# Improving Viterbi is Hard: Better Runtimes Imply Faster Clique Algorithms 

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#### Abstract

The classic algorithm of Viterbi computes the most likely path in a Hidden Markov Model (HMM) that results in a given sequence of observations. It runs in time $O\left(T n^{2}\right)$ given a sequence of $T$ observations from a HMM with $n$ states. Despite significant interest in the problem and prolonged effort by different communities, no known algorithm achieves more than a polylogarithmic speedup.

In this paper, we explain this difficulty by providing matching conditional lower bounds. We show that the Viterbi algorithm runtime is optimal up to subpolynomial factors even when the number of distinct observations is small. Our lower bounds are based on assumptions that the best known algorithms for the All-Pairs Shortest Paths problem (APSP) and for the Max-Weight $k$-Clique problem in edge-weighted graphs are essentially tight.

Finally, using a recent algorithm by Green Larsen and Williams for online Boolean matrixvector multiplication, we get a $2^{\Omega(\sqrt{\log n})}$ speedup for the Viterbi algorithm when there are few distinct transition probabilities in the HMM.


## 1 Introduction

A Hidden Markov Model (HMM) is a simple model that describes a random process for generating a sequence of observations. A random walk is performed on an underlying graph (Markov Chain) and, at each step, an observation is drawn from a probability distribution that depends only on the current state (the node in the graph). HMMs are a fundamental statistical tool and have found wide applicability in a number of fields such as Computational Biology [HE96, KLVHS01, PBvHN11], Signal Processing [Gal98, HAJ90, Kup92], Machine Learning and Computer Vision [SWP98, AK93, ZBS01].

One of the most important questions in these applications is computing the most likely sequence of states visited by the random walk in the HMM given the sequence of observations. Andrew Viterbi proposed an algorithm [Vit67] for this problem that computes the solution in $O\left(T n^{2}\right)$ time for any HMM with $n$ states and an observation sequence of length $T$. This algorithm is known as the Viterbi algorithm and the problem of computing the most likely sequence of states is also known as the Viterbi Path problem.

The quadratic dependence of the algorithm's runtime on the number of states is a long-standing bottleneck that limits its applicability to problems with large state spaces, particularly when the number of observations is large. A lot of effort has been put into improving the Viterbi algorithm to lower either the time or space complexity. Many works achieve speedups by requiring structure in the input, either explicitly by considering restricted classes of HMMs [FHK04] or implicitly by using heuristics that improve runtime in certain cases [ER09, KFYK10]. For the general case, in [LMWZU09, MS11] it is shown how to speed up the Viterbi algorithm by $O(\log n)$ when the number of distinct observations is constant using the Four Russians method or similar ideas. More recently, in [CFR16], the same logarithmic speed-up was shown to be possible for the general case. Despite significant effort, only logarithmic improvements are known other than in very special cases. In contrast, the memory complexity can be reduced to almost linear in the number of states without significant overhead in the runtime [GHS97, TH98, CWH08].

In this work, we attempt to explain this apparent barrier for faster runtimes by giving evidence of the inherent hardness of the Viterbi Path problem. In particular, we show that getting a polynomial speedup ${ }^{1}$ would imply a breakthrough for fundamental graph problems. Our lower bounds are based on standard hardness assumptions for the All-Pairs Shortest Paths and the Min-Weight $k$-Clique problems and apply even in cases where the number of distinct observations is small.

We complement our lower bounds with an algorithm for Viterbi Path that achieves speedup $2^{\Omega(\sqrt{\log n})}$ when there are few distinct transition probabilities in the underlying HMM.

Our results and techniques Our first lower bound shows that the Viterbi Path problem cannot be computed in time $O\left(T n^{2}\right)^{1-\varepsilon}$ for a constant $\varepsilon>0$ unless the APSP conjecture is false. The APSP conjecture states that there is no algorithm for the All-Pairs Shortest Paths problem that runs in truly subcubic ${ }^{2}$ time in the number of vertices of the graph. We obtain the following theorem:

Theorem 1. The Viterbi Path problem requires $\Omega\left(T n^{2}\right)^{1-o(1)}$ time assuming the APSP Conjecture.
The proof of the theorem gives a reduction from All-Pairs Shortest Paths to the Viterbi Path problem. This is done by encoding the weights of the graph of the APSP instance as transition probabilities of the HMM or as probabilities of seeing observations from different states. The proof requires a large alphabet size, i.e. a large number of distinct observations, which can be as large as the number of total steps $T$.

[^0]A natural question question to ask is whether there is a faster algorithm that solves the Viterbi Path problem when the alphabet size is much smaller than $T$, say when $T=n^{2}$ and the alphabet size is $n$. We observe that in such a case, the input size to the Viterbi Path problem is only $O\left(n^{2}\right)$ : we only need to specify the transition probabilities of the HMM, the probabilities of each observation in each state and the sequence of observations. The Viterbi algorithm in this setting runs in $\Theta\left(T n^{2}\right)=\Theta\left(n^{4}\right)$ time. Showing a matching APSP based lower bound seems difficult because the runtime in this setting is quadratic in the input size while the APSP conjecture gives only $N^{1.5}$ hardness for input size $N$. To our best knowledge, all existing reduction techniques based on the APSP conjecture do not achieve such an amplification of hardness. In order to get a lower bound for smaller alphabet sizes, we need to use a different hardness assumption.

For this purpose, we consider the $k$-Clique conjecture. It is a popular hardness assumption which states that it is not possible to compute a minimum weight $k$-clique on an edge-weighted graph with $n$ vertices in time $O\left(n^{k-\varepsilon}\right)$ for constant $k$ and $\varepsilon>0$. With this assumption, we are able to extend Theorem 1 and get the following lower bound for the Viterbi Path problem on very small alphabets:

Theorem 2. For any $C, \varepsilon>0$, the Viterbi Path problem on $T=\Theta\left(n^{C}\right)$ observations from an alphabet of size $\Theta\left(n^{\varepsilon}\right)$ requires $\Omega\left(T n^{2}\right)^{1-o(1)}$ time assuming the $k$-Clique Conjecture for $k=\left\lceil\frac{C}{\varepsilon}\right\rceil+2$.

To show the theorem, we perform a reduction from the Min-Weight $k$-Clique problem. Given a Min-Weight $k$-Clique instance, we create an HMM with two special nodes, a start node and an end node, and enforce the following behavior of the optimal Viterbi path: Most of the time it stays in the start or end node, except for a small number of steps, during which it traverses the rest of the graph to move from the start to the end node. The time at which the traversal happens corresponds to a clique in the original graph of the Min-Weight $k$-Clique instance. We penalize the traversal according to the weight of the corresponding $k$-clique and thus the optimal path will find the minimum weight $k$-clique. Transition probabilities of the HMM and probabilities of seeing observations from different states encode edge-weights of the Min-Weight $k$-Clique instance. Further, we encode the weights of smaller cliques into the sequence of observations according to the binary expansion of the weights.

