6.890 Lecture 18 Baur-Strassen Thm and Applications
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1 Baur-Strassen

We first introduce the notion of straight-line programs (SLP), also known as arithmetic circuits. Let K be a field with binary operations $\odot \in \{+, \cdot, /\}$. Let $F(x_1, ..., x_n)$ be a rational function with variables $x_1, ..., x_n \in K$.

Definition 1.1. *A straight-line program computing F is a sequence of operations $\{g_1, ..., g_s\}$ such that each g_i is of one of the following forms: (1) $g_i \leftarrow g_j \odot g_k$ for $j, k < i$, (2) $g_i \leftarrow g_j \odot c$ for $j < i$ and $c \in K$, (3) $g_i \leftarrow x_k \odot x_l$, (4) $g_i \leftarrow x_k \odot c$ for $c \in K$ or (5) $g_i \leftarrow g_j \odot x_k$ for $j < i$. The output from g_s gives $F(x_1, ..., x_n)$.*

The definition easily extends to the case when F outputs a set of rational functions by having multiple output gates.

Consider the following example of an SLP. We wish to compute $F(x_1, x_2, x_3) = 5 - x_1x_2^2/x_3$. Define the following sequence of operations:

$$g_1 \leftarrow x_2 \cdot x_2$$

$$g_2 \leftarrow x_1 \cdot g_1$$

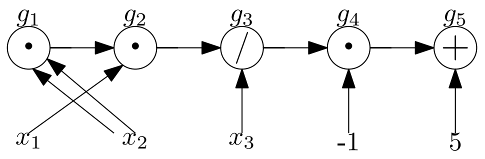
$$g_3 \leftarrow g_2/x_3$$

$$g_4 \leftarrow -1 \cdot g_3$$

$$g_5 \leftarrow 5 + g_4$$

Then the above sequence is a straight-line program computing F.

A nice property of SLPs is that they're easily representable by circuits where the g_i's are the gates in the circuit. For example, the above SLP is represented by the following circuit.



Now we will state the Baur-Strassen theorem, which essentially says that if you have an SLP of size s that computes F, then you can convert it into an SLP of size $5s$ that computes all of the partial derivatives of F.

Theorem 1.1. **[Baur-Strassen]** *Let $F(x_1, ..., x_n)$ be a rational function over a field K. Consider the set $F' = \{\frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_n}\}$. Then $T(F') \leq 6T(F)$, where $T(F)$ is the minumum number of operations needed in a straight-line program computing F as defined above.*

Proof. Suppose $\{g_1, ..., g_s\}$ is a minimum-length SLP that computes $F(x_1, ..., x_n)$. Let $F^{(i)}$ be the function computed by $g_i, \ldots, g_s$. The variable set for $F^{(i)}$ is $\{x_1, \ldots, x_n, g_1, \ldots g_{i-1}\}$ and the number of operations is $s - i + 1$. For ease of notation, we rename the variables to $\{z_1, \ldots z_{n+s}\}$ where $z_j = x_j$ for $j \leq n$ and $z_j = g_{j-n}$ for $j > n$.

Now we will use induction. We will assume that we have computed all the derivatives of $F^{(i+1)}$ with at most $5(s - i)$ operations. Then, we will show how to compute all the derivatives of $F^{(i)}$ with at most $5(s - i + 1)$ operations.

First, the base case: $i = s$. Note that $F^{(s+1)}(x_1, \ldots, x_n, g_1, \ldots g_s) = g_s$. Thus, $\frac{\partial F^{(s+1)}}{\partial g_s} = 1$ and for all $z_j \neq g_s$, $\frac{\partial F^{(s+1)}}{\partial z_j} = 0$. This requires zero operations.

Suppose inductively that we can compute for all j, $\frac{\partial F^{(i+1)}}{\partial z_j}$ with $5(s-i)$ operations. We will show that only 5 additional operations are needed to compute for all j, $\frac{\partial F^{(i)}}{\partial z_j}$.

