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THE PROJECTIVE THEORY OF RELATIVE ORIENTATION [1]

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SUMMARY

The paper shows how the problem of relative orientation may be treated in the plane if the three space coordinates are regarded as homogeneous plane coordinates. The elementary theory of this approach is developed and the paper concludes by deriving a form for the relative orientation matrix in terms of the coordinates of te two epipoles. This matrix seems to be the simplest so far obtained in many attempts to solve this intractable problem.

PROJECTIVE TRANSFORMATION

Central projection of space to a plane, which is the geometrical model of photography, is not strictly a projective transformation, because the matrix of the transformation is singular when regarded in projective terms. We can see this in Fig.1.

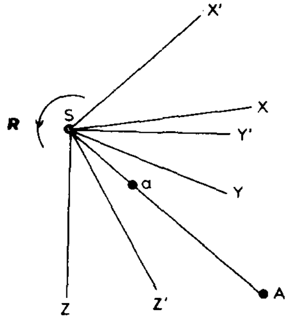

Fig.1.

In this figure a is the image in the (positive) picture plane of a point A in space with coordinates X, Y, Z. In terms of a picture coordinate system, in

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which the axis SZ' is normal to the picture plane, the coordinates of the image are given by:

$$
\begin{aligned}
X' &= f\,\frac{\gamma_{11}X + \gamma_{21}Y + \gamma_{31}Z}{\gamma_{13}X + \gamma_{23}Y + \gamma_{33}Z} \\
Y' &= f\,\frac{\gamma_{12}X + \gamma_{22}Y + \gamma_{32}Z}{\gamma_{13}X + \gamma_{23}Y + \gamma_{33}Z} \\
Z' &= f
\end{aligned}
\tag{1}
$$

where γ_{ij} is a typical element of the orthogonal matrix $\mathbf{R}$ that represents the rotation of the picture coordinate system with respect to the unaccented system; and f is the principal distance.

If we use homogeneous coordinates, putting $X = x/\omega$ etc., then eq.1 is equivalent to:

$$
\begin{pmatrix} x' \\ y' \\ z' \\ \omega' \end{pmatrix} = f \begin{pmatrix} \gamma_{11} & \gamma_{21} & \gamma_{31} & 0 \\ \gamma_{12} & \gamma_{22} & \gamma_{32} & 0 \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & 0 \\ \frac{1}{f}\gamma_{13} & \frac{1}{f}\gamma_{23} & \frac{1}{f}\gamma_{33} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \omega \end{pmatrix}
$$

in which the matrix of the transformation is clearly singular. It is, I think, for this reason that the treatment given to photogrammetric problems is not usually on projective lines, although it will have been noticed that authors of text books seem unable to avoid reference to projective transformation. This present paper takes the matter a little further forward and gives a projective theory of relative orientation. In particular the projective significance of the epipoles is emphasised and if this raises the criticism that they have no practical significance it is one that might have been made over a number of years to authors of photogrammetric text books who cannot resist mentioning them without making the slightest use of them. At least some use is made of them here.

Although the present treatment seems to have mainly a theoretical interest, the paper concludes with an application to the calculation of a relative orientation. In addition perhaps these few researches will revive an interest in a subject that must surely have some importance to photogrammetry.

RELATIVE ORIENTATION CONDITION

It has been shown (THOMPSON, 1959) that the condition that a pair of rays lie in a plane containing the base (epipolar or basal plane) can be put in the form:

$$
(X'\ Y'\ Z') \begin{pmatrix} 0 & -B_Z & B_Y \\ B_Z & 0 & -B_X \\ -B_Y & B_X & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0 \tag{2}
$$