Our results of Theorems 1 and 2 imply that the Viterbi algorithm is essentially optimal even for small alphabets. We also study the extreme case of the Viterbi Path problem with unary alphabet where the only information available is the total number of steps $T$. We show a surprising behavior: when $T \leq n$ the Viterbi algorithm is essentially optimal, while there is a simple much faster algorithm when $T>n$. See Section 7 for more details.

We complement our lower bounds with an algorithm for Viterbi Path that achieves speedup $2^{\Omega(\sqrt{\log n})}$ when there are few distinct transition probabilities in the underlying HMM. Such a restriction is mild in applications where one can round the transition probabilities to a small number of distinct values.

Theorem 3. When there are fewer than $2^{\varepsilon \sqrt{\log n}}$ distinct transition probabilities for a constant $\varepsilon>0$, there is a $T n^{2} / 2^{\Omega(\sqrt{\log n})}$ randomized algorithm for the Viterbi Path problem that succeeds whp.

We achieve this result by developing an algorithm for online ( $\mathrm{min},+$ ) matrix-vector multiplication for the case when the matrix has few distinct values. Our algorithm is based on a recent result for online Boolean matrix-vector multiplication by Green Larsen and Williams [LW16].

Finally, we provide an algorithm that runs in $O\left(T n^{2}\right)^{1-\alpha}$ non-deterministic time for a constant $\alpha>0$. This provides an evidence that lower bounds for Viterbi Path based on Strong Exponential Time Hypothesis (SETH) under deterministic reductions are not possible. See Section 6 for more details.

The results we presented above hold for dense HMMs. For sparse HMMs that have at most $m$ edges out of the $n^{2}$ possible ones, i.e. the transition matrix has at most $m$ non-zero probabilities, the

Viterbi Path problem can be easily solved in $O(T m)$ time. The lower bounds that we presented above can be adapted directly for this case to show that no faster algorithm exists that runs in time $O(T m)^{1-\varepsilon}$. See the corresponding discussion in the appendix.

Hardness assumptions There is a long list of works showing conditional hardness for various problems based on the All-Pairs Shortest Paths problem hardness assumption [RZ04, WW10, AW14, AGW15, AWY15]. Among other results, [WW10] showed that finding a triangle of minimum weight in a weighted graph is equivalent to the All-Pairs Shortest Paths problem meaning that a strongly subcubic algorithm for the Minimum Triangle problem implies a strongly subcubic algorithm for the All-Pairs Shortest Paths problem and the other way around. Computing a min-weight triangle is a special case of the problem of computing a min-weight $k$-clique in a graph for a fixed integer $k$. This is a very well studied computational problem and despite serious efforts, the best known algorithm for this problem still runs in time $O\left(n^{k-o(1)}\right)$, which matches the runtime of the trivial algorithm up to subpolynomial factors. The assumption that there is no $O\left(n^{k-\varepsilon}\right)$ time algorithm for this problem, has served as a basis for showing conditional hardness results for several problems on sequences [ABW15, AWW14] and computational geometry [BDT16].

## 2 Preliminaries

Notation For an integer $m$, we denote the set $\{1,2, \ldots, m\}$ by $[m]$.
Definition 1 (Hidden Markov Model). A Hidden Markov Model (HMM) consists of a directed graph with $n$ distinct hidden states $[n]$ with transition probabilities $\tilde{A}(u, v)$ of going from state $u$ to state $v$. In any given state, there is a probability distribution of symbols that can be observed and $\tilde{B}(u, s)$ gives the probability of seeing symbol s on state $u$. The symbols come from an alphabet $[\sigma]$ of size $\sigma$. An HMM can thus be represented by a tuple $(\tilde{A}, \tilde{B})$.

### 2.1 The Viterbi Path Problem

Given an HMM and a sequence of $T$ observations, the Viterbi algorithm [Vit67] outputs a sequence of $T$ states that is most likely given the $T$ observations. More precisely, let $S=\left(s_{1}, \ldots, s_{T}\right)$ be the given sequence of $T$ observations where symbol $s_{t} \in[\sigma]$ is observed at time $t=1, \ldots, T$. Let $u_{t} \in[n]$ be the state of the HMM at time $t=1, \ldots, T$. The Viterbi algorithm finds a state sequence $U=$ $\left(u_{0}, u_{1}, \ldots, u_{T}\right)$ starting at $u_{0}=1$ that maximizes $\operatorname{Pr}[U \mid S]$. The problem of finding the sequence $U$ is known as the Viterbi Path problem. In particular, the Viterbi Path problem solves the optimization problem

$$
\underset{u_{0}=1, u_{1}, \ldots, u_{T}}{\arg \max } \prod_{t=1}^{T}\left[\tilde{A}\left(u_{t-1}, u_{t}\right) \cdot \tilde{B}\left(u_{t}, s_{t}\right)\right] .
$$

The Viterbi algorithm solves this problem in $O\left(T n^{2}\right)$ by computing for $t=1 \ldots T$ the best sequence of length $t$ that ends in a given state in a dynamic programming fashion. When run in a word RAM model with $O(\log n)$ bit words, this algorithm is numerically unstable because even representing the probability of reaching a state requires linear number of bits. Therefore, $\log$ probabilities are used for numerical stability since that allows to avoid underflows [YEG ${ }^{+} 97$, AV98, LT09, LDLL07, HAHFBR01]. To maintain numerical stability and understand the underlying combinatorial structure of the problem, we assume that the input is given in the form of $\log$-probabilities, i.e. the input to the problem is $A(u, v)=-\log \tilde{A}(u, v)$ and $B(u, s)=-\log \tilde{B}(u, s)$ and focus our attention on the Viterbi Path problem defined by matrices $A$ and $B$.

Definition 2 (Viterbi Path Problem). The Viterbi Path problem is specified by a tuple ( $A, B, S$ ) where $A$ and $B$ are $n \times n$ and $n \times \sigma$ matrices, respectively, and $S=\left(s_{1}, \ldots, s_{T}\right)$ is a sequence of $T=n^{\Theta(1)}$ observations $s_{1}, \ldots, s_{T} \in[\sigma]$ over an alphabet of size $\sigma$. Given an instance $(A, B, S)$ of the Viterbi Path problem, our goal is to output a sequence of vertices $u_{0}, u_{1}, \ldots, u_{T} \in[n]$ with $u_{0}=1$ that solves

$$
\underset{u_{0}=1, u_{1}, \ldots, u_{T}}{\arg \min } \sum_{t=1}^{T}\left[A\left(u_{t-1}, u_{t}\right)+B\left(u_{t}, s_{t}\right)\right] .
$$

We can assume that $\log$ probabilities in matrices $A$ and $B$ are arbitrary positive numbers without the restriction that the corresponding probabilities must sum to 1. See Appendix A for a discussion.

A simpler special case of the Viterbi Path problem asks to compute the most likely path of length $T$ without any observations.

