

Tensor networks and Probability Amplification for QIP(3) by

Parallel Repetition

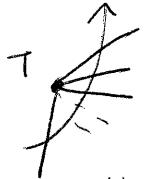
Tensor Network:

A tensor network is one way of visualizing a tensor. Let us consider a tensor $T \in \mathcal{H}^{\otimes k}$, where \mathcal{H} has a fixed basis $|0\rangle, |1\rangle, \dots, |n-1\rangle$. Then T can be written as:

$$T = \sum_{\bar{l}=(l_1, l_2, \dots, l_k)} \alpha_{\bar{l}} |l_1\rangle \langle l_2 | \dots |l_k\rangle$$

where \bar{l} varies over $\{0, 1, 2, \dots, n-1\}^k$.

In the Tensor network notation, we denote this by a vertex T , attached to k edges: k edges, labelled from 1 to k



The Tensor "outputs" the number $\alpha_{\bar{l}}$ when the i th edge gets "input" $|l_i\rangle$, for every tuple \bar{l} .

The utility of this notation is its versatility. For example, we can visualize how the tensor represents a multi-linear map. Consider a tensor $T \in \mathcal{H}^{\otimes 4}$. We see how we can interpret its diagram as a map from $\mathcal{H}^{\otimes 2}$ to $\mathcal{H}^{\otimes 2}$:

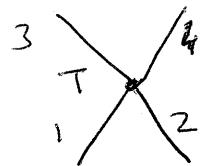


Figure: Representation of $T \in \mathcal{H}^{\otimes 4}$

We define $\langle m_T | i\rangle \otimes | j\rangle = | k\rangle \otimes | l\rangle$ as $T(i, j; k, l)$, which is obtained by "inputting" i, j, k, l on "inputs" 1, 2, 3, 4 above. Now, the map is clearly defined:-

$$m_T(| i\rangle \otimes | j\rangle) = \sum_{k,l} T(i, j; k, l) | k\rangle \otimes | l\rangle$$

Operations with Tensor diagrams-

The ~~wield~~ beauty of the Tensor network approach lies in how these diagrams can be composed in natural ways to visualize tensor operations.

- ① Tensor product: The given tensor diagrams for T and S , the tensor diagram for $T \otimes S$ is obtained by:
- ⊕ Writing the two diagrams together and
- ② Taking the "output" as the product of the "outputs" of T and S when all inputs of both T and S have been set. Example.

$$T: \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 3 \quad 4 \\ \diagdown \quad \diagup \\ 2 \end{array} \otimes S: \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ 1 \quad 2 \\ \diagdown \quad \diagup \\ 3 \end{array} =$$

$$\begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ i \quad j \quad k \quad l \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ 1 \quad 2 \quad 3 \quad 4 \end{array} \quad (T \otimes S)_{i,j,k,l,m,n} = T_{i,j,k,l} S_{m,n}$$

This clearly preserves the semantics of the tensor product.

② Contraction: This operation is a generalized trace, and can be used to represent various operations such as inner products and traces. ~~When~~ It is best illustrated through an example:- Consider tensors v and ω as follows.

$$v \in \mathcal{H}^{\otimes 5}$$

$$\omega \in \mathcal{H}^{\otimes 3}$$

Then a possible contraction is

$$v(l_1, l_2, r_1, r_2, \omega(l_3, r_3)) = s(l_1, l_2, l_4) \delta(l_3, r_3)$$

This has 3 inputs, l_1, r_2 and ~~l_4~~ , s s.t.

$$s(l_1, r_2, r_3) = \sum_{l_3, r_3} v(l_1, l_2, l_3, l_4) s(l_2, r_2, l_3)$$

For our convenience, we also define conjugation, so that ~~v^*~~ in the natural way, so that if ~~v~~ is a v^* is the tensor whose output is the complex conjugate of the output of v on the same input.