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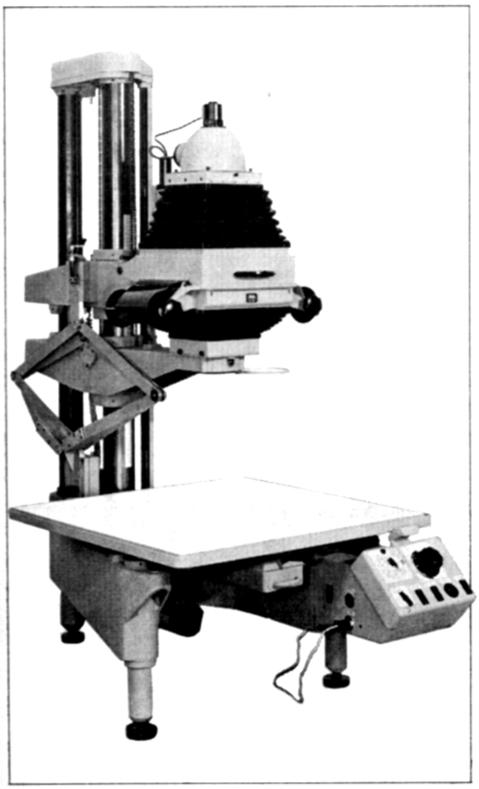

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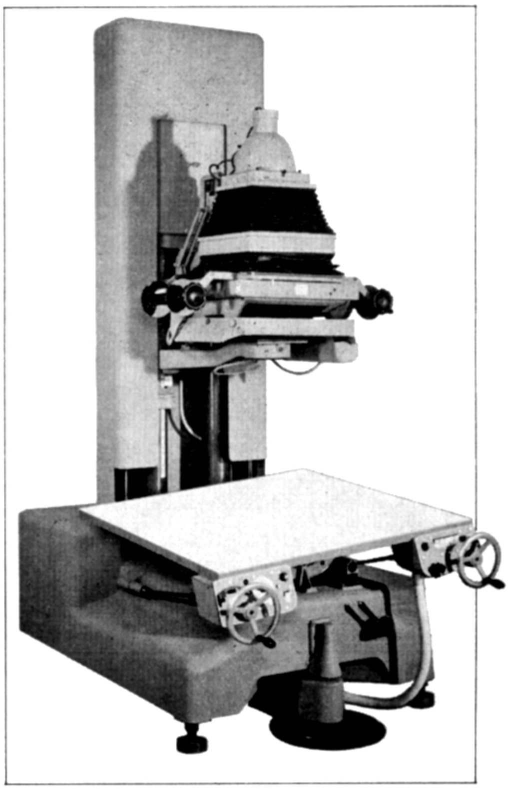

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Members of the Expedition carry out regular scientific observations at six of the 24 continental Antarctic stations; four of the seven scientific stations in the interior of the continent belong to the USSR. All these Soviet stations carry out systematic observations in geology, glaciology, hydrology, geomagnetism, ionospheric physics, telluric currents, auroras, cosmic radiation, seismology, meteorology, actinometry and aerology, and every year, extensive, multidiscipline oceanographic investigations are carried out in the Great Southern Ocean.

The Soviet Antarctic Expedition has collected enormous amounts of new and valuable data. Most of this information has been published in the Russian language in a series of Information Bulletins which have been translated into English, edited, and are now being published. This work was commenced by the Geophysical and Polar Research Center of the University of Wisconsin and continued by the staff of Scripta Technica, Inc., Washington, D.C.

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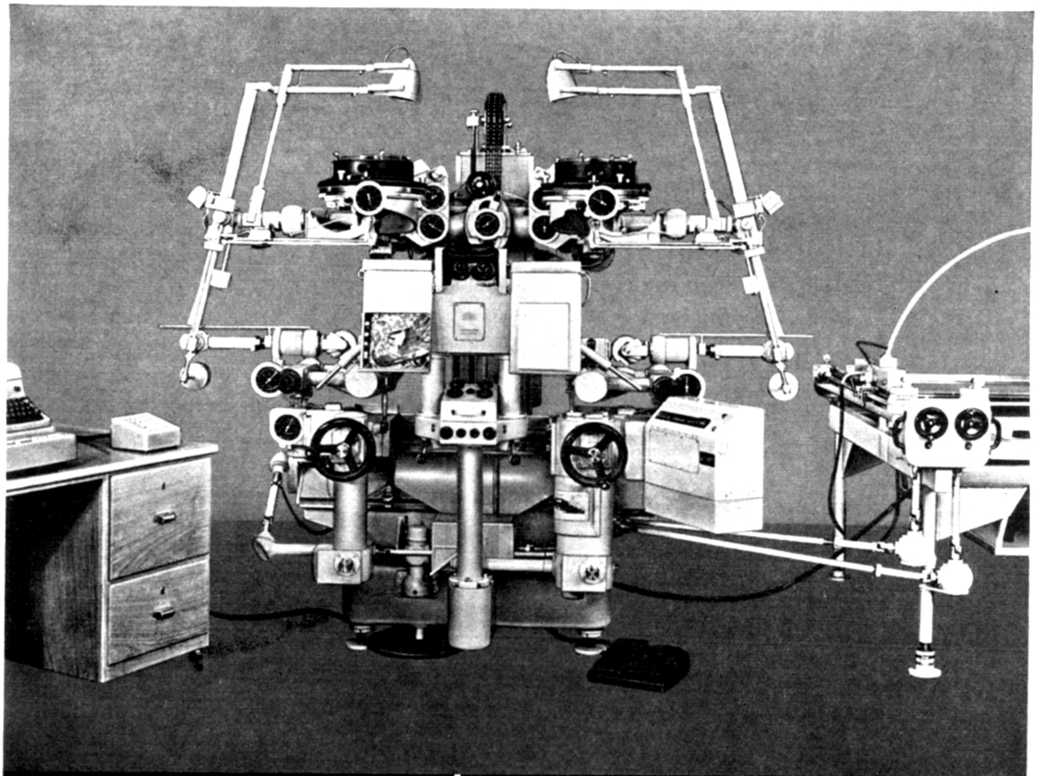

