β_2 by $\beta_1 = \text{atan }2(\sin\beta_1, \cos\beta_1)$ and $\beta_2 = \text{atan }2(\sin\beta_2, \cos\beta_2)$. Since we have more equations than unknowns, from the point of view of linear algebra, we can have a system of inconsistent equations. However, in an ideal environment where there is no noise, the equations must be consistent because they originated from physical situations. Since the linear equations are physically constrained to be consistent, there are either a unique solution or an infinite number or solutions; there are no other possibilities. We will show in Theorem 4 that the solution is unique when $\boldsymbol{k}_{A_1}$ and $\boldsymbol{k}_{A_2}$ are neither parallel or antiparallel to one another and the angles of rotation of A_1 and A_2 are neither 0 nor π. Let us abbreviate (44) to $CY = D$, if rank $(C) = 4$, we can find four linearly independent rows of C to solve for Y uniquely. However, in real applications where noise is present, we can find a least square fit solution $\hat{Y}$ by

$$\hat{Y} = (C^TC)^{-1}C^TD. \tag{45}$$

The translational part of X is constrained by (23); thus we have

$$R_{A_1}P_X + P_{A_1} = R_XP_{B_1} + P_X$$

and

$$R_{A_2}P_X + P_{A_2} = R_XP_{B_2} + P_X.$$

Combining these two equations, we can solve for P_X by

$$\begin{bmatrix} R_{A_1} - I \\ R_{A_2} - I \end{bmatrix} P_X = \begin{bmatrix} R_XP_{B_1} - P_{A_1} \\ R_XP_{B_2} - P_{A_2} \end{bmatrix}. \tag{46}$$

Like the uniqueness conditions for the rotational part, it will be shown that the translational part will have a unique solution if the rotation axes of A_1 and A_2 are neither parallel nor antiparallel to one another and the angles of rotation are neither 0 nor π. Rewriting (46) as $EP_X = F$, a least square fit solution can be calculated by

$$\hat{P}_X = (E^TE)^{-1}E^TF. \tag{47}$$

Before we go into the necessary conditions for uniqueness, we need to prove two more lemmas.

Lemma 6: If R is a 3×3 rotational part of a homogeneous transform and its angle of rotation is neither 0 nor π, any row of $(R - I)$ is a linear combination of the transposes of the two eigenvectors corresponding to the two nonunity eigenvalues of R.

Proof: From (30) we have

$$R - I = [\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \bar{\lambda} - 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1^T \\ \boldsymbol{e}_2^T \\ \boldsymbol{e}_3^T \end{bmatrix} \tag{48}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are the eigenvectors of R corresponding to the eigenvalues 1, λ, and $\bar{\lambda}$. Writing $\boldsymbol{e}_i^T$ as $(e_{i_x}, e_{i_y}, e_{i_z})$ and rearranging (48), we have

$$R - I = (\lambda - 1) \begin{bmatrix} e_{2_x}\boldsymbol{e}_2^T \\ e_{2_y}\boldsymbol{e}_2^T \\ e_{2_z}\boldsymbol{e}_2^T \end{bmatrix} + (\bar{\lambda} - 1) \begin{bmatrix} e_{3_x}\boldsymbol{e}_3^T \\ e_{3_y}\boldsymbol{e}_3^T \\ e_{3_z}\boldsymbol{e}_3^T \end{bmatrix}. \tag{49}$$

□

Lemma 7: For two rotational matrices R_1 and R_2 whose axes of rotation are neither parallel nor antiparallel to one another and whose angles of rotation are neither 0 nor π, it is impossible that the sets of vectors $\{\boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{f}_2\}$ and $\{\boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{f}_3\}$ are both linearly dependent, where $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the eigenvectors of R_1 corresponding to the nonunity eigenvalues of R_1, and $\boldsymbol{f}_2$ and $\boldsymbol{f}_3$ are the eigenvectors of R_2 corresponding to the nonunity eigenvalues of R_2.

