Fast Universal Quantum Computation with Railroad-switch Local Hamiltonians

Reading group summary by Anand Bhaskar

Introduction

Given a circuit $\mathcal{C}$ with gates $U_1, ..., U_T$ acting on the state $|\phi_0\rangle$, we want to simulate the circuit model computation $U_T U_{T-1} ... U_1 |\phi_0\rangle$ in the Hamiltonian Quantum Computing (HQC) model. In the HQC model, computation is performed by starting with a Hamiltonian H (which is a sum of local Hamiltonians) and an initial state $|\psi_0\rangle$ and evolving this initial state according to the Schrödinger equation with H for some time t to get a state $|\psi_t\rangle = e^{-iHt}|\psi_0\rangle$ and then measuring $|\psi_t\rangle$ in some basis to determine the answer of the computation.

It is easy to take a Hamiltonian from the HQC model and implement it in the circuit model by dividing the evolution time t into a large number of intervals and applying the unitary corresponding to each interval (the Lie-Suzuki-Trotter formula is used in this discretization).

The more interesting problem is to create a local Hamiltonian H which can be used in an HQC to simulate a quantum circuit $\mathcal{C}$. In [1], Nagaj showed two such constructions for H. In the first construction, H will be the sum of 3-qubit projector terms, and in the second construction which will be a relatively simple modification of the first, H will be the sum of qubit-qutrit projectors (ie. 6×6 matrices).

A couple of properties of the Hamiltonian H from both constructions:

- $H = \sum_i H_i$ where each H_i is a local projector term.
- H will be frustration-free. This means that if $H = \sum_i H_i$, then the ground state of H is also a ground state for each H_i. The paper seems to indicate that this leads to a large eigenvalue gap above the ground state energy, but this isn't clear from the definition.
- H will be time-invariant. This is a freebie, but is claimed to be a desirable property for HQC.

Overview

We will follow a very similar strategy to what we did for the QMA-completeness reductions. We augment the Hilbert space of the circuit qubits with a clock space and create a Hamiltonian whose ground state is the history state of the circuit. This Hamiltonian will just be the Hamiltonian which ensured correct time propagation of the circuit in the QMA-completeness reductions, namely:

$$H = \sum_{t=1}^{T} \frac{1}{2}(|t\rangle\langle t| + |t-1\rangle\langle t-1| - U_t \otimes |t\rangle\langle t-1| - U_t^\dagger \otimes |t-1\rangle\langle t|)$$

Define the intermediate states of the original circuit as $|\phi_t\rangle = U_t ... U_1 |\phi_0\rangle$. If we define the basis $B = \{|\psi_t\rangle = |\phi_t\rangle \otimes |t\rangle\}$ for the Hilbert space $\mathcal{H}_{\phi_0}$ spanned by B, then the matrix H restricted to the subspace $\mathcal{H}_{\phi_0}$ and written in this basis B looks like that of a symmetric random walk on the states of B arranged in a line.

Connection with quantum random walks

The above connection between Hamiltonians and random walks can indeed be realized algorithmically and was first noticed by Farhi. If we have such a Hamiltonian H above, then we view the Schrödinger evolution e^{-iHt} for some time t as a random walk generated by H on the space $|\phi_0\rangle|0\rangle, |\phi_1\rangle|1\rangle, ..., |\phi_T\rangle|T\rangle$. From the theory of quantum random walks on a line, we will pick a random time $t \leq O(T \log^2 T)$, evolve from $|\phi_0\rangle|t\rangle$ for time t using e^{-iHt}, and measure the clock register. The measurement will be any of $0, 1, ..., T$ with equal probability and thus the work space will be left in any of $|\phi_0\rangle, |\phi_1\rangle, ..., |\phi_T\rangle$ with equal probability $\frac{1}{T+1}$. Since we want to get the output $|\phi_T\rangle$ of the circuit with constant probability, say $2/3$, and the above random walk samples ϕ_T with probability $\frac{1}{T+1}$, we can just append dummy identity gates to make the circuit three times as long, and now with probability $2/3$, the random walk would sample the outcome of the circuit. Note that it is sufficient to restrict the attention of H to the subspace $\mathcal{H}_{\phi_0}$ in the random walk since the initial state $|\phi_0\rangle|0\rangle \in \mathcal{H}_{\phi_0}$, and H (and hence e^{-iHt}) leave the space $\mathcal{H}_{\phi_0}$ invariant.

Clock construction

We still have to specify how to implement the clock in the above construction for H.
In the QMA-completeness results, we used a wall-clock, which used T qubits where t 1's followed by $T-t$ 0's represented the clock state $|t\rangle$. It's easy to imagine an alternative clock with $T+1$ qubits, where $|t\rangle$ is represented by the t-th qubit being 1 and all other qubits being 0. This clock is called the pulse clock. The pulse clock was unsuitable for QMA-completeness results because you can't use local terms to enforce that at least one of the qubits is a 1. But enforcing correctness is not an issue in this HQC construction, so we will use the pulse clock since terms like $|t\rangle\langle t-1|$ are now 2 local (while they would have to be 3 local in the wall clock construction). Since the circuit contains 2-qubit and 1-qubit unitaries, we immediately have a Hamiltonian that is the sum of 4-local qubit projectors.