Definition 3 (Shortest Walk Problem). Given an integer $T$ and a weighted directed graph (with possible self-loops) on $n$ vertices with edge weights specified by a matrix A, the Shortest Walk problem asks to compute a sequence of vertices $u_{0}=1, u_{1}, \ldots, u_{T} \in[n]$ that solves

$$
\underset{u_{0}=1, u_{1}, \ldots, u_{T}}{\arg \min } \sum_{t=1}^{T} A\left(u_{t-1}, u_{t}\right) .
$$

It is easy to see that the Shortest Walk problem corresponds to the Viterbi Path problem when $\sigma=1$ and $B(u, 1)=0$ for all $u \in[n]$.

### 2.2 Hardness assumptions

We use the hardness assumptions of the following problems.
Definition 4 (All-Pairs Shortest Paths (APSP) problem). Given an undirected graph $G=$ $(V, E)$ with $n$ vertices and positive integer weights on the edges, find the shortest path between $u$ and $v$ for every $u, v \in V$.

The APSP conjecture states that the All-Pairs Shortest Paths problem requires $\Omega\left(n^{3}\right)^{1-o(1)}$ time in expectation.

Conjecture 1 (APSP conjecture). The All-Pairs Shortest Paths problem on a graph with $n$ vertices and positive integer edge-weights bounded by $n^{O(1)}$ requires $\Omega\left(n^{3}\right)^{1-o(1)}$ time in expectation.

Definition 5 (Min-Weight $k$-Clique problem). Given a complete graph $G=(V, E)$ with $n$ vertices and positive integer edge-weights, output the minimum total edge-weight of a $k$-clique in the graph.

For any fixed constant $k$, the best known algorithm for the Min-Weight $k$-Clique problem runs in time $O\left(n^{k-o(1)}\right)$ and the $k$-Clique conjecture states that it requires $\Omega\left(n^{k}\right)^{1-o(1)}$ time.

Conjecture 2 ( $k$-Clique conjecture). The Min-Weight $k$-Clique problem on a graph with $n$ vertices and positive integer edge-weights bounded by $n^{O(k)}$ requires $\Omega\left(n^{k}\right)^{1-o(1)}$ time in expectation.

For $k=3$, the Min-Weight 3-Clique problem asks to find the minimum weight triangle in a graph. This problem is also known as the Minimum Triangle problem and under the 3-Clique conjecture it requires $\Omega\left(n^{3}\right)^{1-o(1)}$ time. The latter conjecture is equivalent to the APSP conjecture [WW10].

We often use the following variant of the Min-Weight $k$-Clique problem:

Definition 6 (Min-Weight $k$-Clique problem for $k$-partite graphs). Given a complete $k$-partite graph $G=\left(V_{1} \cup \ldots \cup V_{k}, E\right)$ with $\left|V_{i}\right|=n_{i}$ and positive integer weights on the edges, output the minimum total edge-weight of a $k$-clique in the graph.

If for all $i, j$ we have that $n_{i}=n_{j}^{\Theta(1)}$, it can be shown that the Min-Weight $k$-Clique problem for $k$-partite graphs requires $\Omega\left(\prod_{i=1}^{k} n_{i}\right)^{1-o(1)}$ time assuming the $k$-Clique conjecture. We provide a simple proof of this statement in the appendix.

## 3 Hardness of Viterbi Path

We begin by presenting our main hardness result for the Viterbi Path problem.
Theorem 1. The Viterbi Path problem requires $\Omega\left(T n^{2}\right)^{1-o(1)}$ time assuming the APSP Conjecture.
To show APSP hardness, we will perform a reduction from the Minimum Triangle problem (described in Section 2.2) to the Viterbi Path problem. In the instance of the Minimum Triangle problem, we are given a 3-partite graph $G=\left(V_{1} \cup V_{2} \cup U, E\right)$ such that $\left|V_{1}\right|=\left|V_{2}\right|=n,|U|=m$. We want to find a triangle of minimum weight in the graph $G$. To perform the reduction, we define a weighted directed graph $G^{\prime}=\left(\{1,2\} \cup V_{1} \cup V_{2}, E^{\prime}\right) . E^{\prime}$ contains all the edges of $G$ between $V_{1}$ and $V_{2}$, directed from $V_{1}$ towards $V_{2}$, edges from 1 towards all nodes of $V_{1}$ of weight 0 and edges from all nodes of $V_{2}$ towards 2 of weight 0 . We also add a self-loops at nodes 1 and 2 of weight 0 .

We create an instance of the Viterbi Path problem $(A, B, S)$ as described below. Figure 1 illustrates the construction of the instance.

- Matrix $A$ is the weighted adjacency matrix of $G^{\prime}$ that takes value $+\infty$ (or a sufficiently large integer) for non-existent edges and non-existent self-loops.
- The alphabet of the HMM is $U \cup\left\{\perp, \perp_{F}\right\}$ and thus matrix $B$ has $2 n+2$ rows and $\sigma=m+2$ columns. For all $v \in V_{1} \cup V_{2}$ and $u \in U, B(v, u)$ is equal to the weight of the edge $(v, u)$ in graph $G$. Moreover, for all $v \in V_{1} \cup V_{2}, B(v, \perp)=+\infty$ (or a sufficiently large number) and for all $v \in V_{1} \cup V_{2} \cup\{1\}, B\left(v, \perp_{F}\right)=+\infty$. Finally, all remaining entries corresponding to nodes 1 and 2 are 0 .
- Sequence $S$ of length $T=3 m+1$ is generated by appending the observations $u, u$ and $\perp$ for all $u \in U$ and adding a $\perp_{F}$ observation at the end.

Given the above construction, the theorem statement follows directly from the following claim.
Claim 1. The weight of the solution to the Viterbi Path instance is equal to the weight of the minimum triangle in the graph $G$.

Proof. The optimal path for the Viterbi Path instance begins at node 1. It must end in node 2 since otherwise when observation $\perp_{F}$ arrives we collect cost $+\infty$. Similarly, whenever an observation $\perp$ arrives the path must be either on node 1 or 2 . Thus, the path first loops in node 1 and then goes from node 1 to node 2 during three consecutive observations $u, u$ and $\perp$ for some $u \in U$ and stays in node 2 until the end. Let $v_{1} \in V_{1}$ and $v_{2} \in V_{2}$ be the two nodes visited when moving from node 1 to node 2 . The only two steps of non-zero cost are:

1. Moving from node 1 to node $v_{1}$ at the first observation $u$. This $\operatorname{costs} A\left(1, v_{1}\right)+B\left(v_{1}, u\right)=$ $B\left(v_{1}, u\right)$.


| Node | $u \in U$ | $\perp$ | $\perp_{F}$ |
| :---: | :---: | :---: | :---: |
| 1 | 0 | 0 | $\infty$ |
| $v \in V_{1} \cup V_{2}$ | $w_{v, u}$ | $\infty$ | $\infty$ |
| 2 | 0 | 0 | 0 |

(a) The graph specified by transition matrix $A$. Every edge $\left(v_{1}, v_{2}\right)$ in $V_{1} \times V_{2}$ has the original edge-weight as in graph $G$.
(b) The cost of seeing a symbol at every node given by matrix $B$.