Where X, Y, Z are the coordinates of *any* point on the left-hand ray; X', Y', Z' are the coordinates of any point on the right-hand ray; and B_x, B_y, B_z are the base components. The origins are respectively the left and right-hand vertices.

If $\mathbf{R}$ and $\mathbf{R}'$ are orthogonal matrices representing the rotations of the left and right-hand pictures respectively then eq.2 becomes:

$$(x'\ y'\ z')\ \mathbf{R}'^T\ \mathbf{B}\ \mathbf{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{3}$$

where the coordinates are now referred to the picture coordinate systems respectively, and $\mathbf{B}$ is the base component matrix.

We now note that eq.3 is homogeneous in the three coordinates of a point and in the three base components. We may therefore regard these equations as setting up a condition *in a plane* in which the coordinates and base components are to be taken as the *homogeneous* coordinates of points in this plane. If this is an allowable point of view, then it should be possible to interpret eq.3 in these terms. This was done in an earlier, not very accessible, paper (THOMPSON, 1964) and it is repeated here for completeness.

The transformation:

$$\mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4}$$

when $\mathbf{A}$ is a matrix of order 3×3, may be regarded in two ways. Usually it is looked upon as transforming a point, whose homogeneous coordinates are x, y, z, to a new *point*. The transformation is, of course, projective and is known as a *collineation* since, under it, straight lines transform to straight lines. Owing, however, to the duality of projective geometry eq.4 may also be regarded as transforming the point y, y, z to the *line* l', m', n', where l', m', n' are given by:

$$\begin{pmatrix} l' \\ m' \\ n' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Such a transformation is known as a *correlation*. If a point x', y', z' lies on the line l', m', n' then we have:

$$l'\,x' + m'\,y' + n'\,z' = 0$$

that is:

$$(x'\ y'\ z')\ \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{5}$$

which is eq.3 again.

The condition eq.5 may thus be interpreted in projective terms as the condition that the point (x' y' z') lies on the line induced by a correlation whose matrix is **A**. Alternatively it may be interpreted as the condition that the point x, y, z lies on the line given by:

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \mathbf{A}^T \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

In the relative orientation problem the *points* of the correlation are clearly the image points on the photographs. What are the lines?

We note, first, that the matrix $\mathbf{R}'^T \mathbf{B} \mathbf{R}$ in eq.3 is singular and has rank 2, for $\mathbf{R}'$ and $\mathbf{R}$ are non-singular while $\mathbf{B}$ is singular and of rank 2; and the rank of a matrix remains unchanged when multiplied by a non-singular matrix. Let us therefore investigate the properties of eq.5 when **A** is singular and of rank 2.