Proof: For any rotational matrix R and its Hermitian R^H, $RR^H = R^HR = I$; hence R is a normal matrix [27]. Given that the angle of rotation of R is neither 0 nor π, R must have distinct eigenvalues. From [27, Key Theorem 9.2] in Nobel's text, a matrix formed by 3 column eigenvectors of a normal matrix with distinct eigenvalues is Hermitian. Hence any eigenvector matrix of R is Hermitian. Let $\boldsymbol{e}_1$ be the eigenvector of R_1 corresponding to the unity eigenvalue. Note that $\boldsymbol{e}_1 \cdot \boldsymbol{f}_2$ and $\boldsymbol{e}_1 \cdot \boldsymbol{f}_3$ cannot be zero simultaneously. If they are simultaneously zero, we will have a system of two linearly independent equations which will constrain $\boldsymbol{e}_1$ except for a scaling factor. Since the eigenvectors of R_2 are Hermitian, $\boldsymbol{f}_1 \cdot \boldsymbol{f}_2$ and $\boldsymbol{f}_1 \cdot \boldsymbol{f}_3$ are zero. Similarly, this will constrain $\boldsymbol{f}_1$ up to a scaling factor. Thus $\boldsymbol{f}_1$ and $\boldsymbol{e}_1$ must be scalar products of one another. However, this contradicts the assumption that the axes of rotation (e_1 and f_1) are neither parallel nor antiparallel to one another. Therefore, the two dot products cannot be zero simultaneously. To prove that $\{e_2, e_3, f_2\}$ is linearly independent, we need to prove that $k_1 = k_2 = k_3 = 0$ if

$$k_1 e_2 + k_2 e_3 + k_3 f_2 = 0. \tag{50}$$

Taking the dot product of both sides of (50) with e_1 and using the fact that eigenvectors of a normal matrix with distinct eigenvalues are orthogonal to each other, we will have $k_3 e_1 \cdot f_2 = 0$. If $e_1 \cdot f_2 \neq 0$, then $k_3 = 0$. Equation (50) simplifies to

$$k_1 e_2 + k_2 e_3 = 0. \tag{51}$$

Since e_2 are e_3 are linearly independent, we have $k_1 = k_2 = 0$. Therefore, $\{e_2, e_3, f_2\}$ are linearly independent if $e_1 \cdot f_2 \neq 0$. When $e_1 \cdot f_2 = 0$, $e_1 \cdot f_3$ must be nonzero, from a previous argument in this proof. In this case, we can use a similar method to prove that $\{e_2, e_3, f_3\}$ is linearly independent. □

Theorem 4: A consistent system of two solvable homogeneous transform equations of the form $A_1 X = XB_1$ and $A_2 X = XB_2$ has a unique solution if the axes of rotation for A_1 and A_2 are neither parallel nor antiparallel to one another and the angles of rotations of A_1 and A_2 are neither 0 nor π.

Proof for the Rotational Part: We have already seen that the general solution to $AX = XB$ has one degree of rotational freedom when the angle of rotation of A is neither 0 nor π; any solution revolving about k_A is still a solution. The solution to the system of homogeneous transform equations $A_1 X = XB_1$ and $A_2 X = XB_2$ is found by equating the solutions of the 2 individual equations, as shown in (42). Since (42) is independent of the choices of the particular solutions, we can simplify it by choosing a particular solution which is a solution to both equations; i.e., $R_{XP_0} = R_{XP_1} = R_{XP_2}$. After replacing R_{XP_1} and R_{XP_2} in (44) by R_{XP_0}, R_{XP_0} cancels out and we have