Figure 1: The construction of matrices $A$ and $B$ for the reduction in the proof of Theorem 1. The notation $w_{v, u}$ denotes the weight of the edge $(v, u)$ in the original graph $G$.
2. Moving from node $v_{1}$ to node $v_{2}$ at the second observation $u$. This costs $A\left(v_{1}, v_{2}\right)+B\left(v_{2}, u\right)$.

Thus, the overall cost of the path is equal to $B\left(v_{1}, u\right)+A\left(v_{1}, v_{2}\right)+B\left(v_{2}, u\right)$, which is equal to the weight of the triangle $\left(v_{1}, v_{2}, u\right)$ in $G$. Minimizing the cost of the path in this instance is therefore the same as finding the minimum weight triangle in $G$.

## 4 Hardness of Viterbi Path with small alphabet

The proof of Theorem 1 requires a large alphabet size, which can be as large as the number of total steps $T$. In this section, we show how to get a lower bound for the Viterbi Path problem on alphabets of small size by using a different hardness assumption.
Theorem 2. For any $C, \varepsilon>0$, the Viterbi Path problem on $T=\Theta\left(n^{C}\right)$ observations from an alphabet of size $\Theta\left(n^{\varepsilon}\right)$ requires $\Omega\left(T n^{2}\right)^{1-o(1)}$ time assuming the $k$-Clique Conjecture for $k=\left\lceil\frac{C}{\varepsilon}\right\rceil+2$.

Reduction Throughout the proof, we set $p=\left\lceil\frac{C}{\varepsilon}\right\rceil$ and $\alpha=\frac{C}{p} \leq \varepsilon$.
We will perform a reduction from the Min-Weight $k$-Clique problem for $k=p+2$ to the Viterbi Path problem. In the instance of the Min-Weight $k$-Clique problem, we are given a $k$-partite graph $G=\left(V_{1} \cup V_{2} \cup U_{1} \ldots \cup U_{p}, E\right)$ such that $\left|V_{1}\right|=\left|V_{2}\right|=n$ and $\left|U_{1}\right|=\ldots=\left|U_{p}\right|=$ $m=\Theta\left(n^{\alpha}\right)$. We want to find a clique of minimum weight in the graph $G$. Before describing our final Viterbi Path instance, we first define a weighted directed graph $G^{\prime}=\left(\{1,2,3\} \cup V_{1} \cup V_{2}, E^{\prime}\right)$ similar to the graph in the proof of Theorem 1. $E^{\prime}$ contains all the edges of $G$ between $V_{1}$ and $V_{2}$, directed from $V_{1}$ towards $V_{2}$, edges from node 1 towards all nodes in $V_{1}$ of weight 0 and edges from all nodes in $V_{2}$ towards node 2 of weight 0 . We also add a self-loop at nodes 1 and 3 of weight 0 as well as an edge of weight 0 from node 2 towards node 3 . We obtain the final graph $G^{\prime \prime}$ as follows:

- For every node $v \in V_{1}$, we replace the directed edge $(1, v)$ with a path $1 \rightarrow a_{v, 1} \rightarrow \ldots \rightarrow a_{v, p} \rightarrow v$ by adding $p$ intermediate nodes. All edges on the path have weight 0 .
- For every node $v \in V_{2}$, we replace the directed edge $(v, 2)$ with a path $v \rightarrow b_{v, 1} \rightarrow \ldots \rightarrow b_{v, p} \rightarrow 2$ by adding $p$ intermediate nodes. All edges on the path have weight 0 .
- Finally, we replace the directed edge $(2,3)$ with a path $2 \rightarrow c_{1} \rightarrow \ldots \rightarrow c_{Z} \rightarrow 3$ by adding $Z$ intermediate nodes, where $2^{Z}$ is a strict upper bound on the weight of any $k$-clique ${ }^{3}$. All edges on the path have weight 0 .

We create an instance of the Viterbi Path problem $(A, B, S)$ as described below. Figure 2 illustrates the construction of the instance.

- Matrix $A$ is the weighted adjacency matrix of $G^{\prime \prime}$ that takes value $+\infty$ (or a sufficiently large integer) for non-existent edges and non-existent self-loops.
- The alphabet of the HMM is $U_{1} \cup \ldots \cup U_{p} \cup\left\{\perp, \perp_{0}, \perp_{1}, \perp_{F}\right\}$ and thus matrix $B$ has $O(n)$ rows and $\sigma=p \cdot m+4=O\left(n^{\alpha}\right)$ columns.
For all $v \in V_{1}$, every $i \in[p]$ and every $u \in U_{i}, B\left(a_{v, i}, u\right)$ is equal to the weight of the edge $(v, u)$ in graph $G$. Similarly, for all $v \in V_{2}$, every $j \in[p]$ and every $u \in U_{j}, B\left(b_{v, j}, u\right)$ is equal to the weight of the edge $(v, u)$ in graph $G$.
Moreover, for all $i \in\{1, \ldots, Z\}, B\left(c_{i}, \perp_{1}\right)=2^{i-1}$ and $B\left(c_{i}, \perp_{0}\right)=0$. Finally, $B(v, \perp)=+\infty$ for all nodes $v \notin\{1,3\}$ while $B\left(v, \perp_{F}\right)=+\infty$ for all nodes $v \neq 3$. All remaining entries of matrix $B$ are 0 .
- Sequence $S$ is generated by appending for every tuple $\left(u_{1}, \ldots, u_{p}\right) \in U_{1} \times \ldots \times U_{p}$ the following observations in this order: Initially we add the observations $\left(u_{1}, \ldots, u_{p}, \perp_{0}, \perp_{0}, u_{1}, \ldots, u_{p}\right.$, $\left.\perp_{0}\right)$. Moreover, let $W$ be the total weight of the clique ( $u_{1}, \ldots, u_{p}$ ) in the graph $G$. We add $Z$ observations encoding $W$ in binary ${ }^{4}$ starting with the least significant bit. For example, if $W=11$ and $Z=5$, the binary representation is $01011_{2}$ and the observations we add are $\perp_{1}, \perp_{1}, \perp_{0}, \perp_{1}, \perp_{0}$ in that order. Finally, we append a $\perp$ observation at the end.
Notice, that for each tuple, we append exactly $Z+2 p+4=Z+2 k$ observations. Thus, the total number of observations is $m^{p}(Z+2 k)$. We add a final $\perp_{F}$ observation at the end and set $T=m^{p}(Z+2 k)+1$.

Correctness of the reduction Since the Min-Weight $k$-Clique instance requires $\Omega\left(\left|V_{1}\right| \cdot\left|V_{2}\right| \cdot \prod_{i=1}^{p}\left|U_{i}\right|\right)^{1-o(1)}=\Omega\left(T n^{2}\right)^{1-o(1)}$ time, the following claim implies that the above Viterbi Path instances require $\Omega\left(T n^{2}\right)^{1-o(1)}$ time. The alphabet size used is at most $O\left(n^{\alpha}\right)$ and $\alpha \leq \varepsilon$ and the theorem follows.

