Using Bloch's Expansions for reducing locality of Hamiltonians

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Showing that a given Hamiltonian can be approximated by another Hamiltonian of smaller locality is the central step in the proof of the universality of adiabatic quantum computation with 2-local Hamiltonians and in proofs of the promise QMA-completeness of computing ground state energies of local Hamiltonians[KKR06].

Perturbative gadgets, also introduced in [KKR06], provide a way for achieving such reductions. Jordan and Farhi[JF08] use a perturbative expansion due to Bloch[Blo58] to come up with such an approximate reduction from a general k-local Hamiltonian to a 2-local Hamiltonian. In this note we briefly describe their results.

Notation

An arbitrary k-local Hamiltonian H^{comp} can be expressed in the Pauli basis as follows:

$$H^{\text{comp}} = \sum_{i=1}^{r} c_i H_i,$$

where each $H_i = \sigma_{i,i_1}\sigma_{i,i_2}\sigma_{i,i_3}\ldots\sigma_{i,i_k}$ acts only on qbits indexed $i_1, i_2, i_3, \ldots i_k$. The idea is to augment the system with some ancilla qbits, and to find two local Hamiltonians H^{anc}, which acts only on the ancilla qbits, and V, which acts on both ancilla qbits and the computation bits, so that a perturbation to H^{anc} by a small multiple of V produces a Hamiltonian, which when restricted to a suitable state of the ancilla qbits is close to H^{comp}. Intuitively, we want the Hamiltonian H^{comp} to occur as one of the higher order terms in the perturbative expansion of H^{gad}, and we expect to achieve this by designing V appropriately so that the form of the perturbative expansion gives rise to the required terms.

Perturbative Expansions

We first describe the general perturbative expansion due to Bloch[Blo58]. Let $H^{(0)}$ be a Hamiltonian with a d-dimensional degenerate ground space of energy 0, and suppose V is a perturbation to $H^{(0)}$, scaled by a small real number λ. Let γ be the second smallest eigenvalue of $H^{(0)}$. Suppose we want to estimate $H^{\text{eff}}(d)$, which is H restricted to its d lowest eigenstates. Bloch showed that when $\lambda\,||V|| < \frac{\gamma}{4}$, one can expand $H^{\text{eff}}(d)$ in powers of λ, in terms of the operator $\mathcal{A}$ as follows:

$$\begin{aligned} H^{\text{eff}} &= (P_0 + O(\lambda)) \left(\sum_{m=1}^{\infty} \mathcal{A}^{(m)} \right) (P_0 + O(\lambda)) \\ \mathcal{A}^{(m)} &= \lambda^m \sum P_0 V S^{l_1} V S^{l_2} \ldots V S^{l_{m-1}} V P_0 \end{aligned} \tag{1}$$

Here P_0 is the projector to the ground space of $H^{(0)}$. S is defined to be the pseudo-inverse of $-H^{(0)}$, and S^0 is taken by convention to be $-P_0$. The sum is over all $(m-1)$-tuples satisfying $l_1 + \cdots + l_{m-1} = m-1$, and $l_1 + \cdots + l_p \geq p$, for $p \in \{1, 2, \ldots, m-2\}$.

Reducing Locality

To illustrate the above ideas, we consider the k-local Hamiltonian $H^{\text{comp}} = \sigma_1\sigma_2\sigma_3\ldots\sigma_k$. We introduce k ancilla qbits and introduce a Hamiltonian H^{anc} whose ground state corresponds to the state where all ancilla qbits are either in the $|0\rangle$ or $|1\rangle$ state, and which penalises each disagreement by increasing the energy of the state by 1. If Z_i is the Pauli Z operator on the ith ancilla bit then we can define H^{anc} as:

$$H^{\text{anc}} = \sum_{1\le i<j\le k} \frac{1}{2}(I - Z_iZ_j)$$

Note that the second lowest eigenvalue of H^{anc} is $k-1$. V is designed so that it "bumps" the ground state into the next state, and can bring it back to the ground only after at least k applications, and also, when this happens, the product is proportional to H^{comp}. V is defined as:

$$V = \sum_{i=1}^{k} \sigma_i X_i \tag{2}$$

We now take H^{eff} to be $H^{\text{anc}} + \lambda V$, restricted to its lowest 2^k states(this is to ensure that the whole of the spectrum of H^{comp} is approximated), and further restricted to the $\frac{1}{\sqrt{2}}\left(\left|0^k\right\rangle + \left|1^k\right\rangle\right)$ state of the ancilla qbits. Notice that the last restriction is legal because H^{anc} and V commute with $X = \prod_{i=1}^{k} X_i$, and hence the restriction just corresponds to taking the block corresponding to the $+1$ eigenvalue of X in the block diagonalization of $H^{\text{anc}} + \lambda V$ with respect to the eigen-basis of X. The reason for this restriction will become clear later.

Notice that V^k contains terms of the form $\prod_{i=1}^{k} \sigma_i \prod_{i=1}^{k} X_i$, which are proportional to H^{comp} on the computational qbits and preserve the ground state of H^{anc}. On the other hand, any lower order application of V moves the ground state of H^{anc}. Thus, in the expansion 1, all terms before the k-th order either vanish, or are just proportional to P_0. Also, notice that in the k-th order term, only the terms where $l_i = 1$ for each i survive. This is because if any of the $l_i's$ are 0, the corresponding term vanishes or becomes proportional to P_0, because it has less than k applications of V sandwiched between P_0 operators. Taking all this into account, [JF08] get the following approximation for H^{eff}:

$$H^{\text{eff}} = f(\lambda)P_0 + (-1)^{k-1}\frac{k\lambda^k}{(k-1)!}P_0(H^{\text{comp}} \otimes \prod_{i=1}^{n} X_i)P_0 + O(\lambda^{k+1}),$$

Here f is a function known from the calculation. To handle the $\prod_{i=1}^{n} X_i$ operator, we recall that we considered H^{eff} restricted to the $\frac{1}{\sqrt{2}}\left(\left|0^k\right\rangle + \left|1^k\right\rangle\right)$ state of the ancilla qbits, and hence we can replace it by just a projection P_+ on that space. Since, H^{eff} is a 2-local Hamiltonian by construction, we have the expression:

$$H^{\text{eff}} = f(\lambda)P_0 + (-1)^{k-1}\frac{k\lambda^k}{(k-1)!}(H^{\text{comp}} \otimes P_+) + O(\lambda^{k+1}),$$

, which shows that the spectrum of H^{comp} is approximated by the spectrum of $(-1)^{k-1}\frac{(k-1)!}{k\lambda^k}H^{\text{eff}}$ up to $O(\lambda)$ terms.

Results and Comments

When k is constant, the above approximation is quite versatile. Notice that since it approximates the whole spectrum of H^{comp}, we can use it both to reduce locality in QMA-completeness proofs, as well as in proofs of the universality of Adiabatic Quantum Computation. Typically, a proof would proceed by taking Hamiltonians of higher locality(which mush be a constant) for which QMA-completeness, or universality of adiabatic computation are known, and would then apply this construction to get a Hamiltonian of lower locality such that the lowest energy(for QMA completeness), or the lowest energy gap(for universality of adiabatic quantum computation) blow up only by a constant factor. The latter context is the motivation in the work of Jordan and Farhi[JF08].

However, when k is *not* a constant, this method does not work quite well: the blow up in the norm of the resulting Hamiltonian is exponential in k. This suggests the interesting problem of trying to find perturbative gadgets which are comparable in their generality to the above construction of Jordan and Farhi, but lose a smaller factor in the approximation.

References

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