$$\begin{bmatrix} 1-k_{x_1}^2 & 0 & k_{x_2}^2-1 & 0 \\ -k_{x_1}k_{y_1} & -k_{z_1} & k_{x_2}k_{y_2} & k_{z_2} \\ -k_{x_1}k_{z_1} & k_{y_1} & k_{x_2}k_{z_2} & -k_{y_2} \\ -k_{x_1}k_{y_1} & k_{z_1} & k_{x_2}k_{y_2} & -k_{z_2} \\ 1-k_{y_1}^2 & 0 & k_{y_2}^2-1 & 0 \\ -k_{y_1}k_{z_1} & -k_{x_1} & k_{y_2}k_{z_2} & k_{x_2} \\ -k_{x_1}k_{z_1} & -k_{y_1} & k_{x_2}k_{z_2} & k_{y_2} \\ -k_{y_1}k_{z_1} & k_{x_1} & k_{y_2}k_{z_2} & -k_{x_2} \\ 1-k_{z_1}^2 & 0 & k_{z_2}^2-1 & 0 \end{bmatrix} \begin{bmatrix} \cos\beta_1 \\ \sin\beta_1 \\ \cos\beta_2 \\ \sin\beta_2 \end{bmatrix} = \begin{bmatrix} k_{x_2}^2-k_{x_1}^2 \\ k_{x_2}k_{y_2}-k_{x_1}k_{y_1} \\ k_{x_2}k_{z_2}-k_{x_1}k_{z_1} \\ k_{x_2}k_{y_2}-k_{x_1}k_{y_1} \\ k_{y_2}-k_{y_1} \\ k_{y_2}k_{z_2}-k_{y_1}k_{z_1} \\ k_{x_2}k_{z_2}-k_{x_1}k_{z_1} \\ k_{y_2}k_{z_2}-k_{y_1}k_{z_1} \\ k_{x_2}^2-k_{z_1}^2 \end{bmatrix}. \tag{52}$$

Let us abbreviate (52) as $C'Y' = D'$. With the assumption of consistency, a unique solution exists if and only if the rank of Y' is 4, in which case we can pick 4 linearly rows to form 4 equations to solve for the same number of unknowns. Since the rank of C' is the same as the rank of C'^TC' and that the later is a 4×4 matrix, C' has a rank of 4 if and only if C'^TC' has full rank. Thus we will have a unique solution iff the determinant of C'^TC' is not equal to zero. We have used the SMP program [19] to express the determinant of C'^TC' in symbolic form and have simplified it by making the following substitutions:

1) $k_{x_i}^2 + k_{y_i}^2 + k_{z_i}^2 = 1, \qquad i = 1, 2.$
2) $k_{x_1}k_{x_2} + k_{y_1}k_{y_2} + k_{z_1}k_{z_2} = k_{A_1} \cdot k_{A_2}.$
3) $1 - k_{x_1}^2 k_{x_2}^2 - k_{y_1}^2 k_{y_2}^2 - k_{z_1}^2 k_{z_2}^2 - 2k_{x_1}k_{x_2}k_{y_1}k_{y_2} - 2k_{x_1}k_{x_2}k_{z_1}k_{z_2} - 2k_{y_1}k_{y_2}k_{z_1}k_{z_2} = \sin^2\theta_{12}.$

The third substitution comes from the fact that $|k_{A_1} \times k_{A_2}|$ equals $|k_{A_1}||k_{A_2}| \sin\theta_{12}$. The determinant is finally simplified to

$$\det(C'^TC') = 4\sin^2\theta_{12}(\sin^2\theta_{12} - 4)(k_{A_1} \cdot k_{A_2} + 1) \cdot (k_{A_1} \cdot k_{A_2} - 1). \tag{53}$$

The determinant is zero when $\sin\theta_{12} = \pm 2$, which is impossible, when $\sin\theta_{12} = 0$, and when $k_{A_1} \cdot k_{A_2} = \pm 1$. Thus we will have a nonunique solution only when k_{A_1} and k_{A_2} are parallel or antiparallel to one another. □