We indicate the (homogeneous) coordinate vector of a point by $\mathbf{p}$ or $\mathbf{p}'$ and of a line by $\mathbf{l}$ or $\mathbf{l}'$. Consider three non-collinear points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ to which correspond the lines $\mathbf{l}_1'$, $\mathbf{l}_2'$, $\mathbf{l}_3'$. We have:

$$(\mathbf{l}_1' \, \mathbf{l}_2' \, \mathbf{l}_3') = \mathbf{A} \, (\mathbf{p}_1 \, \mathbf{p}_2 \, \mathbf{p}_3)$$

Suppose $\mathbf{p}_0'$ is a point that lies on all three lines, then:

$$(\mathbf{l}_1' \, \mathbf{l}_2' \, \mathbf{l}_3')^T \, \mathbf{p}_0' = 0$$

and this equation will have solutions for $\mathbf{p}_0'$, other than $(0 \;\; 0 \;\; 0)^T$ which is inadmissible, if and only if $(\mathbf{l}_1' \, \mathbf{l}_2' \, \mathbf{l}_3')$ in singular. But this matrix is given by $\mathbf{A} \, (\mathbf{p}_1 \, \mathbf{p}_2 \, \mathbf{p}_3)$ and **A** is singular. Moreover **A** has rang 2 and $(\mathbf{p}_1 \, \mathbf{p}_2 \, \mathbf{p}_3)$ is non-singular, since the three chosen points are not collinear and their coordinate vectors are therefore linearly independent. Thus $(\mathbf{l}_1' \, \mathbf{l}_2' \, \mathbf{l}_3')$ has rank 2 and $\mathbf{p}_0'$ is uniquely determined apart from an arbitrary scalar factor. The three lines $\mathbf{l}_1'$, $\mathbf{l}_2'$, $\mathbf{l}_3'$ are thus distinct and are concurrent in a determinate point. It follows that all the lines of the correlation are concurrent in this same point which is, of course, the right-hand (accented) epipole, any line $\mathbf{l}'$ being a right-hand epipolar ray. In the same way we may consider three non-collinear points $\mathbf{p}_1'$, $\mathbf{p}_2'$, $\mathbf{p}_3'$ and their three corresponding lines $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$ given by:

$$(\mathbf{l}_1 \, \mathbf{l}_2 \, \mathbf{l}_3) = \mathbf{A}^T \, (\mathbf{p}_1' \, \mathbf{p}_2' \, \mathbf{p}_3')$$

The left-hand, unaccented, epipole will then be the solution of:

$$(\mathbf{l}_1 \, \mathbf{l}_2 \, \mathbf{l}_3)^T \, \mathbf{p}_0 = \mathbf{0}$$

Evidently the two epipoles will have the same coordinates if, and only if, $\mathbf{A} = \pm \, \mathbf{A}^T$, that is to say if **A** is symmetric or skew symmetric.

We introduced the three arbitrary non-collinear points to demonstrate the existence of the epipoles in the correlation. Since, however, epipoles are points at which all lines are concurrent we can choose lines which are the transforms of special points and thereby set up simple equations from which to find the epipoles.

Since any line $\mathbf{Ap}$ passes through the accented epipole $\mathbf{p}_0'$ we have $\mathbf{p}_0'^T\, \mathbf{Ap} = 0$ for all $\mathbf{p}$.

Put $\mathbf{p}^T$ successively equal to (0 0 1), (1 0 0) and (0 1 0), i.e., choose as our three points, the origin and the points at infinity in the direction of the x and y axes respectively.

We have at once:

$$\mathbf{A}^T\, \mathbf{p}_0' = \mathbf{0} \tag{6}$$

Similarly, for the unaccented epipole:

$$\mathbf{A}\, \mathbf{p}_0 = \mathbf{0} \tag{7}$$

THE EPIPOLES IN SPECIAL CASES

There are two cases of special interest in relative orientation. In the first the x-axis coincides with the base and the skew-symmetric matrix in eq.2 takes the form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and eq.6 becomes:

$$\mathbf{R}^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \mathbf{R}'\, \mathbf{p}_0' = \mathbf{0}$$

But $\mathbf{R}$ is non-singular and may be removed and the equation to be solved is then:

$$\begin{pmatrix} 0 & 0 & 0 \\ \gamma_{31}' & \gamma_{32}' & \gamma_{33}' \\ -\gamma_{21}' & -\gamma_{22}' & -\gamma_{23}' \end{pmatrix} \mathbf{p}_0' = \mathbf{0}$$

where γ_{ij}' is a typical element of $\mathbf{R}'$.

