The Gowers Norm in the Testing of Boolean Functions

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April 23, 2009

Outline

Property testing

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- algorithm makes few queries into the input
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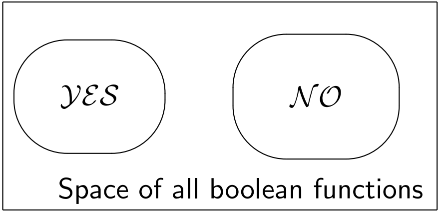
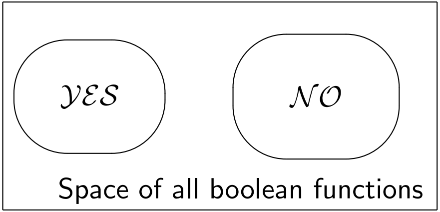
$f(s_1)$	$f(s_2)$	$f(s_3)$	$\cdots$	$f(s_{2^n})$

$$\begin{array}{c} \downarrow \\ \text{Alg} \end{array}$$

Definition of a testing algorithm





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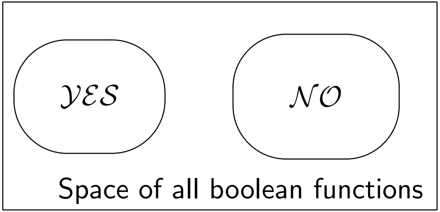

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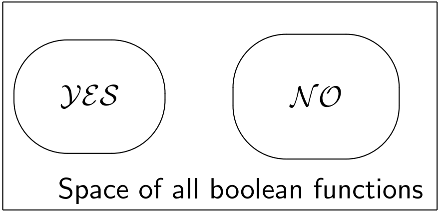

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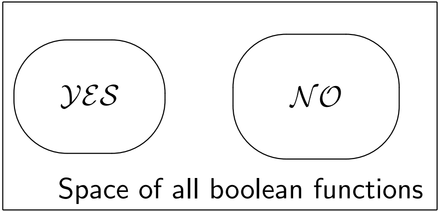

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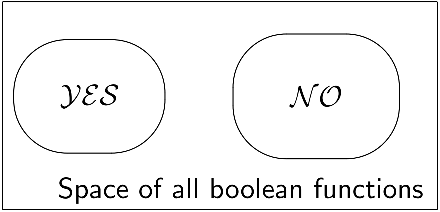

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- q fixed. if $f \in \mathcal{NO}$, f must be far from $\mathcal{YES}$

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"meta theorem"

Under the unique games conjecture, a (q, c, s) dictatorship test translates into a PCP system with q queries, completeness $0.99c$, and soundness s.

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- [DS,KKMO]: one that has an "influential variable"

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  - $f(x) = x_1 + \mathrm{Maj}(x_2, \ldots, x_n)$. $\mathrm{I}_1(T_\rho(f)) > 0$.
- Upshot: any function has a finite number of variables with positive ρ-attenuated influence

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Theorem (C08)

A q-query (adaptive) dictatorship test with completeness 1 and soundness $O(q^3 \cdot 2^{-q})$.

Outline

1. Introduction
   - Dictatorship testing
   - Testing graph free
2. Technique overview
3. Design & analysis of dictatorship test
4. Design & analysis of graph free test

A graph-free test

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  - $f \in \mathcal{YES}_{C_3}$ if f has no Schur triple ("triangle")
    i.e., no x, y such that $f(x) = f(y) = f(x + y) = 1$

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  - $f \in \mathcal{YES}_{C_3}$ if f has no Schur triple ("triangle")
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Our result: joint with Bhattacharyya, Sudan, Xie

Let G be a graph with k edges. There is a k-query "G-free" test with completeness 1 and soundness < 1.

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- No. Property of triangle-free is linear-invariant & non-linear

Motivation: combinatorics

Removal lemma for graphs

(folklore) If a graph on n vertices has $o(n^3)$ triangles, then $o(n^2)$ edges can be removed so that the graph has no triangle.

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- main tool: Szemerédi's regularity lemma
- applications: additive combinatorics, e.g. arithmetic progressions

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Theorem (Green)

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- motivated by a number theoretic conjecture

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- similiarities: regularity lemmas, notions of pseudorandomness

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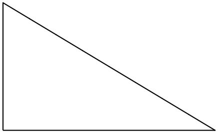
C_3



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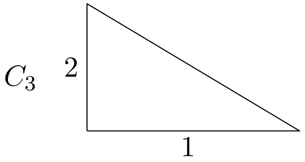

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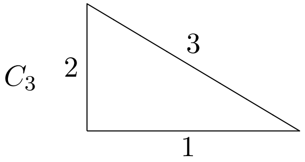

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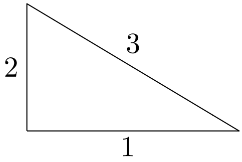

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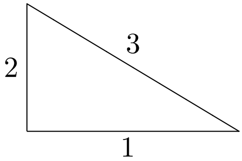

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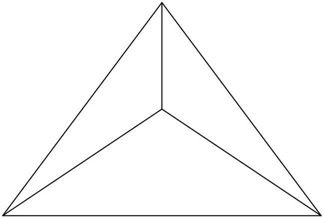