Proof for the Translational Part: Since E is a 6×3 matrix, we have 6 equations and 3 unknowns. We know that these equations cannot be inconsistent since they originated from physical conditions. Therefore, we have a unique solution for P_X if and only if matrix E has a rank of 3, in which case we can pick 3 linearly independent rows for E to solve for P_X. From Lemma 6, any row of $(R_{A_1} - I)$ is a linear combination of the transposes of the eigenvectors e_2^T and e_3^T corresponding to the nonunity eigenvalues, and any row of $(R_{A_2} - I)$ is a linear combination of the transposes of the eigenvectors f_2^T and f_3^T corresponding to the nonunity eigenvalues. Since the rank of R_{A_1} is two (from the proof of Theorem 2), we can pick two linear independent rows from it, both are linear combinations of e_2 and e_3. We can also pick a row from R_{A_2}, which is a linear combination of f_2 and f_3, and combine it with the two rows from R_{A_1}. Since we know that if k_{A_1} is not aligned with k_{A_2}, from Lemma 7, at least one of f_2^T and f_3^T must be linearly independent from e_2^T and e_3^T. Say a row from R_{X_2} is $af_2^T + bf_3^T$. We can always pick a row where $a \neq 0$ or a row where $b \neq 0$ since rank $(R_{A_2}) = 2$. Thus we can always find a row from R_{A_2} and combine it with two rows from R_{A_1} to form three linearly independent rows. We can use the corresponding three equations from (46) to solve for a unique P_X. □

V. An Example

We have written a program calling IMSL routines [18] to test our method. A single-precision version is used on a VAX

780 machine. We will solve for the sensor position relative to the robot wrist by moving the robot twice and observing the changes in the sensor positions. The two robot movements must have distinct axes of rotation and their angles of rotation must not be 0 or π in order to ensure a unique solution. Let A_1 and B_1 be the first robot movement and B_1 be the resulting motion of the sensor, and let A_2 be the second robot movement and B_2 the resulting sensor motion. Two equations relating the motions and the sensor-mounting position will result:

$$A_1X = XB_1 \tag{54}$$

$$A_2X = XB_2. \tag{55}$$

B_1 and B_2 are determined by A_1 and A_2 and the actual sensor mounting position. Let X_{act} be the actual sensor mounting position, then

$$B_1 = X_{act}^{-1} A_1 X_{act} \tag{56}$$

$$B_2 = X_{act}^{-1} A_2 X_{act}. \tag{57}$$

The above two equations are only used for simulations. In an actual robot application, B_1 and B_2 are found by the sensor system; however, A_1 and B_1, and A_2 and B_2 are still related by (56) and (57).

Assume the actual sensor mounting position and two robot motions are as follows:

$$X_{act} = \text{Trans}\ (10\ \text{mm},\ 50\ \text{mm},\ 100\ \text{mm}) \cdot \text{Rot}\ ([1,\ 0,\ 0]^T,\ 0.2\ \text{rad}) \tag{58}$$

$$A_1 = \text{Trans}\ (0\ \text{mm},\ 0\ \text{mm},\ 0\ \text{mm}) \cdot \text{Rot}\ ([0,\ 0,\ 1]^T,\ 3.0\ \text{rad}) \tag{59}$$

and

$$A_2 = \text{Trans}\ (-400\ \text{mm},\ 0\ \text{mm},\ 400\ \text{mm}) \cdot \text{Rot}\ ([0,\ 1,\ 0]^T,\ 1.5\ \text{rad}). \tag{60}$$

The above parameters are chosen to match the setup in our laboratory. The camera coordinate frame (X_{act}) is nearly parallel to the robot wrist frame but is angled slightly towards the gripper. The first robot motion (A_1) is approximately a rotation of 3 rad (172°) about the camera's line of sight, so that the upside-down camera is still pointing to the general direction of the object. Notice that we did not choose 180° because our theorems do not apply to that case. However, we chose a value close to 180° because that minimizes the noise sensitivities. How close to 180° we should choose depends on how accurate our system (robot and vision system) is. For example, if we know that the system has a maximum angular error of 2°, we must choose the robot motion to be less than 178°. The second motion (A_2) is a rotation of 1.5 rad (86°) about the y-axis of the robot wrist and the translation is chosen such that the fixed object is still in the camera's view.