Note that $F^{(i)}$ is equal to $F^{(i+1)}$ with g_i substituted for one of the operations in the definition of SLP, i.e. g_i is substituted for $s_b \odot s_\ell$ where $s_b, s_\ell \in K \cup \{g_r : r < i\} \cup \{x_t\}_{t\in[n]}$. Note that $\frac{\partial F^{(i)}}{\partial z_j} = 0$ for $z_j = g_p$ with $p > i$. For the rest of the z_j, the chain rule implies that

$$\frac{\partial F^{(i)}}{\partial z_j} = \frac{\partial F^{(i+1)}}{\partial z_j} + \left(\frac{\partial F^{(i+1)}}{\partial g_i}\right)\left(\frac{\partial g_i}{\partial z_j}\right).$$

Thus, if $z_j \notin \{s_b, s_\ell\}$ then $\frac{\partial F^{(i)}}{\partial z_j} = \frac{\partial F^{(i+1)}}{\partial z_j}$, and if $z_j = s_b$ then $\frac{\partial F^{(i)}}{\partial z_j} = \frac{\partial F^{(i+1)}}{\partial s_b} + \left(\frac{\partial F^{(i+1)}}{\partial g_i}\right)\left(\frac{\partial g_i}{\partial s_b}\right)$, and analogously for s_ℓ.

Consider the following cases for $g_i = s_b \odot s_\ell$.

Case 1: $g_i = z_t + z_{t'}$. In this case we only need to compute the following derivatives:

$$\frac{\partial F^{(i)}}{\partial z_t} = \frac{\partial F^{(i+1)}}{\partial z_t} + \frac{\partial F^{(i+1)}}{\partial g_i}$$

$$\frac{\partial F^{(i)}}{\partial z_{t'}} = \frac{\partial F^{(i+1)}}{\partial z_{t'}} + \frac{\partial F^{(i+1)}}{\partial g_i}$$

Each of these two computations only requires 1 operation, for a total of 2 operations.

Case 2: $g_i = z_t \cdot z_{t'}$. In this case we only need to compute the following derivatives:

$$\frac{\partial F^{(i)}}{\partial z_t} = \frac{\partial F^{(i+1)}}{\partial z_t} + \frac{\partial F^{(i+1)}}{\partial g_i} \cdot z_{t'}$$

$$\frac{\partial F^{(i)}}{\partial z_{t'}} = \frac{\partial F^{(i+1)}}{\partial z_{t'}} + \frac{\partial F^{(i+1)}}{\partial g_i} \cdot z_t$$

Each of these two computations require 2 operation, for a total of 4 operations.

Case 3: $g_i \leftarrow z_t / z_{t'}$. In this case we only need to compute the following derivatives:

$$\frac{\partial F^{(i)}}{\partial z_t} = \frac{\partial F^{(i+1)}}{\partial z_t} + \frac{\partial F^{(i+1)}}{\partial g_i} \Big/ z_{t'}$$

$$\frac{\partial F^{(i)}}{\partial z_{t'}} = \frac{\partial F^{(i+1)}}{\partial z_{t'}} + \frac{\partial F^{(i+1)}}{\partial g_i} \cdot \left(\frac{-z_t}{z_{t'}^2}\right)$$

These can be computed as follows: (1) compute $a \leftarrow \frac{\partial F^{(i+1)}}{\partial g_i} \Big/ z_{t'}$, (2) compute $b \leftarrow a \cdot (-1)$, (3) compute $c \leftarrow b \cdot g_i$, here $c = -\frac{\partial F^{(i+1)}}{\partial g_i}(z_t)/z_{t'}^2$, (4) compute $\frac{\partial F^{(i+1)}}{\partial z_t} + a$, (5) compute $\frac{\partial F^{(i+1)}}{\partial z_t} + c$.

This gives a total number of 5 operations.

These three cases also cover the case when s_b or s_ℓ is a scalar, since this is strictly easier. $\square$

Here's an example for our earlier function. Below's the SLP for the partial derivatives of $F(x_1, x_2, x_3) = 5 - x_1 x_2^2 / x_3$. The partial derivatives should be $\partial F / \partial x_1 = -x_2^2 / x_3$, $\partial F / \partial x_2 = -2 x_1 x_2 / x_3$, and $\partial F / \partial x_3 = x_1 x_2^2 / x_3^2$.

