Quantum Correlations in the Two Party Case

Reading Group Summary

Goal: We will show that if two distant parties are locally quantum and the no signaling principle is obeyed, all correlations between the two parties can be described using quantum mechanics.

Assumptions:

(1) *Locally Quantum*
We assume that Alice can perform all quantum measurements $M_A = \{Q_a\}_a$ (POVM), where $\sum_a Q_a = \mathbb{1}$, on a Hilbert space $\mathcal{H}_A$ of dimension d. The probability that she obtains outcome a for measurement M_A is given by a function $\mathcal{B}(\mathcal{H}_A) \to [0, 1]$, where $\mathcal{B}(\mathcal{H}_A)$ is the set of bounded operators acting on $\mathcal{H}_A$. The same holds for Bob, and we denote a possible measurement by $M_B = \{R_b\}_b$.

(2) *No Signaling*
We make no assumptions about the joint system held by Alice and Bob, so they can share any function ω such that $\Pr(a, b|M_A, M_B) = \omega(Q_a, R_b)$. We assume that the no signaling principle is obeyed - the marginal probability distribution observed by Alice must be independent of the measurements performed by Bob and vice versa.

Proof Idea: Our goal is to show that there exists a quantum state which reproduces the function ω shared by Alice and Bob. This is equivalent to showing that there exists a Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, a state $\rho_{AB} \in \mathcal{B}(\mathcal{H}_{AB})$ and measurements $\tilde{M}_A = \{\tilde{Q}_a\}_a$ and $M_B = \{R_b\}_b$ for Alice and Bob such that $\Pr(a, b|M_A, M_B) = tr((\tilde{Q}_a \otimes R_b)\rho_{AB})$. The proof will rely on Gleason's theorem and the Choi-Jamiolkowski isomorphism, which are included below. The first step will be to produce a matrix W_{AB} such that $\Pr(a, b|M_A, M_B) = tr((Q_a \otimes R_b)W_{AB})$. Since W_{AB} is only guaranteed to be positive on pure product states, the second step will use the CJ isomorphism to produce the final quantum state and operators as mentioned above.

Step 1 Finding W_{AB}: The idea in this step is to create a suitable linear map T to which we can apply the CJ isomorphism, obtaining the operator W_{AB}. Using the assumption that marginal probabilites are well defined, we apply Gleason's theorem to find unnormalized quantum states corresponding to each POVM element on both sides. These states preserve information from the shared preparation ω. We can then produce a linear operator $\hat{\omega}$ mapping POVM elements on Alice's side to unnormalized quantum states on Bob's side. By altering this linear operator slightly, we obtain T. Applying the CJ isomorphism to T gives W_{AB}.

Step 2 Quantum Correlation: We obtained W_{AB} in the previous step by using our linear map T and acting on Bob's side of the projection on the maximally entangled state $|\Phi\rangle$. Now if we can shift the action of T to a POVM element instead, we can set $\rho_{AB} = |\Phi\rangle\langle\Phi|$. Because W_{AB} is positive on pure tensor products, T is a positive map. Then if $T(\mathbb{1}) = \mathbb{1}$, T must map POVM elements to POVM elements. We can shift the action of T to Q_a to obtain $\tilde{Q}_a$, and we now have $\Pr(a, b|M_A, M_B) = tr((\tilde{Q}_a \otimes R_b)\rho_{AB})$. If T is not unital ($T(\mathbb{1}) \neq \mathbb{1}$), we can split T into a unital map T_1 and another map T_2. In this case, we will obtain ρ_{AB} by the action of T_2 on $|\Phi\rangle\langle\Phi|$ and we will obtain $\tilde{Q}_a$ by the action of T_1 on Q_a.

Open Problems: It would be interesting to understand why locally quantum behavior does not generally imply quantum correlations — for more than 2 parties, the theorem described above fails [2]. Another open issue is to understand why there exist no signaling correlations which cannot be obtained by locally quantum parties — is there a set of principles which would exactly characterize quantum correlations?

Gleason's Theorem: A frame function is a function $f : \mathcal{P}(\mathcal{H}) \to [0,1]$, where $\mathcal{H}$ is a Hilbert space and $\mathcal{P}(\mathcal{H})$ is the set of positive operators acting on $\mathcal{H}$, such that for all $X \subset \mathcal{P}(\mathcal{H})$, if $\sum_{E\in X} E = \mathbb{1}$ then $\sum_{E\in X} f(E) = 1$. Gleason's theorem states that for all frame functions there exists a unit trace positive operator W such that for $E \in \mathcal{P}(\mathcal{H})$, $f(E) = tr(WE)$.

Choi-Jamiolkowski Isomorphism: This is an isomorphism between linear maps $T : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ and operators $W_{AB} \in \mathcal{B}(\mathcal{H}_{AB})$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces of the same dimension and $\mathcal{B}(\mathcal{H}_A)$ is the set of bounded operators acting on $\mathcal{H}_A$. The map used here is as follows: $W_{AB} = \mathbb{1} \otimes T(|\Phi\rangle\langle\Phi|)$.

References:

[1] H. Barnum *et al.*, Phys. Rev. Lett. **82**, 5385 (1999).

[2] A. Acín *et al.*, Phys. Rev. Lett. **104**, 140404 (2010).