We can find the numerical values of the A_1, B_1, A_2, B_2, and X_{act} using (6), (56), and (57).

$$A_1 = \begin{bmatrix} -0.989992 & -0.141120 & 0.000000 & 0 \\ 0.141120 & -0.989992 & 0.000000 & 0 \\ 0.000000 & 0.000000 & 1.000000 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{61}$$

$$B_1 = \begin{bmatrix} -0.989992 & -0.138307 & 0.028036 & -26.9559 \\ 0.138307 & -0.911449 & 0.387470 & -96.1332 \\ -0.028036 & 0.387470 & 0.921456 & 19.4872 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{62}$$

$$A_2 = \begin{bmatrix} 0.070737 & 0.000000 & 0.997495 & -400.000 \\ 0.000000 & 1.000000 & 0.000000 & 0.000000 \\ -0.997495 & 0.000000 & 0.070737 & 400.000 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{63}$$

$$B_2 = \begin{bmatrix} 0.070737 & 0.198172 & 0.977612 & -309.543 \\ -0.198172 & 0.963323 & -0.180936 & 59.0244 \\ -0.977612 & -0.180936 & 0.107415 & 291.177 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{64}$$

$$X_{act} = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 10 \\ 0.000000 & 0.980067 & -0.198669 & 50 \\ 0.000000 & 0.198669 & 0.980067 & 100 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{65}$$

Now we can find the axes of rotations of A_1, B_1, A_2, and B_2 by (7), (10)–(12)