Some examples of the use of these operations:

① Inner product:

Let $v \in \mathcal{H}$, $w \in \mathcal{H}$. So, their diagrams are

$$v \xrightarrow{*} 1 \quad w \xrightarrow{*} 1 \quad \text{the contraction}$$

The inner product $\langle v | w \rangle$ is thus / $v^* \xrightarrow{*} w$

② Matrix trace:

Let $v \in \mathcal{H} \otimes \mathcal{H}$ be the matrix tensor denoting

the corresponding matrix: $v \swarrow_j$

Then $\text{Tr}(v)$ is clearly denoted by ~~$\text{Tr}(v) = \sum_j$~~ and the contraction

$$\text{• Tr}(v) = \sum_j$$

③ Suppose a tensor X has the Schmidt decomposition

$$X = \sum \alpha_i |v_i\rangle \langle w_i|, \text{ then}$$

$$l_1 \swarrow l_2 \quad X \quad l_{2k}$$

=

$$l_2 \swarrow l_1 \quad l_{n+1} \quad l_{n+2} \\ l_k \quad v \quad a \quad l_{2k}$$

Where $a_{ii} = \alpha_i$
and $a_{ij} = 0$ if $i \neq j$

This is the Schmidt decomposition.

④ Matrix multiplication: Let $u \in H \otimes H$, $v \in H \otimes H$ be tensors representing matrices:

$$l_1 \xrightarrow[u]{} r_1 \quad l_2 \xrightarrow[v]{} r_2$$

Then the tensor $s = uv$ is

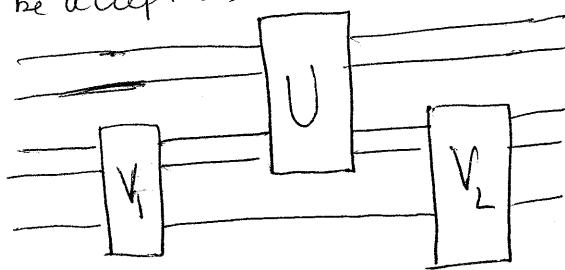
$$\begin{array}{c} l_1 \xrightarrow[s]{} r_2 \\ l_1 \xrightarrow{s} r_2 \end{array} = \begin{array}{c} l_1 \xrightarrow[u]{} v \xrightarrow[r_2]{} \end{array}$$

Parallel Repetition Repetition

We will now see how to use the Tensor Network framework to do probability amplification for QIP(3). Specifically, we want to show that if a given QIP(3) protocol has soundness s , then repeating it twice gives a soundness of s^2 .

Consider a QIP(3) protocol: (Assume the input x , which is used to define V_1, V_2 , is not to be accepted).

Fig ①

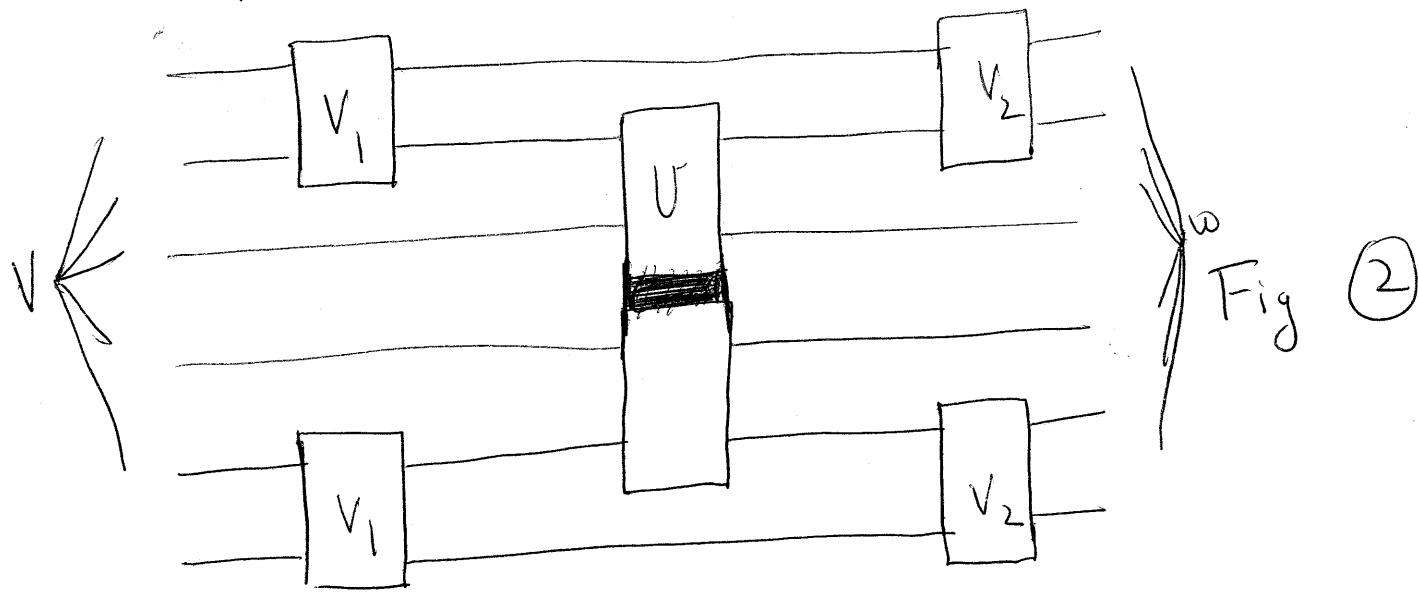


(V_1, V_2) are not necessarily unitaries, and may involve projections (such as projecting part of input to 10)