Claim 2. The weight of the solution to the Viterbi Path instance is equal to the minimum weight of a $k$-clique in the graph $G$.

Proof. The optimal path for the Viterbi Path instance begins at node 1. It must end in node 3 since otherwise when observation $\perp_{F}$ arrives we collect cost $+\infty$. Similarly, whenever an observation $\perp$ arrives the path must be either on node 1 or 3 . Thus, the path first loops in node 1 and then goes from node 1 to node 3 during the sequence of $Z+2 k$ consecutive observations corresponding to some tuple $\left(u_{1}, \ldots, u_{p}\right) \in U_{1} \times \ldots \times U_{p}$ and stays in node 3 until the end. Let $v_{1}$ and $v_{2}$ be the nodes in $V_{1}$ and $V_{2}$, respectively, that are visited when moving from node 1 to node 3 . The only steps of non-zero cost happen during the subsequence of observations corresponding to the tuple $\left(u_{1}, \ldots, u_{p}\right)$ :

[^1]

Figure 2: The construction of matrices $A$ and $B$ for the reduction in the proof of Theorem 2.

1. When the subsequence begins with $u_{1}$, the path jumps to node $a_{v_{1}, 1}$ which has a cost $B\left(a_{v_{1}, 1}, u_{1}\right)$ equal to the edge-weight $\left(v_{1}, u_{1}\right)$ in graph $G$. It then continues on to nodes $a_{v_{1}, 2}, \ldots, a_{v_{1}, p}$ when seeing observations $u_{2}, \ldots, u_{p}$. The total cost of these steps is $\sum_{i=1}^{p} B\left(a_{v_{1}, i}, u_{i}\right)$ which is the total weight of edges $\left(v_{1}, u_{1}\right), \ldots,\left(v_{1}, u_{p}\right)$ in graph $G$.
2. For the next two observations $\perp_{0}, \perp_{0}$, the path jumps to nodes $v_{1}$ and $v_{2}$. The first jump has no cost while the latter has cost $A\left(v_{1}, v_{2}\right)$ equal to the weight of the edge $\left(v_{1}, v_{2}\right)$ in $G$.
3. The subsequence continues with observations $u_{1}, \ldots, u_{p}$ and the path jumps to nodes $b_{v_{2}, 1}, \ldots, b_{v_{2}, p}$ which has a total cost $\sum_{i=1}^{p} B\left(b_{v_{2}, i}, u_{i}\right)$ which is equal to the total weight of edges $\left(v_{2}, u_{1}\right), \ldots,\left(v_{2}, u_{p}\right)$ in graph $G$.
4. The path then jumps to node 2 at no cost at observation $\perp_{0}$.
5. The path then moves on to the nodes $c_{1}, \ldots, c_{Z}$. The total cost of those moves is equal to the total weight of the clique $\left(u_{1}, \ldots, u_{p}\right)$ since the observations $\perp_{0}$ and $\perp_{1}$ that follow encode that weight in binary.

The overall cost of the path is exactly equal to the weight of the $k$-clique $\left(v_{1}, v_{2}, u_{1}, \ldots, u_{p}\right)$ in $G$. Minimizing the cost of the path in this instance is therefore the same as finding the minimum weight $k$-clique in $G$.

## 5 A faster Viterbi Path algorithm

In this section, we present a faster algorithm for the Viterbi Path problem, when there are only few distinct transition probabilities in the underlying HMM.

Theorem 3. When there are fewer than $2^{\varepsilon \sqrt{\log n}}$ distinct transition probabilities for a constant $\varepsilon>0$, there is a $T n^{2} / 2^{\Omega(\sqrt{\log n})}$ randomized algorithm for the Viterbi Path problem that succeeds whp.

The number of distinct transition probabilities is equal to the number of distinct entries in matrix $\tilde{A}$ in Definition 1. The same is true for matrix $A$ in the additive version of Viterbi Path, in Definition 2. So, from the theorem statement we can assume that matrix $A$ has at most $2^{\varepsilon \sqrt{\log n}}$ different entries for some constant $\varepsilon>0$.

To present our algorithm, we revisit the definition of Viterbi Path. We want to compute a path $u_{0}=1, u_{1}, \ldots, u_{T}$ that minimizes the quantity:

$$
\begin{equation*}
\min _{u_{0}=1, u_{1}, \ldots, u_{T}} \sum_{t=1}^{T}\left[A\left(u_{t-1}, u_{t}\right)+B\left(u_{t}, s_{t}\right)\right] . \tag{1}
\end{equation*}
$$

Defining the vectors $b_{t}=B\left(\cdot, s_{t}\right)$, we note that (1) is equal to the minimum entry in the vector obtained by a sequence of $T(\mathrm{~min},+)$ matrix-vector products ${ }^{5}$ as follows:

$$
\begin{equation*}
A \oplus\left(\ldots\left(A \oplus\left(A \oplus\left(A \oplus z+b_{1}\right)+b_{2}\right)+b_{3}\right) \ldots\right)+b_{T} \tag{2}
\end{equation*}
$$

where $z$ is a vector with entries $z_{1}=0$ and $z_{i}=\infty$ for all $i \neq 1$. Vector $z$ represents the cost of being at node $i$ at time 0 . Vector $\left(A \oplus z+b_{1}\right)$ represents the minimum cost of reaching each node at time 1 after seeing observation $s_{1}$. After $T$ steps, every entry $i$ of vector (2) represents the minimum minimum cost of a path that starts at $u_{0}=1$ and ends at $u_{T}=i$ after $T$ observations. Taking the minimum of all entries gives the cost of the solution to the Viterbi Path instance.

To evaluate (2), we design an online (min,+ ) matrix-vector multiplication algorithm. In the online matrix-vector multiplication problem, we are given a matrix and a sequence of vectors in online fashion. We are required to output the result of every matrix-vector product before receiving the next vector. Our algorithm for online ( $\mathrm{min},+$ ) matrix-vector multiplication is based on a recent algorithm for online Boolean matrix-vector multiplication by Green Larsen and Williams [LW16]:
Theorem 4 (Green Larsen and Williams [LW16]). For any matrix $M \in\{0,1\}^{n \times n}$ and any sequence of $T=2^{\omega(\sqrt{\log n)}}$ vectors $v_{1}, \ldots, v_{T} \in\{0,1\}^{n}$, online Boolean matrix-vector multiplication of $M$ and $v_{i}$ can be performed in $n^{2} / 2^{\Omega(\sqrt{\log n})}$ amortized time whp. No preprocessing is required.