$$k_{A_1} = [0.000000,\ 0.000000,\ 1.000000]^T \tag{66}$$

$$k_{B_1} = [0.000000,\ 0.198669,\ 0.980067]^T \tag{67}$$

$$k_{A_2} = [0.000000,\ 1.000000,\ 0.000000]^T \tag{68}$$

$$k_{B_2} = [0.000000,\ 0.980067,\ -0.198669]^T. \tag{69}$$

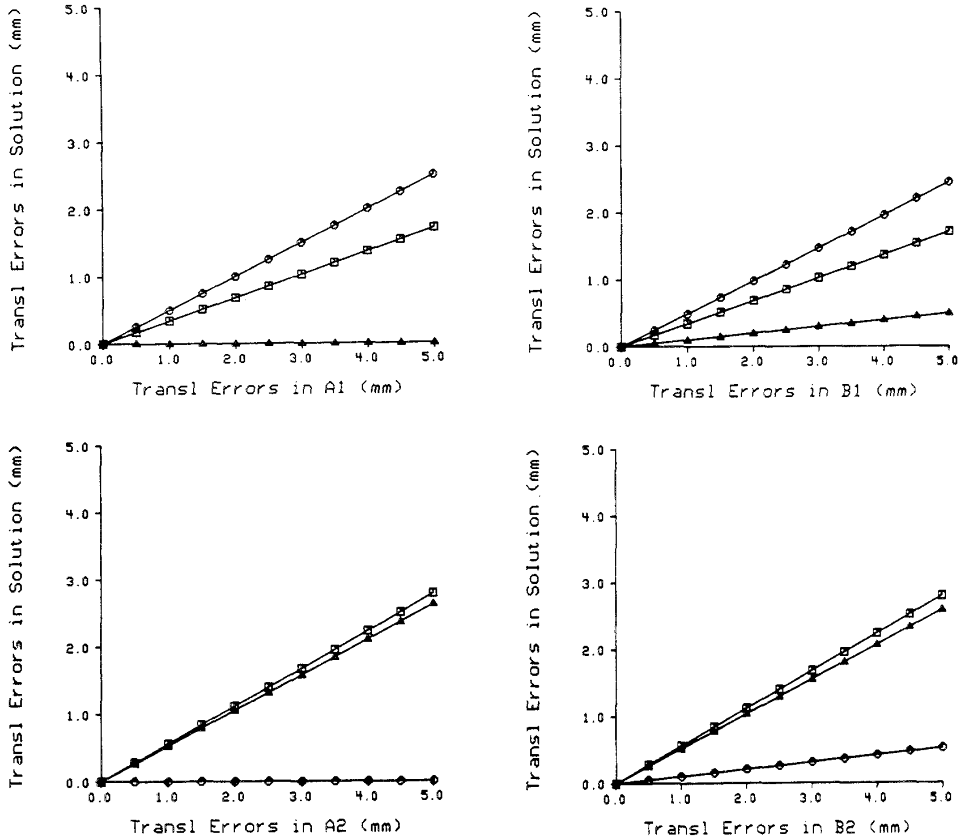

Fig. 6. Translational noise sensitivities due to translational perturbations of robot motion measurements and sensor motion measurements. Errors due to perturbations in x, y, and z directions are marked by □, ○, and △, respectively.

From the above four axes of rotations and from (37)–(39) we can find R_{XP_1} and R_{XP_2}, which are the particular solutions to the rotational parts of (54) and (55), respectively. The numerical values of these two rotational matrices are

$$R_{XP_1} = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.980067 & -0.198669 \\ 0.000000 & 0.198669 & 0.980067 \end{bmatrix} \quad (70)$$

and

$$R_{XP_2} = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.980067 & -0.198669 \\ 0.000000 & 0.198669 & 0.980067 \end{bmatrix}. \quad (71)$$

Notice that the two particular solutions in this example are the same and are both equal to the final solution. This is merely a coincidence. When other X, A_1, and A_2 are used, the particular solutions are generally different from the final solution.

From Theorem 1, the solution is either Rot $(\boldsymbol{k}_{A_1}, \beta_1)R_{XP_1}$ or Rot $(\boldsymbol{k}_{A_2}, \beta_2)R_{XP_2}$. We can solve for β_1 and β_2 from (45) and (46) and from $\beta_i = \text{atan } 2(\sin \beta_i, \cos \beta_i)$, $i = 1, 2$. We found θ_1 to be 0. The rotational part of $X(R_X)$ can be found by computing the numerical values of Rot $(\boldsymbol{k}_{A_1}, \beta_1)R_{XP_1}$

$$R_X = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.980067 & -0.198669 \\ 0.000000 & 0.198669 & 0.980067 \end{bmatrix}. \quad (72)$$

This solution is correct because it is the same as the rotational part of the actual sensor position (X_{act}).

To find the translational part of the solution, we use (47) and (48); it is found to be $[10.0000, 50.0000, 100.000]^T$, which is the same as that of the actual sensor position.

VI. Noise Sensitivities

To measure the noise sensitivity of our calibration method, it is necessary to compare true measurements of the sensor mounting position with experimental results using the method discussed. However, true measurements are difficult or expensive to obtain. In this paper, we will simulate the noise sensitivities by perturbing the robot motions (A_1 and A_2) and the sensor motions (B_1 and B_2), and observing the resulting errors in the sensor mounting position (X). In the rest of this section, noise sensitivity will refer to error in the solution per unit perturbation, e.g., 0.6-mm solution error per 1-mm perturbation.

Noise sensitivities are configuration-dependent. We will use the set of values given in last section's example, which are chosen realistically for our laboratory setup. Noise sensitivities are also dependent on the direction of perturbation. Since a homogeneous transform has six degrees of freedoms, we will perturb the translations in x, y, and z directions and the rotations about the x, y, and z axes.

Fig. 6 shows the translational noise sensitivities due to translational perturbations of robot motion measurements and