$$g_1 \leftarrow x_2 \cdot x_2$$
$$g_2 \leftarrow x_1 \cdot g_1$$
$$g_3 \leftarrow g_2/x_3$$
$$g_4 \leftarrow -1 \cdot g_3$$
$$g_5 \leftarrow 5 + g_4$$
$$Dg_5Dg_4 \leftarrow 1$$
$$Dg_5Dg_3 \leftarrow -1$$
$$Dg_5Dg_2 \leftarrow Dg_5Dg_3/x_3 = -1/x_3$$
$$Dg_5Dx_3 \leftarrow Dg_5Dg_3 \cdot g_3 \cdot Dg_5Dg_2 = g_2/x_3^2 = x_1x_2^2/x_3^2 \text{ (two ops)}$$
$$Dg_5Dx_1 \leftarrow Dg_5Dg_2 \cdot g_1 = -g_1/x_3 = -x_2^2/x_3$$
$$Dg_5Dg_1 \leftarrow Dg_5Dg_2 \cdot x_1 = -x_1/x_3$$
$$Dg_5Dx_2 \leftarrow Dg_5Dg_1 \cdot 2x_2 = -2x_1x_2/x_3$$

Interestingly, as far as we know the Baur-Strassen theorem only works for first derivatives, not second derivatives. In fact, if it worked for second derivatives then ω would be 2.

2 Shortest Cycle

Let G be a directed graph on n nodes with edge weights in $\{1, \dots, M\}$. Our goal is to find the cycle in G with the minimum total weight.

Remark 1. *We can do this in $O(n^3)$ time. Perform Dijkstra's from every node and stop when we find the shortest path back to the original node.*

Remark 2. *If there exists a $\tilde{O}(n^{(3-\epsilon)} \log^c M)$ algorithm for some $\epsilon > 0$, then APSP can be done in $O(n^{(3-\epsilon)} \log Mn)$ time. Furthermore, this result holds in both directions.*

We will present a $O(Mn^\omega)$ algorithm. This algorithm will use the framework from [1]. First we will show how to get the weight of the cycle and then we will use the Baur-Strassen theorem to find the cycle.

2.1 Finding the weight of the shortest cycle

Theorem 2.1. *In $\tilde{O}(Mn^{\omega})$ time we can compute the weight of the shortest cycle.*

Proof. We will define a symbolic matrix, similar to last lecture. The matrix will have a variable x_{ij} for each edge (i,j) as well as a variable y. Like last lecture we will plug in random values for the x_{ij}, however y will remain a variable.

Let $G = (V, E)$, with weights $w : E \to \{1, \dots M\}$. Define x_{ij} as a formal variable defined for each $(i,j) \in E$. Define $A[y]$ as

$$A[y](i,j) = \begin{cases} x_{ij} y^{w(i,j)} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We will consider the polynomial that is the determinant of the sum of $A[y]$ and the identity matrix. First, we introduce some notation concerning polynomials P.

1. Let $d^*(P)$ be the minimum degree d^* such that y^d has a nonzero coefficient in P. For example, if $P(y,x) = y^{10} + 7x^3y^2 + x$. Then $d^*(P) = 0$ since the y^0 term has a nonzero coefficient.
2. Let $f^*(P)$ be the coefficient in front of y^{d^*} in P.

Claim. *The minimum weight of a cycle in G is $d^*(\det(A + I) - 1)$.*

Later, we will use this claim together with the Schwartz-Zippel lemma to find d^*.

Proof. Recall that

$$\det(A + I) = \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}(\sigma)} \prod_{k=1}^{n} (A + I)[k, \sigma(k)].$$

Let $g(\sigma)$ denote the term of the summation corresponding to σ.
Note that when $\sigma = \text{id}$ (the identity permutation), $g(\sigma) = 1$.
Thus,

$$\det(A + I) - 1 = \sum_{\sigma \in S_n, \sigma \neq \text{id}} g(\sigma).$$

If $\sigma \neq id$, note that its cycle decomposition has at least one cycle $C = (i_1, i_2, \dots, i_k)$ that is not a singleton. C contributes $x_{i_1 i_2} \cdot x_{i_2 i_3} \cdot \dots \cdot x_{i_k i_1} \cdot y^{w(i_1,i_2) + \dots + w(i_k,i_1)}$ to the product of $(A + I)[k, \sigma(k)]$. More generally, every non-singleton cycle contributes an analogous term.

A *cycle packing* of a graph G is a set of vertex-disjoint (possibly singleton) cycles that cover all of the vertices of G.

Thus, we have

$$g(\sigma) = \pm \prod_{\text{cycle packing } \{C_1, \dots, C_\ell\}} \prod_{i=1}^{\ell} \left(\prod_{(a,b) \in C_i} x_{ab} \right) y^{\sum_{i=1}^{\ell} w(C_i)}.$$

Since the degree of each term in the polynomial is a sum of cycle weights, the term with the minimum nonzero degree must correspond to the cycle packing consisting the union of the minimum weight cycle with singleton cycles to cover the rest of the graph. That is, $(d^*(\det(A + I) - 1)$ is exactly the weight of the minimum weight cycle in the graph.