We show the following theorem for online ( $\min ,+$ ) matrix-vector multiplication, which gives the promised runtime for the Viterbi Path problem ${ }^{6}$ since we are interested in the case where $T$ and $n$ are polynomially related, i.e. $T=n^{\Theta(1)}$.
Theorem 5. Let $A \in \mathbb{R}^{n \times n}$ be a matrix with at most $2^{\varepsilon \sqrt{\log n}}$ distinct entries for a constant $\varepsilon>0$. For any sequence of $T=2^{\omega(\sqrt{\log n})}$ vectors $v_{1}, \ldots, v_{T} \in \mathbb{R}^{n}$, online ( $\mathrm{min},+$ ) matrix-vector multiplication of $A$ and $v_{i}$ can be performed in $n^{2} / 2^{\Omega(\sqrt{\log n})}$ amortized time whp. No preprocessing is required.
Proof. We will show the theorem for the case where $A \in\{0,+\infty\}^{n \times n}$. The general case where matrix $A$ has $d \leq 2^{\varepsilon \sqrt{\log n}}$ distinct values $a^{1}, \ldots, a^{d}$ can be handled by creating $d$ matrices $A^{1}, \ldots, A^{d}$, where each matrix $A^{k}$ has entries $A_{i j}^{k}=0$ if $A_{i j}=a^{k}$ and $+\infty$ otherwise. Then, vector $r=A \oplus v$ can be computed by computing $r^{k}=A^{k} \oplus v$ for every $k$ and setting $r_{i}=\min _{k}\left(r_{i}^{k}+a^{k}\right)$. This introduces a factor of $2^{\varepsilon \sqrt{\log n}}$ in amortized runtime but the final amortized runtime remains $n^{2} / 2^{\Omega(\sqrt{\log n})}$ if $\varepsilon>0$ is sufficiently small. From now on we assume that $A \in\{0,+\infty\}^{n \times n}$ and define the matrix $\bar{A} \in\{0,1\}^{n \times n}$ whose every entry is 1 if the corresponding entry at matrix $A$ is 0 and 0 otherwise.

For every query vector $v$, we perform the following:

[^2]- Sort indices $i_{1}, \ldots, i_{n}$ such that $v_{i_{1}} \leq \ldots \leq v_{i_{n}}$ in $O(n \log n)$ time.
- Partition the indices into $p=2^{\alpha \sqrt{\log n}}$ sets, where set $S_{k}$ contains indices $i_{(k-1)\left\lceil\frac{n}{p}\right\rceil+1}, \ldots, i_{k\left\lceil\frac{n}{p}\right\rceil}$.
- Set $r=(\perp, \ldots, \perp)^{T}$, where $\perp$ indicates an undefined value.
- For $k=1 \ldots p$ fill the entries of $r$ as follows:
- Let $\mathbb{I}_{S_{k}}$ be the indicator vector of $S_{k}$ that takes value 1 at index $i$ if $i \in S_{k}$ and 0 otherwise.
- Compute the Boolean matrix-vector product $\pi^{k}=\bar{A} \odot \mathbb{I}_{S_{k}}$ using the algorithm from Theorem 4.
- Set $r_{j}=\min _{i \in S_{k}}\left(A_{j, i}+v_{i}\right)$ for all $j \in[n]$ such that $r_{j}=\perp$ and $\pi_{j}^{k}=1$.
- Return vector $r$.

Runtime of the algorithm per query The algorithm performs $p=2^{\alpha \sqrt{\log n}}$ Boolean matrixvector multiplications, for a total amortized cost of $p \cdot n^{2} / 2^{\Omega(\sqrt{\log n})}=n^{2} / 2^{\Omega(\sqrt{\log n})}$ for a small enough constant $\alpha>0$. Moreover, to fill an entry $r_{j}$ the algorithm requires going through all elements in some set $S_{k}$ for a total runtime of $O\left(\left|S_{k}\right|\right)=n / 2^{\Omega(\sqrt{\log n})}$. Thus, for all entries $p_{j}$ the total time required is $n^{2} / 2^{\Omega(\sqrt{\log n})}$. The runtime of the other steps is dominated by these two operations so the algorithm takes $n^{2} / 2^{\Omega(\sqrt{\log n})}$ amortized time per query.

Correctness of the algorithm To see that the algorithm correctly computes the (min,+ ) product $A \oplus v$, observe that the algorithm fills in the entries of vector $r$ from smallest to largest. Thus, when we set a value to entry $r_{j}$ we never have to change it again. Moreover, if the value $r_{j}$ gets filled at step $k$, it must be the case that $\pi_{j}^{k^{\prime}}=0$ for all $k^{\prime}<k$. This means that for all indices $i \in S_{1} \cup \ldots \cup S_{k-1}$ the corresponding entry $A_{j, i}$ was always $+\infty$.

## 6 Nondeterministic algorithms for the Viterbi Path problem

It is natural question to ask if one could give a conditional lower bound for the Viterbi Path problem by making a different hardness assumption such as the Strong Exponential Time Hypothesis (SETH) ${ }^{7}$ or the 3 -Sum conjecture ${ }^{8}$. In this section we argue that there is no conditional lower bound for the Viterbi Path problem that is based on SETH if the Nondeterministic Strong Exponential Time Hypothesis (NSETH) ${ }^{9}$ is true. NSETH was introduced in [CGI $\left.{ }^{+} 16\right]$ and the authors showed the following statement. If a certain computational problem is in $\operatorname{NTIME}\left(n^{c-\alpha}\right) \cap \operatorname{coNTIME}\left(n^{c-\alpha}\right)$ for a constant $\alpha>0$, then, if NSETH holds, there is no $\Omega\left(n^{c-o(1)}\right)$ conditional lower bound for this problem based on SETH under deterministic reductions. Thus, to rule out $\Omega\left(T n^{2}\right)^{1-o(1)}$ SETH-based lower bound for Viterbi Path under deterministic reductions, it suffices to show that Viterbi Path $\in$ NTIME $\left(\left(T n^{2}\right)^{1-\alpha}\right) \cap \operatorname{coNTIME}\left(\left(T n^{2}\right)^{1-\alpha}\right)$ for a constant $\alpha>0$. This is what we do in the rest of the section.

To show that Viterbi Path $\in \operatorname{NTIME}\left(\left(T n^{2}\right)^{1-\alpha}\right) \cap \operatorname{coNTIME}\left(\left(T n^{2}\right)^{1-\alpha}\right)$, we have to define a decision version of the Viterbi Path problem. A natural candidate is as follows: given the instance

[^3]and a threshold, decide if the cost of the solution is at most the threshold. Clearly, Viterbi Path $\in$ $\operatorname{NTIME}(T) \subseteq \operatorname{NTIME}\left(\left(T n^{2}\right)^{1-\alpha}\right)$ since the nondeterministic algorithm can guess the optimal path and check that the solution is at most the threshold.