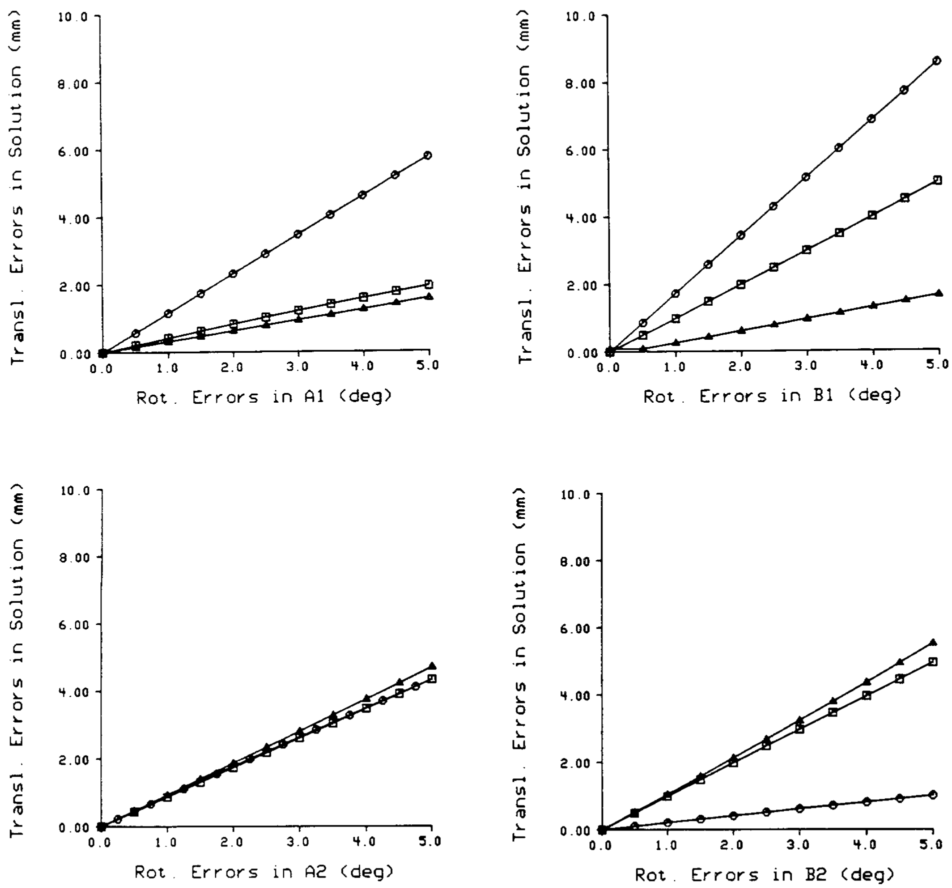

Fig. 7. Translational noise sensitivities due to rotational perturbations of robot motion measurements and sensor motion measurements. Errors due to perturbations about the x, y, and z axes are marked by □, ○, and △, respectively.

sensor motion measurements. The translational components of A_1, B_1, A_2, B_2 are perturbed by adding between 1 and 5 mm to each of the x, y, and z components. The resulting translational errors are then calculated by taking the Euclidean distance between the actual sensor mounting position (X_{act}) and the calculated position (X), where the distance is the magnitude of the $\boldsymbol{p}$ vector (or translation vector) of the compound matrix $X^{-1}X_{act}$. Errors due to perturbations in x, y, and z directions are marked by □, ○, and △, respectively. Rotational errors due to translational perturbations are not plotted because they are always zero.

Figs. 7 and 8 show the translational and rotational noise sensitivities due to rotational perturbations. The rotational parts of A_1, B_1, A_2, and B_2 are perturbed by rotating them around each of their x, y, and z axes by 0° to 5°. Rotational errors are calculated by taking the minimum angle required to align the perturbed solution X to the actual mounting position X_{act} (angle of rotation of the compound matrix $X^{-1}X_{act}$). Errors due to rotational perturbations about the x, y, and z axes are marked by □, ○, and △, respectively.

Notice that noise sensitivities vary greatly, depending on the direction of perturbation. It may be useful to use this information for planning sensor-mount calibration if the error characteristics of the robot and the sensor are known.

VII. Conclusions

We have described a method to find the position of a wrist-mounted sensor relative to a robot wrist, without using direct measurements. This will be useful for calibrating vision systems, range sensing systems, and tactile sensing systems. The process can be automated and does not require any measuring equipment.