Then The Soundness of the protocol is $\leq s$ iff for all unitaries U , and all pure states V, W ,

$$|\langle w | V_2 U V_1 | v \rangle| \leq s. \quad -\textcircled{1}$$

The parallelized protocol looks like:



Thus, the prover can try to cheat by entangling its responses to the two ~~sep~~ parallel runs. This has soundness s' iff for all states $|v\rangle, |w\rangle$ and unitaries U .

$$|\langle w| (V_2 \otimes V_1) U (V_1 \otimes V_1) |v\rangle| \leq \sqrt{s'} \quad \text{--- ②}$$

Now, we notice that inner product corresponds to contraction, so ② above corresponds to the fact that for any two vectors v and w , if we contract their tensors with the tensor of the protocol as shown in the figure ②, then the value of the tensor should be less than $\sqrt{s'}$, for all choices of U . We need to use the fact that for the following tensor network:- (Fig 3)

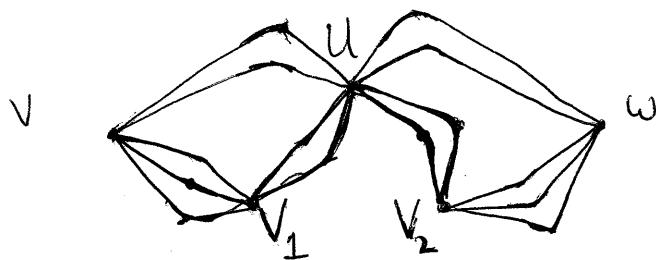
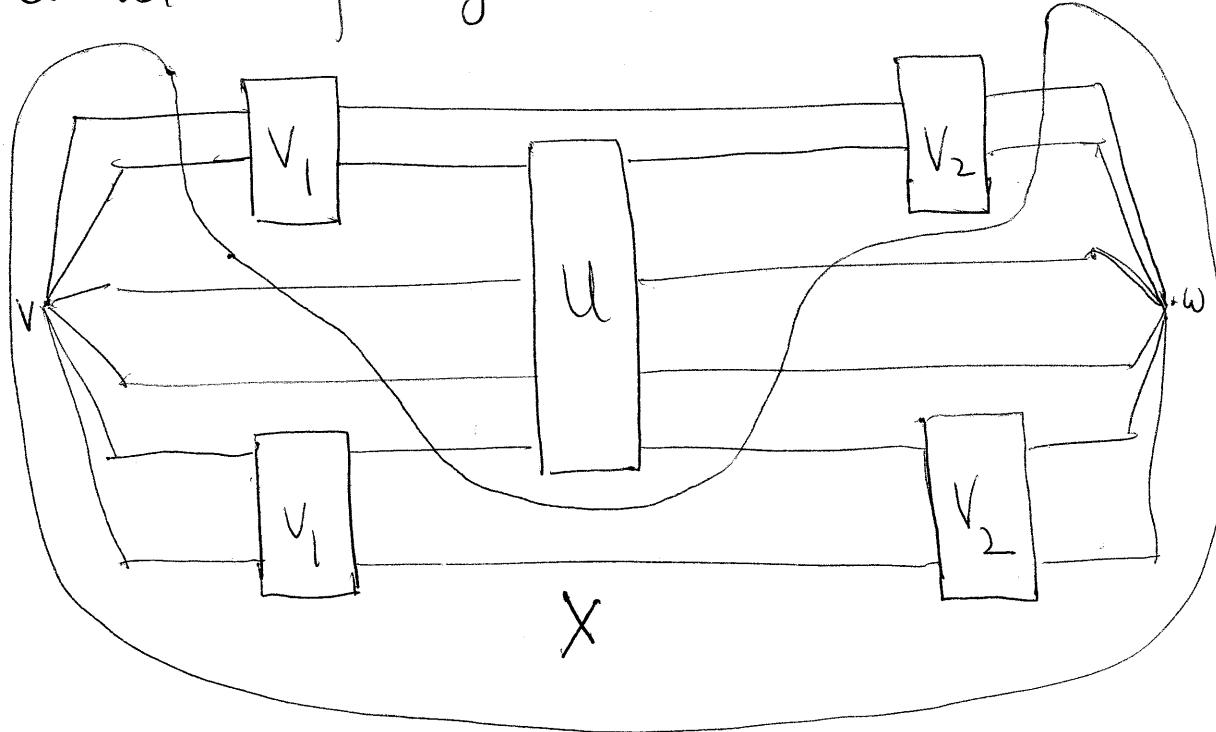