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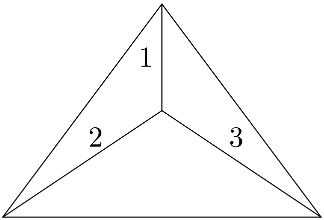

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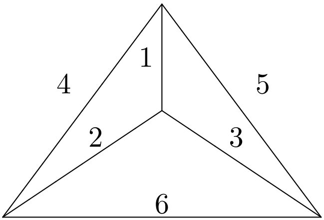

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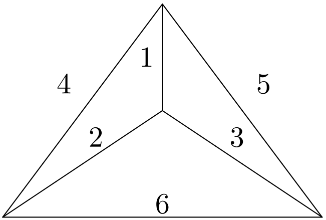
$$M = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$



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Can only guarantee $\tau(\epsilon) \geq 2^{-O(W(\frac{1}{\epsilon}))}$,
where $W(t)$ denotes a tower of twos with height t

Outline

1. Introduction
   - Dictatorship testing
   - Testing graph free
2. Technique overview
3. Design & analysis of dictatorship test
4. Design & analysis of graph free test

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Test T:

- selects a random local structure.
- accepts iff f has no structure.

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where $\Phi(f)$ is some "extension of f" dependent on $(\mathcal{YES}, \mathcal{NO})$

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roughly: due to values of d, field characteristic, roots of unity, etc

The Gowers uniformity norm $\|f\|_{U_d}$

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Definition

- $\|f\|_{U_2} = ||\hat{f}||_4$
  i.e., U_2 norm is the L_4 norm of f's Fourier transform

- Inductively, define $\|f\|_{U_{d+1}}^{2^{d+1}} = \mathbb{E}_h \|\Delta_h f\|_{U_d}^{2^d}$,

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- Expanding, $\|f\|_{U_2}^4 = \mathbb{E}_{x,y,z}[f(x)f(x+y)f(x+z)f(x+y+z)]$
- monotonicity: $\|f\|_{U_2} \leq \|f\|_{U_3} \leq \dots$

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  If $\|1_A - \delta\|_{U_{k-1}} = o(1)$, then A has approximately $\delta^k N^2$ k-term AP.

Generalized von-Neumann type theorems

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- [Samorodnitsky, S.&Trevisan06]: The acceptance probability of the AKKLR test for degree d polynomials is bounded above by the $(d+1)$-th Gowers norm

Generalized von-Neumann type theorems

- statements that show certain "estimations of f" are bounded above by the Gowers norm of f
- [Green&Tao:] A system of linear equations with "complexity d" is bounded above by the $(d+1)$-th Gowers norm

System of linear equations with low complexity

Theorem (Green & Tao)

Suppose the system of linear equations $L_1, \ldots, L_k$ over variables $x_1, \ldots, x_m$ has "complexity d."

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$$\mathop{\mathbb{E}}_{x_1,\ldots,x_m}\left[\prod_{i\in[k]} f_i(L_i(x_1,\ldots,x_m))\right] \le \min_{i\in[k]} \|f_i\|_{U_{d+1}}$$

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Outline

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ST Lemma:

For balanced function f, if for all $i \in [n]$ $\mathrm{I}_i(f) = o(1)$, then for all $d \in \mathbb{Z}^+$, $\|f\|_{U_d} = o(1)$.

Basic test with completeness 1

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- monomial testing: Parnas, Ron, Samorodnitsky 02
- essentially checks if $f(y) \wedge f(z) = f(y \wedge z)$.

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[Håstad & Khot 02]; too inefficient for us

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[Håstad 97]; imperfect completeness

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[GLST 98]

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- Consider a graph $G = ([k], E)$

Running dependent iterations of basic test

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- Pick $x_1, \ldots, x_k \in \{0,1\}^n$.
- Consider a graph $G = ([k], E)$
- Accept iff for each $(i,j) \in E$, $f(x_i)f(x_j) = f(x_i + x_j)$.

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- We will use a complete hypergraph following [ST06]

Design of a hypergraph dictatorship test

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accepts large parities, $f(x) = (-1)^{x_1 + \ldots + x_n}$

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[ST06] test; has imperfect completeness

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"Extension" Φ_ρ

For every $f : \{0,1\}^n \to \{\text{-}1, 1\}$, define $\Phi_\rho(f) : \{0,1\}^{2n} \to [-1, 1]$ to be

$$\Phi_\rho(f)(x, y) = \mathop{\mathbb{E}}_{z \in \mathcal{D}_\rho} f(x + y \wedge z)$$

Analysis continued

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- By generalized von-Neumann,

$$\Pr[k\text{-deg AKKLR test accepts } \Phi_\rho(f)] \leq \|\Phi_\rho(f)\|_{U_{k+1}}$$

Outline

M-free test

Let M be an r by k matrix over $\{0, 1\}$.

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If f is M-free, test always accepts.

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  H is a subspace of $\{0,1\}^n$

  f_{g+H} is the restriction of f to the affine subspace $g + H$

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Let $\epsilon > 0$, $f : \{0,1\}^n \to \{0,1\}$.

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Let $\epsilon > 0$, $f : \{0,1\}^n \to \{0,1\}$.
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If M is graphic, then the rows of M form a complexity 1 system of linear forms.

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- Since H has constant co-dimension, test rejects with positive probability.

Future direction

Open question:

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- prove some form of inverse for Gowers norm

  a low-degree polynomial test with soundness $\frac{1}{2}$.