Our method requires the solution to a homogeneous transform equation of the form $AX = XB$, where the angle of rotation of A is neither 0 nor π. We found that the solution is not unique; it has one degree of rotational freedom and one degree of translational freedom. We propose that we use two simultaneous equations of the form $A_1X = XB_1$ and $A_2X = XB_2$. Physically, this means that we move the robot twice and observe the changes in the sensor frame twice. The necessary condition for a unique solution is that the axes of rotation of A_1 and A_2 are neither parallel nor antiparallel to one another and that the angles of rotation are neither 0 nor π. A computer program is written for the proposed method. We have generated several test cases in which the conditions for uniqueness are satisfied; all the computed solutions are found to be correct. Another program is written to test the noise sensitivity of the method. The matrices A_1, B_1, A_2, and B_2 are perturbed and the errors in the resulting solutions are plotted.

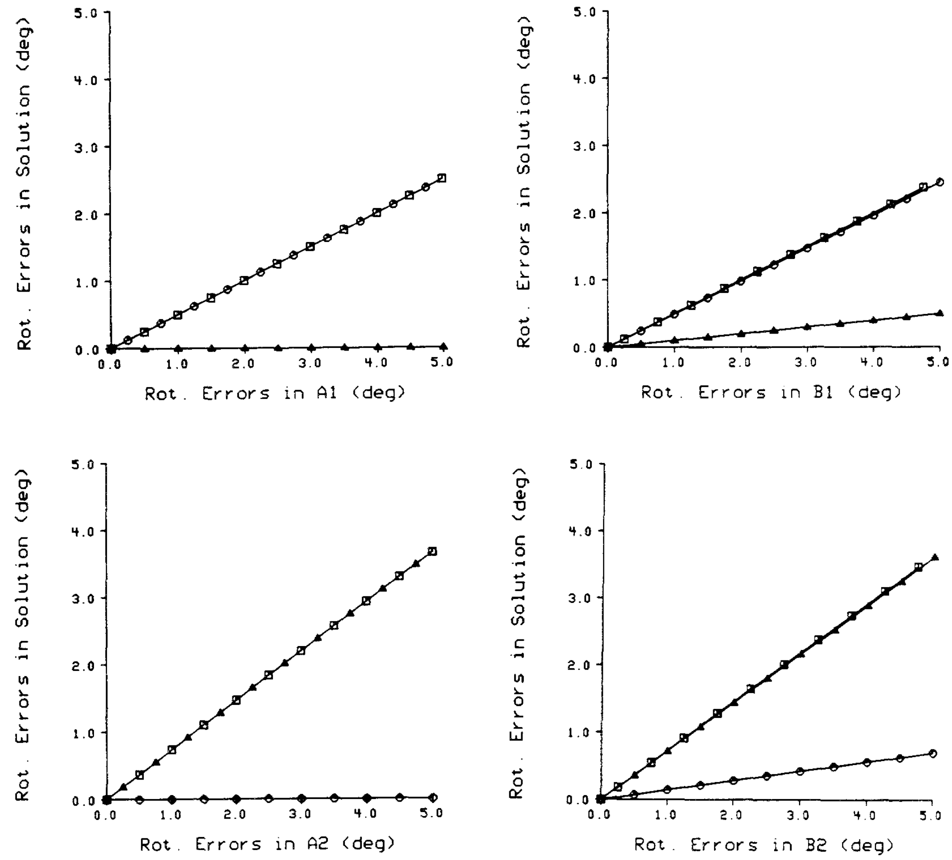

Fig. 8. Rotational noise sensitivities due to rotational perturbations of robot motion measurements and sensor motion measurements. Errors due to perturbations about the x, y, and z axes are marked by □, ○, and △, respectively.

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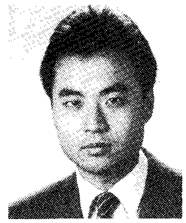

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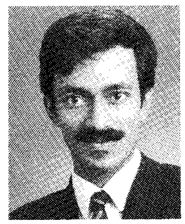

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