If the coordinates of the epipoles are x_0', y_0', z_0' the solution is seen to be:

$$\frac{x_0'}{\gamma_{11}'} = \frac{y_0'}{\gamma_{12}'} = \frac{z_0'}{\gamma_{13}'}$$

after we have taken note of the fact that an element of an orthogonal matrix is equal to its co-factor. In the same way the unaccented epipole will be the solution of:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{R}\, \mathbf{p}_0 = \mathbf{0}$$

i.e., $x_0/\gamma_{11} = y_0/\gamma_{12} = z_0/\gamma_{13}$

In the second case we take $R = I$ and p_0' is the solution of:

$$\begin{pmatrix} 0 & B_Z & -B_Y \\ -B_Z & 0 & B_X \\ B_Y & -B_X & 0 \end{pmatrix} \mathbf{R}' \mathbf{p}_0' = \mathbf{0}$$

which is easily verified as being:

$$\frac{x_0'}{\gamma_{11}' B_X + \gamma_{21}' B_Y + \gamma_{31}' B_Z} = \frac{y_0'}{\gamma_{12}' B_X + \gamma_{22}' B_Y + \gamma_{32}' B_Z} = \frac{z_0'}{\gamma_{13}' B_X + \gamma_{23}' B_Y + \gamma_{33}' B_Z}$$

this being equivalent to:

$$\mathbf{p}_0' = \mathbf{R}'^T \begin{pmatrix} B_X \\ B_Y \\ B_Z \end{pmatrix}$$

for a point to lie on its own corresponding line. Such points will be given by:

$$(x\ y\ z)\ \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

This is the equation of a conic known as the *coincidence conic*.

Readers will remember that in the standard form of the equation of a conic the square matrix is symmetric. It is easily verified that there is no change in eq.8 if we substitute the symmetric matrix $^1/_2\,(\mathbf{A} + \mathbf{A}^T)$ for $\mathbf{A}$. We now show that the meets of corresponding epipolar rays lie on the coincident conic.

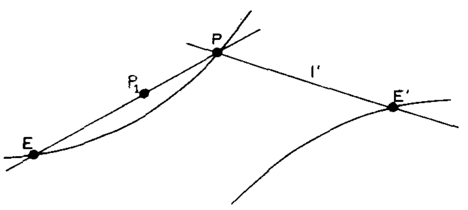

Fig 2.

Consider any point P_1 defining an epipolar ray EP_1, E being the unaccented epipole (Fig.2). If the coordinate vector of P_1 is $\mathbf{p}_1$ and λ is a scalar then $\mathbf{p}_0 + \lambda\,\mathbf{p}_1$ will be the coordinate vector of a point on $\mathbf{EP}_1$. The line corresponding to this point is given by:

$$\mathbf{l}' = \mathbf{A}\,(\mathbf{p}_0 + \lambda\,\mathbf{p}_1)$$

But $\mathbf{Ap}_0 = \mathbf{0}$ and hence:

$$\mathbf{l}' = \lambda\,\mathbf{Ap}_1$$

Thus the single line $\mathbf{l}'$ corresponds to *all points* on EP_1 for multiplication by a scalar λ does not change the *ratio* of the homogeneous coordinates of the line.

If now P is the intersection of EP_1 and $\mathbf{l}'$, the line corresponding to P will be $\mathbf{l}'$ (for $\mathbf{l}'$ corresponds to all points EP_1) and thus P will be a point of the coincident conic since it lies on its corresponding line.

This result could have been anticipated from space considerations. The epipolar rays form two pencils with vertices respectively at E and E'. Moreover the pencils are projectively related since their ways meet by pairs on a line (the meet of the two picture planes). If then these two pencils are arbitrarily placed in the same plane the meets of corresponding pairs of lines will lie on a conic.

Moreover since $\mathbf{A}\,\mathbf{p}_0 = \mathbf{0}$ and $\mathbf{A}^T\,\mathbf{p}_0' = \mathbf{0}$ the coordinates of the epipoles satisfy eq.8 and the epipoles therefore lie on the conic. This is a well-known property of the vertices of two projectively related pencils. The curves shown in Fig.2 are the two branches of a hyperbola.

The above treatment of the coincidence conic draws attention to an important property of the plane treatment of the problem. Pairs of lines, such as pairs of epipolar rays, will always meet in a plane. In space the epipolar rays meet by pairs only when correctly oriented: in the plane they will meet when arbitrarily oriented.

