Hamiltonian Complexity

Warning: what follows is really rudimentary; I barely know anything about the area myself – just enough to want to get into it!

Hamiltonian complexity is an area that is at the interface of quantum computing and condensed matter physics; may interesting questions and methods come from the physicist's side, which raise many interesting algorithmic and complexity-theoretic questions. In this reading group we would go over the fundamental literature in the domain, hopefully eventually building up a good understanding of the more interesting, but hard to read, papers (see Hasting's papers...). As such it'll very much be about reading and understanding: although we'll be guided by many questions, they won't turn into proper research problems before we can have some understanding of the topic.

This blog post is probably a good place to start reading on the subject (see the accompanying slides), but here are some additional references that might help give some guidance through the vast literature. The first section concerns hardness results, and the second structure results and algorithms. Almost all papers are available on the arXiv: just search for their title.

1 QMA-completeness of the local Hamiltonian problem

A hermitian matrix $H = \sum_i H_i$ is called a k-local Hamiltonian if each H_i acts only on a k qubits (or, more generally, qudits: elementary systems of constant dimension d). H is said to be local 1-dimensional if the particles are arranged on a line, and each H_i only acts on pairs of nearest-neighbors.

One can consider various types of promises on H, and it is important not to mix them up. Usually, one either wants to estimate the smallest eigenvalue of H (sometimes called the ground state energy), or the corresponding eigenvector (the ground state). The spectral gap is the difference between the second smallest and smallest eigenvalues. The size of the Hamiltonians is an important parameter; usually all terms H_i are assumed to be of polynomial size.

Kitaev [7] initially showed that approximating the ground state energy of a 5-local Hamiltonian within 1/poly is QMA-complete (the input size is the description length of the Hamiltonian; to make this into a decision problem one has a promise that either the energy is $\leq E$ or $\geq E + E_0$ for a inverse-polynomial E_0). [6] later improved this to QMA-completeness for the 2-local Hamiltonians. Note that this was mildly surprising because the local Hamiltonian can be seen as a quantum analogue of 3-SAT, where each Hamiltonian is a constraint (clause), but 2-SAT is in P.

This line of results was stretched even further in[3] , where it is shown that even 1-dimensional systems are QMA-complete. An open question is to show the same hardness when the promise on the gap is that it is constant with respect to the total energy, i.e. the maximum eigenvalue of the total Hamiltonian H. The "open problems" section of that paper contains a brief discussion of the complexity of various Hamiltonian problems, depending on the size, whether there is a gap or not, etc.

Finally, the recent paper [2] proves "the detectability lemma": a relationship between the total energy of the system and the number of constraints violated. This can be seen as a first attempt towards a quantum PCP theorem.

2 Structure of gapped systems

Physicists are mostly interested in the properties of ground states of *gapped* local Hamiltonians, i.e. those for which there is a constant gap between the lowest and second-lowest eigenvalues. Apparently these are the states that arise in practice (in particular in adiabatic computing, the largest the eigenvalue gap the more efficient the computation). For a long time physicists have experimentally known that such ground states can be found and approximately described much more efficiently than one would expect. Only recently have these (heuristic) methods been formalized, and some precise results explaining them started to appear.

Matrix Product States (MPS) are a fundamental tool in the area. They were introduced in [8] as a generalization of Valence Bond States [1]. However, the best reference to start is maybe [11], who showed that any quantum state had a representation as a MPS. The size of the representation is governed by the amount of entanglement across any "cut" of the quantum state. [4] is also a good introduction to MPS representations. There are many subsequent, more elaborate, representations: see for instance [10].

Various methods are used in order to efficiently find the ground state of gapped Hamiltonians. The most popular seems to be the Density Matrix Renormalization Group (DMRG) methods: see http://www.itp.uni-hannover.de/ jeckelm/dmrg/ for extensive references.

In order to explain the success of these simulation methods and representations, some structural results have been recently proved for the ground states of local Hamiltonians. We would study and try to understand some of these properties in a CS-way during the reading group:

- Lieb-Robinson bounds. These govern in some way the propagation of information, or correlations, through the ground state. See this talk, or [9]
- Area Laws. These usually state that, in a ground state, the amount of entanglement going from one region to the other should be roughly proportional to the surface area (as opposed to the volume). See Hasting's paper [5] for the original result. See also this thesis for a more detailed exposition.
- Ground states exhibit many locality properties that enable their efficient simulation. This paper could be a start, but there are many others.

References

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