We now present an algorithm which shows that Viterbi Path $\in \operatorname{coNTIME}\left(\left(T n^{2}\right)^{1-\alpha}\right)$. Consider the formula (2). We can rewrite it by defining a sequence of vectors $v_{0}, v_{1}, \ldots, v_{T}$ such that:

$$
v_{0}=z \text { and } v_{t}=A \oplus v_{t-1}+b_{t}
$$

Thus, we can solve the Viterbi Path instance by computing $v_{T}$ and calculating its minimum entry. Notice that the above can be rewritten in a matrix form as:

$$
\begin{equation*}
\left[v_{1}, \ldots, v_{T}\right]=A \oplus\left[v_{0}, \ldots, v_{T-1}\right]+\left[b_{1}, \ldots, b_{T}\right] \tag{3}
\end{equation*}
$$

Using this equation, the nondeterministic algorithm can check that the solution to Viterbi Path is greater than the threshold by directly guessing all vectors $v_{1}, \ldots, v_{T}$ and computing the minimum entry of $v_{T}$. The algorithm can verify that these guesses are correct by checking that equation (3) holds. The most computationally demanding step is verifying that the (min,+ )-product of matrices $A$ and $\left[v_{0}, \ldots, v_{T-1}\right]$ is accurate but it is known that MinPlusProduct $\in \operatorname{NTIME}\left(\left(T n^{2}\right)^{1-\alpha}\right) \cap$ coNTIME $\left(\left(T n^{2}\right)^{1-\alpha}\right)$ for some constant $\alpha>0\left[\mathrm{CGI}^{+} 16\right]$. Therefore, it follows that also Viterbi Path $\in$ NTIME $\left(\left(T n^{2}\right)^{1-\alpha}\right) \cap \operatorname{coNTIME}\left(\left(T n^{2}\right)^{1-\alpha}\right)$.

## 7 Complexity of Viterbi Path for unary alphabet

In this section, we focus on the extreme case of Viterbi Path with unary alphabet.
Theorem 6. The Viterbi Path problem requires $\Omega\left(T n^{2}\right)^{1-o(1)}$ time when $T \leq n$ even if the size of the alphabet is $\sigma=1$, assuming the APSP Conjecture.

The above theorem follows from APSP-hardness of the Shortest Walk problem that we present next.

Theorem 7. The ShORTEST WALK problem requires $\Omega\left(T n^{2}\right)^{1-o(1)}$ time when $T \leq n$, assuming the APSP Conjecture.

Proof. We will perform a reduction from the Minimum Triangle problem to the Viterbi Path problem. In the instance of the Minimum Triangle problem, we are given a 3 -partite undirected graph $G=\left(V_{1} \cup V_{2} \cup U, E\right)$ with positive edge weights such that $\left|V_{1}\right|=\left|V_{2}\right|=n,|U|=m$. We want to find a triangle of minimum weight in the graph $G$. To perform the reduction, we define a weighted directed and acyclic graph $G^{\prime}=\left(\{1,2\} \cup V_{1} \cup V_{2} \cup U \cup U^{\prime}, E^{\prime}\right)$. Nodes in $U^{\prime}$ are in one-to-one correspondence with nodes in $U$ and $\left|U^{\prime}\right|=m$. $E^{\prime}$ is defined as follows. We add all edges of $G$ between nodes in $U$ and $V_{1}$ directed from $U$ towards $V_{1}$ and similarly, we add all edges of $G$ between nodes in $V_{1}$ and $V_{2}$ directed from $V_{1}$ towards $V_{2}$. Instead of having edges between nodes in $V_{2}$ and $U$, we add the corresponding edges of $G$ between nodes in $V_{2}$ and $U^{\prime}$ directed from $V_{2}$ towards $U^{\prime}$. Moreover, we add additional edges of weight 0 to create a path $P$ of $m+1$ nodes, starting from node 1 and going through all nodes in $U$ in some order. Finally, we create another path $P^{\prime}$ of $m+1$ nodes going through all nodes in $U^{\prime}$ in the same order as their counterparts on path $P$ and ending at node 2 . These edges have weight 0 apart from the last one, entering node 2 , which has weight $-C$ (a sufficiently large negative constant) ${ }^{10}$.

[^4]We create an instance of the Shortest Walk problem by setting $T=m+4$ and $A$ to be the weighted adjacency matrix of $G^{\prime}$ that takes value $+\infty$ (or a sufficiently large integer) for non-existent edges and self-loops.

The optimal walk of the Shortest Walk instance must include the edge of weight $-C$ entering node 2 since otherwise the cost will be non-negative. Moreover, the walk must reach node 2 exactly at the last step since otherwise the cost will be $+\infty$ as there are no outgoing edges from node 2 . By the choice of $T$, the walk leaves path $P$ at some node $u \in U$, then visits nodes $v_{1}$ and $v_{2}$ in $V_{1}$ and $V_{2}$, respectively, and subsequently moves to node $u^{\prime} \in U^{\prime}$ where $u^{\prime}$ is the counterpart of $u$ on path $P^{\prime}$. The total cost of the walk is thus the weight of the triangle $\left(u, v_{1}, v_{2}\right)$ in $G$, minus $C$. Therefore, the optimal walk has cost equal to the weight of the minimum triangle up to the additive constant $C$.

Notice that when $T>n$, the runtime of the Viterbi algorithm is no longer optimal. Equation 2 for the general Viterbi Path problem reduces, in the case of unary alphabet, to computing ( $\mathrm{min},+$ ) matrix-vector product $T$ times: $A \oplus A \oplus \ldots \oplus A \oplus z$. However, this can equivalently be performed by computing $A^{\oplus T}$ using exponentiation with repeated squaring. This requires only $O(\log T)$ matrix ( $\mathrm{min},+$ )-multiplications. Using the currently best algorithm for (min, + ) matrix product [Wil14], we get an algorithm with a total running time $\log T \cdot n^{3} / 2^{\Omega(\sqrt{\log n})}$.

## Acknowledgments

We thank Piotr Indyk for many helpful discussions and for comments on an earlier version of the writeup. This work was supported in part by an IBM PhD Fellowship, the NSF and the Simons Foundation.

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## A Sum of Probabilities

In the definition of the additive version of Viterbi Path, we didn't impose any constraint on the weights. In the multiplicative version where weights correspond to probabilities, we have the restriction that probabilities of transition from each vertex sum to 1.

To convert an instance $\mathcal{I}_{\text {Add }}$ of the additive Viterbi Path formulation to an equivalent instance $\mathcal{I}_{M u l}$ in the multiplicative setting, we add a shift of $\log n$ to all entries of $A$ and a shift of $\log T$ to entries of matrix $B$. This doesn't change the optimal solution but only changes its value by an additive shift of $T \log n+T \log T$. This transformation makes all probabilities in the $\mathcal{I}_{M u l}$ instance small enough such that transition probabilities sum to less than 1 and similarly probabilities of outputting observations sum to less than 1. To handle the remaining probability, we introduce an additional node $\alpha$ and an additional symbol $\gamma$ in the alphabet of observations. Every original transitions to node $\alpha$ with its remaining transition probability and outputs observation $\gamma$ with its remaining transition probability. We require that node $\alpha$ outputs observation $\gamma$ with $100 \%$ probability. As we never observe $\gamma$ in the sequence of observations, the optimal solution must never go through node $\alpha$ and thus the optimal solution remains the same.

The transformation above requires introducing an additional symbol in the alphabet. For our reduction of All-Pairs Shortest Paths to Viterbi Path when $\sigma=1$, we don't want the alphabet to increase. We describe an alternative transformation for $\sigma=1$ that doesn't introduce additional symbols. This case corresponds to the Shortest Walk instance and matrix $B$ is irrelevant.