Also, $f^*(\det(A + I) - 1) = \sum_{\text{cycles } C \text{ of min weight}} (\pm 1) \cdot \prod_{(i,j) \in C} x_{ij}$ □

Now, we will plug into $f^*(\det(A+I) - 1)$ random values for x_{ij} in the range $\{1, \dots, n^2\}$ and use the field $\mathbb{F}_p$ for some prime $p > n^2$. Using the Schwartz-Zippel lemma, we have that with high probability $(\geq 1 - 1/n)$, $f^*(\det(A + I) - 1)$ does not evaluate to 0. Furthermore, d^* of the evaluation of $f^*(\det(A + I) - 1)$ is the weight of the shortest cycle with high probability.

Now, we need to argue that we can compute $\det(A + I)$ after having only substituted values in for the x_{ij}s (not y). To do this, we will apply Storjohann's theorem:

Theorem 2.2 (Storjohann's theorem)**.** *Let $A[y]$ be an $n \times n$ matrix where $\forall i, j$, $A[y](i,j)$ is a polynomial over y with coefficients in a field K and has degree at most M. Then we can compute* $\det(A[y])$ *using an SLP in $\tilde{O}(Mn^\omega)$ operations.*

Let $A[y] = A + I$. Then, Storjohann's theorem allows us to compute $\det(A+I)$ in $\tilde{O}(Mn^\omega)$ time. This completes the proof. $\square$

2.2 Finding the shortest cycle

To find the shortest cycle, we will find an edge (u,v) on a shortest cycle, and then run Dijkstra's algorithm from v to find $d(v,u)$.

To find such an edge, we will use Storjohann's theorem and the Baur-Strassen theorem.

Claim. *(u,v) is on the shortest cycle if and only if $\frac{\partial f^*(\det(A+I)-1)}{\partial x_{uv}} \neq 0$.*

Proof. Recall $f^*(\det(A+I)-1)$ is a sum of terms corresponding to cycles of minimum weight d^*. Consider a cycle $C = \{u, v, i_1, \ldots, i_k\}$ of weight d^*. Then in $f^*(\det(A+I)-1)$ we have the term $x_{uv} \cdot x_{vi_1} \cdot x_{i_1 i_2} \cdot \cdots \cdot x_{i_k u}$ with nonzero coefficient. This means that $\frac{\partial f^*(\det(A+I)-1)}{\partial x_{uv}} \neq 0$.

This works in the opposite direction as well. That is, if $\frac{\partial f^*(\det(A+I)-1)}{\partial x_{uv}} \neq 0$ then there is some term in $f^*(\det(A+I)-1)$ containing x_{uv} with a nonzero coefficient, which implies that (u,v) is in a shortest cycle. $\square$

Storjohann's theorem gives us an SLP of length $\tilde{O}(Mn^\omega)$ which can evaluate $\det(A+I)-1$ for a random choice of each x_{ij}. In particular, this SLP computes $f^*(\det(A+I)-1)$ for our choice of x_{ij}'s.

The input to this SLP is for every $i, j \in [n]$ and every $\ell \leq M$, the coefficient in front of y^ℓ in $(A+I)(i,j)$.

We will prepend to the SLP instructions which sets the coefficient in front of y^ℓ in $(A+I)(i,j)$ to be x_{ij} if (i,j) is an edge of weight ℓ, 1 if $i = j$, and 0 otherwise. The number of instructions here is at most Mn^2, so the size of this new SLP is still $\tilde{O}(Mn^\omega)$.

Now we apply the Baur-Strassen theorem on this new SLP to compute each $\frac{\partial f^*(\det(A+I)-1)}{\partial x_{uv}}$. In particular, we can pick random values for the x_{ij}'s and evaluate each $\frac{\partial f^*(\det(A+I)-1)}{\partial x_{uv}}$ efficiently, and therefore check which terms are non-zero. This allows us to obtain an edge (u,v) such that such that $\frac{\partial f^*(\det(A+I)-1)}{\partial x_{uv}} \neq 0$, and thus (u,v) is on a shortest cycle.

References

[1] Cygan, M., Gabow, H., Sankowski, P. *Algorithmic Applications of Baur-Strassen's Theorem: Shortest Cycles, Diameter, and Matchings* (2015) J. ACM, 62(4), 28:1-28:30.