Fig 3 :- This is the tensor for the akt in Fig ① with arbitrary inputs v and output w and unitary U .

v, w are restricted to be unit vectors.

The fact that the single round protocol in Fig ① has \pm soundness s is equivalent to the fact that the value of the above tensor is $\pm s$ for any choice of U unitary, and v, w unit norm vectors.

Now we want to 'break' the tensor of the parallelized protocol in Fig ② to get be able to use fig ③. Consider the following cut.

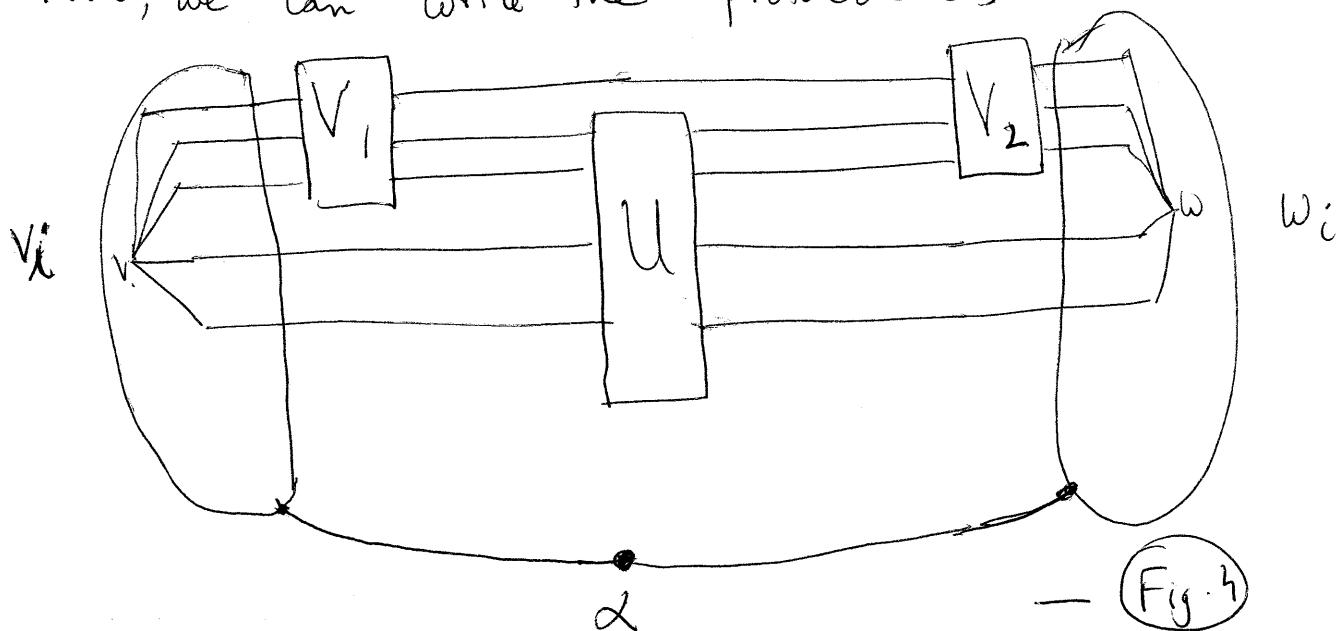


Let us look at the tensor X inside the curve drawn above, and suppose that its Schmidt decomposition is

$$X = \sum \alpha_i |v_i\rangle \otimes |w_i\rangle$$

where α_i 's are ~~non-negative~~ ^{positive} reals.