3-qubit projectors

Assume that the circuit is composed of 1-qubit and C-Not gates. The 1-qubit gates only introduce 3-local projectors to H (1 for the unitary and 2 for the clock terms like $|t\rangle\langle t-1|$). Whenever we have a C-Not, say at time t, with control qubit q_1 and target qubit q_2, we expand the space in the HQC model to handle the branching of the two cases of when $q_1 = 0$ and $q_1 = 1$.
We write the state before the C-Not, $|\psi_{t-1}\rangle = |\phi_{t-1}\rangle|t-1\rangle = (a|\varphi_0\rangle + b|\varphi_1\rangle) \otimes |t-1\rangle$ where $|\varphi_0\rangle$ is the subspace of $|\phi_{t-1}\rangle$ where $q_1 = 0$ and $|\varphi_1\rangle$ is the subspace where $q_1 = 1$. Then, if we define $|\psi'_{t-1}\rangle = a|\varphi_0\rangle|l_1\rangle + b|\varphi_1\rangle|u_1\rangle$, we can enforce transitions between $|\psi_{t-1}\rangle$ and $|\psi'_{t-1}\rangle$ by a suitable projector term which is 3-local (this term essentially projects onto the space spanned by the 0-subspace of q_1 tensored with the l_1 clock subspace, and the 1-subspace of q_1 tensored with the u_1 subspace). This is the splitting part of the railroad-switch gadget construction where $|\phi_{t-1}\rangle$ splits into two tracks based on the previously described projection. The upshot of this is that we can directly apply a $\sigma^x_{q_2}$ gate on the upper track (ie. when in the u_1 clock subspace) since we know that $q_1 = 1$ here, and apply an identity on the lower track (ie. when in the l_1 clock subspace) since we know that $q_1 = 0$ here. This gate application can again be written as a 3-local projector, enforcing movement from $|\psi'_{t-1}\rangle$ to $|\psi''_{t-1}\rangle$ where $|\psi''_{t-1}\rangle = a|\varphi_0\rangle|l_2\rangle + b\sigma^x_{q_2}|\varphi_1\rangle|u_2\rangle$. Finally, we join the two tracks using another projector which will enforce movement from $|\psi''_{t-1}\rangle$ to $|\psi_t\rangle$.

Refer to figure 3 in the paper for a better picture.
If we now augment the basis B described previously with these states $|\psi_{t-1}\rangle'$ and $|\psi_{t-1}\rangle''$ whenever U_t is a C-Not, and replace the projector terms corresponding to C-Not gates in H with the new projector terms coming from the railroad-switch gadget, we can see that we still have a random walk but now on a slightly larger space due to these extra basis states from the railroad-switch gadgets. But now we've reduced the Hamiltonian to be a sum of 3-qubit projectors.

Qubit-qutrit projectors

In the railroad-switch gadget for C-Not with control qubit q_1 and target qubit q_2, we needed 3-qubit projectors because when we performed the projection onto $|0\rangle\langle 0|_{q_1}$, we also moved on the clock space from time $|t-1\rangle$ to $|l_1\rangle$. But now suppose we create a qutrit clock state and decide that whenever we perform a 1-qubit projection on the work space we will only move from one state of the qutrit to another state of the same qutrit, then we can encode the projection $|0\rangle\langle 0|_{q_1}$ along with the clock space change as a qubit-qutrit projector. This same argument applies for the 3-qubit projector term involving the $|1\rangle\langle 1|_{q_1}$ projector for moving to the upper track and for the $\sigma^x_{q_2}$ gate applied on the upper track. The only problem is that there is no way to ensure that only the $q_1 = 0$ part of the state moves to the lower track and only the $q_1 = 1$ part of the state moves to the upper track, since we've moved the $|0\rangle\langle 0|_{q_1}$ and $|1\rangle\langle 1|_{q_1}$ projectors onto qutrit states on the lower and upper tracks respectively. But if the wrong value for q_1 entered a particular track, the projector $|0\rangle\langle 0|_{q_1}$ or $|1\rangle\langle 1|_{q_1}$ (as the case may be) would kill the state in the next step, so the random walk is still almost a random walk on a line, but with an extra edge sticking out from some points on the line corresponding to the wrong value of q_1 entering a track. In the paper, they actually convert the line state graph to a circle for ease of analysis and make the C-Not's evenly spaced so that the state graph looks like a circle with edges spiking out periodically (they call it a necklace graph). They claim that the mixing time is $\tilde{O}(T^2)$ for this necklace graph which is of length $O(T)$ here.
Refer to figure 4 in the paper for a good picture of how the projectors act.

References

[1] D. Nagaj, Fast Universal Quantum Computation with Railroad-switch Local Hamiltonians, arXiv:0908.4219