THE EPIPOLAR MATRIX

It is of some interest to develop a general matrix whose elements are composed so far as possible of the coordinates of the epipoles. We have, in fact, to find a matrix $\mathbf{A}$ such that:

$$\mathbf{A}\,\mathbf{p}_0 = 0 \text{ and } \mathbf{A}^T\,\mathbf{p}_0' = 0$$

where $\mathbf{p}_0$ and $\mathbf{p}_0'$ are regarded as given.

It is possibly more convenient to work in the non-homogeneous coordinates of the epipoles, say X_0, Y_0 and X_0', Y_0'.

We may then write the equations:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} X_0' \\ Y_0' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These six equations may be used to determine the nine elements a_{ij} but clearly we cannot do this uniquely and we have a choice. Let us attempt to obtain five of the elements in terms of a_{22}, a_{32}, a_{23}, a_{33} and the four epipolar coordinates. This we shall see is, in general, possible. We rearrange the above equations in the form:

$$\begin{pmatrix} X_0 & Y_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_0 & 0 & Y_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_0 & 0 & 0 & Y_0 & 1 \\ X_0' & 0 & 0 & Y_0' & 1 & 0 & 0 & 0 & 0 \\ 0 & X_0' & 0 & 0 & 0 & Y_0' & 0 & 1 & 0 \\ 0 & 0 & X_0' & 0 & 0 & 0 & Y_0' & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{31} \\ a_{22} \\ a_{23} \\ a_{32} \\ a_{33} \end{pmatrix} = 0$$

After systematic elimination in the usual way and rearrangement to illustrate the truncated triangular form of the resultant matrix we have:

$$\begin{pmatrix} X_0 & Y_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{Y_0 X_0'}{X_0} & -\frac{X_0'}{X_0} & Y_0' & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{X_0'}{Y_0} & \frac{X_0 Y_0'}{Y_0} & \frac{X_0}{Y_0} & Y_0' & 0 & 1 & 0 \\ 0 & 0 & 0 & X_0 & 0 & Y_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_0 & 0 & 0 & Y_0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{31} \\ a_{22} \\ a_{23} \\ a_{32} \\ a_{33} \end{pmatrix} = 0$$

which shows that the matrix has, in general, rank 5 and that the first five unknowns may be expressed in terms of the remaining four. The results of the back-substitution are:

$$a_{11} = \frac{Y_0}{X_0 X_0'} (a_{22} Y_0' + a_{32}) + \frac{1}{X_0 X_0'} (a_{23} Y_0' + a_{33})$$

$$a_{12} = -\frac{1}{X_0'} (a_{22} Y_0' + a_{32})$$

$$a_{13} = -\frac{1}{X_0'} (a_{23} Y_0' + a_{33})$$

$$a_{21} = -\frac{1}{X_0} (a_{22} Y_0 + a_{23})$$

$$a_{31} = -\frac{1}{X_0} (a_{32} Y_0 + a_{33})$$

The matrix A then takes the form:

$$\begin{pmatrix} \frac{Y_0}{X_0 X_0'}(a_{22} Y_0' + a_{32}) + \frac{1}{X_0 X_0'}(a_{23} Y_0' + a_{33}) & -\frac{1}{X_0'}(a_{22} Y_0' + a_{32}) & -\frac{1}{X_0'}(a_{23} Y_0' + a_{33}) \\ -\frac{1}{X_0}(a_{22} Y_0 + a_{23}) & a_{22} & a_{23} \\ -\frac{1}{X_0}(a_{32} Y_0 + a_{33}) & a_{32} & a_{33} \end{pmatrix}$$

This matrix is easily seen to factorise into:

$$\begin{pmatrix} 1 & -\frac{Y_0'}{X_0'} & -\frac{1}{X_0'} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{Y_0}{X_0} & 1 & 0 \\ -\frac{1}{X_0} & 0 & 1 \end{pmatrix}$$

The first and third matrices are always non-singular and hence the whole matrix must, at most, have rank 2 and will have rank 1 unless:

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0$$

With this restriction then the four a_{ij} are at our disposal, but choices that make the centre matrix relevant to the relative orientation problem are obviously severely restricted.

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