6.890 HOMEWORK 5 **Due: November 23, 2021**
Submit by email to `6890mit@gmail.com` Handed out: November 10, 2021

Instructions: Everyone needs to submit their own write-up. If you work together with other students, indicate their names on your write-up.

The maximum number of points in the problem set is 16.

Problem 1 (4 points)

Recall the Tutte matrix of a graph $G = (V, E)$ is the matrix T with

$$T = \begin{cases} 0 & \text{if } (i,j) \notin E \\ x_{ij} & \text{if } (i,j) \in E \text{ and } i < j \\ -x_{ji} & \text{if } (i,j) \in E \text{ and } i > j \end{cases}$$

Above, the x_{ij} are the $|E|$ distinct variables corresponding to the edges. Show that the rank of T is twice the size of a maximum matching of G.

Hint: Use the following fact (without proof): Let A be an $n \times n$ skew-symmetric matrix of rank r. For any two sets $S, T \in \{1, \dots, n\}$, denote by $A[S,T]$ the submatrix of A consisting of the rows indexed by S and the columns indexed by T. Then, for any two sets $S, T \in \{1, \dots, n\}$ of size r,

$$det(A[S,S]) \times det(A[T,T]) = det(A[S,T]) \times det(A[T,S]).$$

Problem 2 (each subproblem 2 points)

Let $G = (V, E)$ be a directed graph with edge weights in $\{1, \dots, M\}$. The *diameter* of G is the quantity $D = \max_{u,v \in V} d(u, v)$, where $d(u, v)$ is the distance between u and v in G.

Here we will show that D can be computed in $\tilde{O}(Mn^{\omega})$ time.

(a) Let $N = O(M)$. Show that one can compute $d(u, v)$ for all pairs for which $d(u, v) \leq N$ in $\tilde{O}(Mn^{\omega})$ time. (Hint: successive squaring.)

(b) Let $k \leq Mn$ and $N = O(M)$ be given integers. Show that one can compute $d(u, v)$ for all pairs u, v for which $d(u, v) \in [k - N, k + N]$ in $\tilde{O}(Mn^{\omega})$ time.

(Hint: Use recursion and distance product on matrices with entries in $[-O(M), O(M)]$, based on the following Fact: any st-path of weight w has a middle edge (u, v) such that $d(s, u), d(v, t) \in [\lfloor w/2 \rfloor - M, \lfloor w/2 \rfloor]$.)

(c) Let $k \leq Mn$ be a given integer. Show that in $\tilde{O}(Mn^{\omega})$ time one can compute a matrix B such that $B[i, j] = 1$ if and only if $d(i, j) \leq k$.

(Hint: use (a),(b), Boolean matrix multiplication and the middle-edge fact from above.)

(d) Conclude that using (c) one can compute the diameter in $\tilde{O}(Mn^{\omega})$ time overall.

Problem 3 (4 points)

Let A be an arithmetic algorithm if the only operations the algorithm is allowed to perform is (1) multiplication of an already computed value by a scalar, (2) multiplication of two already computed values, (3) dividing a value (or a scalar) by a nonzero value, (4) adding two values.

Let f be a polynomial function over N variables, $x_1, \ldots, x_N$ over some field K. Let $L(f)$ be the minimum number of operations that an arithmetic algorithm uses to evaluate f on an arbitrary given point. Let $L(f_1, \ldots, f_k)$ be the minimum number of operations that an arithmetic algorithm uses to evaluate all of $f_1, \ldots, f_k$ on the same given (arbitrary) point. Baur-Strassen's theorem showed that

$$L(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_N) \leq 6L(f).$$

Let $F_f = \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} \right\}_{ij}$ be the set of all the second order partial derivatives of f. Suppose now that

$$L\left(f, F_f\right) \leq O(L(f)),$$

for all polynomials f. Show that then matrix multiplication is in $O(n^2)$ time.

Hint: Consider $f(u_1, \ldots, u_n, \{a_{ij}\}, \{b_{ij}\}, v_1, \ldots, v_n) = \sum_{ijk} u_i a_{ij} b_{jk} v_k$.