We first scale all weights in matrices $A$, by dividing by some large weight $W$, so that all values are between 0 and 1 and then add a shift of $\log n$ to all of them. This doesn't change the optimal solution but only changes its value by a multiplicative factor $W$ and an additive shift of $T \log n$. After this transformation all values are between $\log n$ and $1+\log n$ and thus the corresponding probabilities in the $\mathcal{I}_{M u l}$ instance are at most $1 / n$. This causes transition probabilities to sum to less than 1 . To assign the remaining probability, we introduce a clique of $4 n$ additional nodes. All nodes in the clique have probability $\frac{1}{4 n}$ of transition to any other node in the clique and 0 probability of transition to any of the original node. For every original node, we spread its remaining transition probability evenly to the $4 n$ nodes of the clique. It is easy to see that the optimal solution to the Viterbi Path problem will not change, as it is never optimal to visit any of the nodes in the clique. This is because all edges in the original graph have weight at most $1+\log n$ while if a node in the clique is visited the path must stay in the clique at a cost of $2+\log n$ per edge.

## B Reduction from Min-Weight $k$-Clique to Min-Weight $k$-Clique in $k$-partite graphs

In this section, we show the following lemma using standard techniques.

Lemma 1. Consider the Min-Weight $k$-Clique problem in $k$-partite graphs $G=\left(V_{1} \cup \ldots \cup V_{k}, E\right)$ with $\left|V_{i}\right|=n_{i}$. If for all $i, j$ we have that $n_{i}=n_{j}^{\Theta(1)}$, then the Min-Weight $k$-Clique problem for this class of instances requires $\Omega\left(\prod_{i=1}^{k} n_{i}\right)^{1-o(1)}$ time assuming the $k$-Clique conjecture.
Proof. Without loss of generality assume that $n_{1} \geq n_{i}$ for all $i$ and let $n=n_{1}$. Assume, that there is an $O\left(\prod_{i} n_{i}\right)^{1-\varepsilon}$ algorithm that finds a minimum weight $k$-clique in $k$-partite graphs with $\left|V_{i}\right|=n_{i}$ for all $i$. We can use this faster algorithm to find a $k$-clique in a graph $G=(V, E)$ where $|V|=n$, as follows: Let $\mathcal{V}^{i}$ be a partition of $V$ into $\frac{n}{n_{i}}$ sets of size $n_{i}$. For all $\left(V_{1}, \ldots, V_{k}\right) \in \mathcal{V}^{1} \times \cdots \times \mathcal{V}^{k}$, we create a $k$-partite graph $G^{\prime}=\left(V_{1} \cup \ldots \cup V_{k}, E^{\prime}\right)$ by adding edges corresponding to the edges of graph $G$ between nodes across partitions and find the minimum weight $k$-clique in the graph $G^{\prime}$ using the faster algorithm. Computing the minimum weight $k$-clique out of all the graphs we consider gives the solution to the Min-Weight $k$-Clique instance on $G$. The total runtime is $\prod_{i=1}^{k} \frac{n}{n_{i}} \cdot O\left(\prod_{i=1}^{k} n_{i}\right)^{1-\varepsilon}=n^{k-\Omega(\varepsilon)}$ which would violate the $k$-Clique conjecture. The previous equality holds because of the assumption that $n_{i}=n_{j}^{\Theta(1)}$.

## C Hardness for sparse HMMs

For sparse HMMs that have at most $m$ edges out of the $n^{2}$ possible ones, i.e. the transition matrix has at most $m$ non-zero probabilities, the Viterbi Path problem can be easily solved in $O(T m)$ time. The lower bounds that we presented in the paper can be adapted directly for this case to show that no faster algorithm exists that runs in time $O(T m)^{1-\varepsilon}$. This can be easily seen via a padding argument. Consider a hard instance for Viterbi Path on a dense HMM with $\sqrt{m}$ states and $m$ edges. Adding $n-\sqrt{m}$ additional states with self-loops, we obtain a sparse instance with $n$ states and $m+n-\sqrt{m}=O(m)$ edges. Thus, any algorithm that computes the optimal Viterbi Path in $O(T m)^{1-\varepsilon}$ time for the resulting instance would solve the original instance with $\sqrt{m}$ states in $O\left(T(\sqrt{m})^{2}\right)^{1-\varepsilon}$ time contradicting the corresponding lower bound.

This observation directly gives the following lower bounds for Viterbi Path problem, parametrized by the number $m$ of edges in an HMM with $n$ states.

Theorem 8. The Viterbi Path problem requires $\Omega(T m)^{1-o(1)}$ time for an HMM with $m$ edges and $n$ states, assuming the APSP Conjecture.
Theorem 9. For any $C, \varepsilon>0$, the Viterbi Path problem on $T=\Theta\left(m^{C}\right)$ observations from an alphabet of size $\Theta\left(m^{\varepsilon}\right)$ requires $\Omega(T m)^{1-o(1)}$ time assuming the $k$-Clique Conjecture for $k=\left\lceil\frac{C}{\varepsilon}\right\rceil+2$.

Theorem 10. The Viterbi Path problem requires $\Omega(T m)^{1-o(1)}$ time when $T \leq \sqrt{m}$ even if the size of the alphabet is $\sigma=1$, assuming the APSP Conjecture.


[^0]:    ${ }^{1}$ Getting an algorithm running in time, say $O\left(T n^{1.99}\right)$.
    ${ }^{2}$ Truly subcubic means $O\left(n^{3-\delta}\right)$ for constant $\delta>0$.

[^1]:    ${ }^{3} \mathrm{~A}$ trivial such upper bound is $k^{2}$ times the weight of the maximum edge.
    ${ }^{4}$ Since $2^{Z}$ is a strict upper-bound on the clique size at most $Z$ digits are required.

[^2]:    ${ }^{5} \mathrm{~A}(\min ,+)$ product between a matrix $M$ and a vector $v$ is denoted by $M \oplus v$ and is equal to a vector $u$ where $u_{i}=\min _{j}\left(M_{i, j}+v_{j}\right)$.
    ${ }^{6}$ Even though computing all (min, + ) products does not directly give a path for the Viterbi Path problem, we can obtain one at no additional cost by storing back pointers. This is standard and we omit the details.

[^3]:    ${ }^{7}$ A direct implication of SETH is that satisfiability of CNFs can't be decided in time $O\left(2^{n(1-\varepsilon)}\right)$ for any constant $\varepsilon>0$.
    ${ }^{8}$ The 3 -Sum conjecture states that given a set of $n$ integers, deciding if it contains three integers that sum to 0 requires $\Omega\left(n^{2-o(1)}\right)$ time.
    ${ }^{9}$ The NSETH is an extension of SETH and it states that deciding the language of unsatisfiable CNFs requires $\Omega\left(2^{n(1-\varepsilon)}\right)$ nondeterministic time.

[^4]:    ${ }^{10}$ Since the definition of Shortest Walk doesn't allow negative weights, we can equivalently set its weight to be 0 and add $C$ to all the other edge weights.

