6.S078: Fine-Grained Algorithms and Complexity **MIT**
Problem Set 1 **Due: October 5, 11:59pm (Monday)**
Ryan and Virginia **Fall 2020**

Rules: You may collaborate with other students, but please do not consult research papers, textbooks, or the internet in general. All submitted solutions must be your own, written in your own words. Write down the set of students you collaborated with, at the top of your homework submission. Email your pset to your TA, nwein at mit.edu

1 SAT With Few Clauses (10 points)

Recall CNF-SAT-CLAUSES $= \{(F, k) \mid F$ is a satisfiable CNF with k clauses$\}$.

Prove that CNF-SAT-CLAUSES is solvable in $O^{\star}(2^k)$ time (recall the $O^{\star}$ notation omits polynomial factors).
In fact, the number of SAT assignments to F can be computed in such time.

Hint: You could try the following strategy. Pick an arbitrary clause C. Either (i) remove C from the formula, and recursively count the number of SAT assignments, or (ii) set all literals in C to false, remove C, and recursively count the number of SAT assignments...

2 Some Colorful Problems (15 points, 5 points each)

The following "colorful" variants of canonical fine-grained problems are often very useful to work with in reductions.

(a) Let $OV_{n,d}$ be the problem: *Given n vectors in $\{0,1\}^d$, is there an orthogonal pair?*
Let Colorful-$OV_{n,d}$ be the problem: *Given n red vectors and n blue vectors in $\{0,1\}^d$, is there a red-blue pair which is orthogonal?*

Let $t(n,d) \leq O(n^2 \cdot d^c)$ for some constant $c \geq 1$. Prove that $OV_{n,d}$ is in $O(t(n,d))$ time if and only if Colorful-$OV_{n,d}$ is in $O(t(n,d))$ time.

(b) Let 3SUM be the problem: *Given n numbers, are there three which sum to zero?*
Let Colorful-3SUM be the problem: *Given n numbers, where each number is colored either red, green, or blue, is there a red number r, a blue number b, and a green number g which all sum to zero?*

Let $t(n) \leq O(n^2)$. Prove that if 3SUM is solvable in $O(t(n))$ time, then Colorful-3SUM is solvable in $O(t(n))$ time. (You may assume that addition and multiplication operations on the n numbers take only constant-time.)

Extra credit: Give a reduction from 3SUM to Colorful-3SUM. Be careful: the "natural" ideas don't quite work.

(c) Let Zero-Triangle be the problem: *Given an n-node graph with weights on the edges, are there three edges of the form $(a,b),(b,c),(c,a)$ such that the weights of the three sum to zero?*
Let Colorful-Zero-Triangle be the problem: *Given an n-node edge-weighted graph where each node is colored either red, green, or blue, is there a zero-triangle with a red, green, and a blue node?*

Let $t(n) \leq O(n^3)$. Prove that Zero-Triangle is solvable in $O(t(n))$ time if and only if Colorful-Zero-Triangle is solvable in $O(t(n))$ time.

3 Equivalences With OV (15 points)

In this problem, we say two problems P_1 and P_2 (parameterized by n and d) are *equivalent* if, for any function $T(n, d)$ that is non-decreasing in n and d,

- If P_1 is solvable in time $T(n, d)$ then there are constants $c_1 > 0$ and $c_2 > 0$ such that P_2 is solvable in time $O(T(c_1 n, c_2 d))$.
- If P_2 is solvable in time $T(n, d)$ then there are constants $c_1 > 0$ and $c_2 > 0$ such that P_1 is solvable in time $O(T(c_1 n, c_2 d))$.

Prove that the following three problems are all *equivalent*, in the above sense. (Note you only need to give three reductions.)

- Orthogonal Vectors (OV): Given a size-n set of vectors $V \subseteq \{0, 1\}^d$, are there $a, b \in V$ with $a \cdot b = 0$?
- The Exists-Subset Query problem: Given two sets $S = \{S_1, \ldots, S_n\}$ and $T = \{T_1, \ldots, T_n\}$, where all S_i and T_i are subsets of $[d] := \{1, \ldots, d\}$, is there a pair $S_i \in S$, $T_j \in T$ such that $S_i \subseteq T_j$?
- The Exists-Partial Match problem: We are given a size-n "database" set $D \subseteq \{0, 1\}^d$, and a size-n "query" set $Q \subseteq \{0, 1, \star\}^d$. (Here, "$\star$" represents a wildcard.) We say that a query $q = (q_1, \ldots, q_d)$ matches a string $x = (x_1, \ldots, x_d)$ if for all $i = 1, \ldots, d$, either $q_i = \star$, or (if $q_i \in \{0, 1\}$) $q_i = x_i$. Determine whether there exists a query $q \in Q$ and a string $x \in D$ such that q_i matches x.

4 Algorithm for OV and Subset Query (10 points)

Recall the Exists-Subset Query problem (from the previous problem): Given two sets $S = \{S_1, \ldots, S_n\}$ and $T = \{T_1, \ldots, T_n\}$, where all S_i and T_i are subsets of $[d] := \{1, \ldots, d\}$, is there a pair $S_i \in S$, $T_j \in T$ such that $S_i \subseteq T_j$?

In this problem, we give an algorithm which runs in subquadratic time when $d \ll 2\log(n)$. (Therefore, the "hard case" for these problems are when $d > 2\log(n)$.)

(a) (9 points) Show that Exists-Subset Query can be solved in $(n + 2^d) \cdot \text{poly}(d)$ time.
You may assume "random access" to tables, i.e., you may assume that in poly$(\log Q)$ *time, you can look-up an entry in a table of size Q, and write an entry in a table of size Q.*

(b) (1 point) Conclude that OV can be solved in the same time as well.