# The complexity of quantum spin systems on a 2D square lattice

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Starting from Kitaev's seminal reduction showing QMA-completeness of the 5-local Hamiltonian problem [3], a number of works have extended that result to more and more specific types of Hamiltonians. In general, the goal is to understand the crucial properties of a Hamiltonian problem that make it QMA-complete, or, to the opposite, solvable in P or BQP. In this respect, one can try to restrict the number of qudits each Hamiltonian acts on, but also their spatial locality, their type (projections or not), and the dimension of the individual systems.

In order to perform such a reduction, one wishes to approximate a given Hamiltonian $H$ as $H \approx H_0 + \ldots + H_\ell$, where the $H_i$ are "simpler" than $H$: they act on less qubits, or have a simpler algebraic or spatial structure, etc. The precise sense in which the approximation is required to hold depends on the applications one has in mind. For the case of QMA, it is only required that the least eigenvalue be preserved, but for applications to the adiabatic theorem one might also wish for the ground state and gap to be preserved. The main tool available to prove such decompositions is perturbation theory (see [1]).

## Main result and proof overview

The main result in [4] is that the 2-local Hamiltonian problem is QMA-complete, even when the graph of interactions is restricted to nearest-neighbor interactions on a 2D square lattice.

The proof starts from the 5-local Hamiltonian obtained from Kitaev's reduction, with a slightly modified clock which ensures that all interactions are of constant range. The main idea is to organize the clock qubits around the circuit qubits so that the clock qubit $c_t$ is close to the computational qubits on which the circuit acts at time $t$.

The resulting graph of interactions has bounded degree and bounded interaction length. To transform it into a 2D lattice, the authors introduce a number of "perturbative gadgets" that can be used to manipulate the locality of a given Hamiltonian. For instance, there is a "subdivision gadget" which replaces a $k$-local Hamiltonian by four $\lceil k/2 \rceil + 1$-local Hamiltonians, acting on the original qubits plus one ancilla, in a way that is spatially local. There is also a "fork gadget", used to reduce the degree of vertices, and a "cross gadget", used to eliminate edge crossings. The appropriate gadgets are applied in parallel to Hamiltonians acting on different parts of the ground state, transforming the whole interaction graph to a nearest-neighbor one on a $2D$ lattice in a small number of sequential steps. This is important as each application of a gadget induces a blow-up in the norm of the Hamiltonians.

## Perturbative gadgets

The correctness of the gadgets is proved through the perturbative framework from [2], based on the use of resolvents. However, simpler proofs with better parameters can be given using the approach developed later in [1]. We try to explain briefly the intuition behind those constructions.

Let $H = \sigma_1 \otimes \cdots \otimes \sigma_\ell$ a $\ell$-local Hamiltonian, with the $\sigma_i$ being Pauli operators. Introduce an auxiliary qubits initialized to $|0\rangle_a$, and let $H_0 = |1\rangle\langle 1|_a$. Our goal is to design a $V$ such that $H_0 + \lambda V \approx \lambda^k H$ in the limit of small $\lambda$ (how small $\lambda$ needs to be is a crucial parameter of the construction, since it governs the size blow-up of the Hamiltonians), and $V$ has a simpler structure than $H$. As shown in [1], for this special setting, if we let $H_\ell$ be the lower $d$-dimensional part of $H_0 + \lambda V$ (where $d$ is the dimension of $H$) then

$$H_\ell = \sum_{m=1}^{\infty} \mathcal{A}^{(m)} \qquad \text{where} \qquad \mathcal{A}^{(m)} = \lambda^m \sum_{(m-1)} P_0 V S^{l_1} V S^{l_2} \cdots V S^{l_{m-1}} V P_0 \tag{1}$$

where $P_0$ is the projector on the ground space of $H_0$, $S = -P_0^\perp$, and $S^0 = -P_0$ by convention. The summation $(m)$ is over all sequences such that $l_1 + \ldots + l_m = m$ and $l_1 + \ldots + l_i \geq i$ for all $1 \leq i \leq m$. The series converges as long as $4\lambda\|V\| < 1$.

In designing $V$, one tries to have most of the terms in this summation be null, excepted the ones of interest (or higher-order terms, that are made negligible by taking $\lambda$ small enough). The idea is to exploit the interplay between $V$ and the $S$ terms: typically, $V$ will have a component which moves states from the ground space of $H_0$ to its excited space and vice-versa, so that for instance $S^0 V S^0 = S^1 V S^1 = 0$.

As an example, we describe the subdivision gadget. The original Hamiltonian is $H = A \otimes B$, where $A$ and $B$ each act on $k/2$ qubits. Let $H_0 = |1\rangle\langle 1|_a$, and $V = V_1 + V_2 = A \otimes \mathrm{Id} \otimes X + \mathrm{Id} \otimes B \otimes X$, where $X$ is a bit-flip on the ancilla qubit. Let us look at the first few terms in the expansion (1).

- The term $\mathcal{A}^{(1)}$ is $\lambda P_0 V P_0$, which is 0 since any state which has its ancilla to $|0\rangle$ gets it switched to $|1\rangle$ by $V$, and is set to 0 by $P_0$.
- $\mathcal{A}^{(2)} = \lambda^2 P_0 V S^1 V P_0 = -\lambda^2 P_0 V P_1 V P_0$. Expanding $V = V_1 + V_2$ leads to 4 terms. Of those, the ones with the same $V$ are $(k/2+1)$-local terms. The other two are identical to $A \otimes B \otimes |0\rangle\langle 0|$.

Setting $\lambda$ small enough will ensure that terms $\mathcal{A}^{(m)}$ for $m \geq 3$ are negligible: one can show that their total norm is bounded by $(4\|V\|\lambda)^3$.

More complicated gadgets, such as the fork gadget, follow a lightly more elaborate construction: there we have $H = A \otimes B \otimes C$, and we take $H_0 = |0\rangle\langle 0|$, $V = V_1 + V_2 + V_3 = C \otimes |1\rangle\langle 1| + B \otimes X - A \otimes X$. The first significant terms will appear at the third order expansion. The general idea is always the same: $V$ moves the ancilla qubits between the different energy levels of $H_0$ in a way that ensures that the lowest-order non-zero terms in (1) involve products of terms that make up the target Hamiltonian.

## Applications

As an application, the authors show how the cluster state, which is used in measurement-based computation, can be written as the ground state of a two-local Hamiltonian. This is important to argue that it is a physically realizable system; the natural definition of the cluster state has it as the ground state of a 5-local Hamiltonian.

As another application, we can use the subdivision gadget to provide a general $k$-to-2 reduction, achieving the same as [1], by proceeding in $\log k$ recursive steps. The benefit of this construction is that the blow-up in norm of the Hamiltonians is smaller ($k^{\log k}$ instead of $k^k$). However, the number of ancillas is larger (roughly $k^2$ instead of $k$).

An interesting open question that is mentioned in the paper is whether a perturbative theory could be developed that involves smaller blow-ups in norm, by only focusing on preserving the result of arbitrary local observable measurements on the ground state of the Hamiltonian, rather than preserving the whole state itself. This would be sufficient for adiabatic computation.

## References


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