Now, we can write the protocol as:



- Fig 4

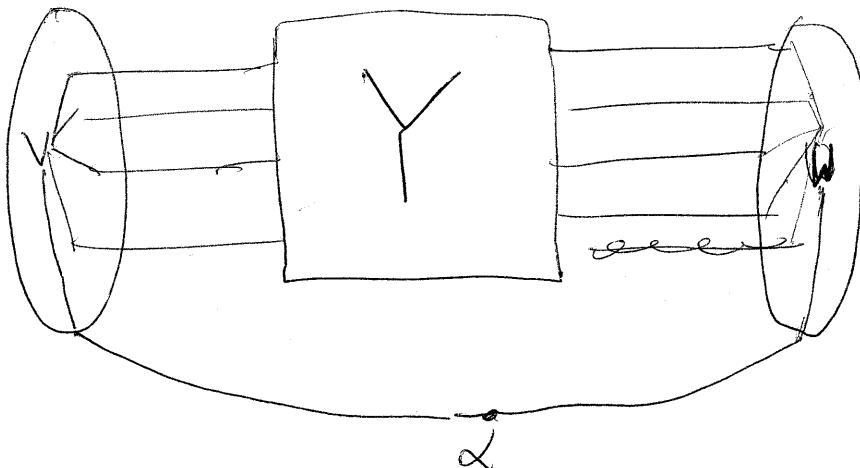
Now we can have exactly the situation in figure ①, and so we can see that the value of this tensor is at most: $\sum \alpha_i \sqrt{S} = \sqrt{S} \left(\sum \alpha_i \right)$, for any choice of U unitary.

Now, we need to look at the ~~#~~ the same tensor in another way to get an estimate on $\sum \alpha_i$.

We do this in two steps:

- ① Notice that the Value of the Fig 5 is maximized when $V_1 U V_2$ actually takes each $|v_i\rangle$ to $|w_i\rangle$. We

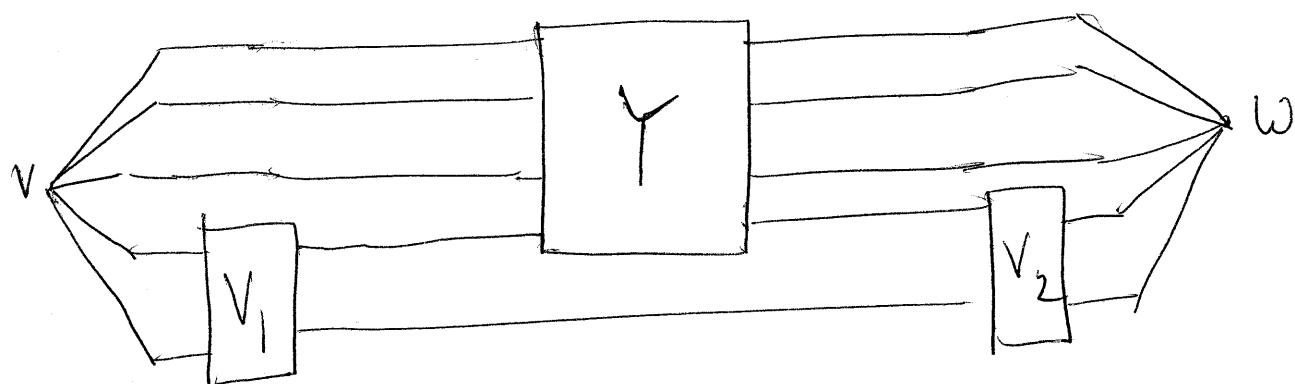
Can thus say that the value of the fig ⑤ is at most the value of below is $\sum \alpha_i$ (Figure 5)



where Y is a unitary which in particular takes $|v_i\rangle$ to $|w_i\rangle$

$$\text{Thus, the Value is } \sum \alpha_i \langle v_i | v_i \rangle \langle v_i | w_i \rangle = \sum \alpha_i$$

However, Fig (S) is by definition equivalent to



And this matches figure 3, so its value is at most \sqrt{s} . So, we get that $\sqrt{s} \geq \sqrt{\sum \alpha_i}$

Thus, we get that the value of fig ④ is at most $\sqrt{s} \cdot \sqrt{s} = s$, and so the value of fig 2 is $\sqrt{s'} = s \Rightarrow s' = s^2$. But s'

was defined to be the soundness of the parallelized protocol, and thus we are done.

We can repeat the same argument by induction, and finally, we get that if we do parallel repetition n times, the soundness drops to $\pi^{\frac{1}{2^n}}$.