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The Probability of Collective Choice with Shared Knowledge Structures

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Complex decision making typically involves many agents attempting to aggregate many alternatives. If agents' preferences are unconstrained, cyclic outcomes are highly probable. In contrast, we show that if agents share similar models of the choice domain, then a stable collective outcome occurs with about 90% probability. These results have implications for systems of interacting agents, such as group decisions by individuals in social settings or interpretations of sense data by competing perceptual modules of a brain.

INTRODUCTION

There are abundant theoretical results from the field of social choice showing that there will almost surely be no single agreement when large numbers of individuals must choose from a large set of alternatives (e.g., Arrow, 1963; Campbell & Tullock, 1965; Jones, Rodcliff, Taber, & Timpore, 1995; Kelly, 1986). These results have significant implications for a variety of complex systems that are engaged in collective decision making using voting-like procedures. The most obvious application is in social settings such as legislatures or committees, where a group of

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individuals must aggregate their diverse opinions or preferences. However, other important applications include the integration of conflicting information provided by a group of experts (e.g., Saari & Haunsperger, 1991), where such experts may be as trivial as different sensory devices reporting to an information integration center (Clarke & Yuille, 1990; Jacobs, Jordan, Nowlan, & Hinton, 1991; Knill & Richards, 1996). Another example is understanding the brain, which can be viewed as consisting of a large number of semiautonomous neural modules acting as agents with potentially conflicting goals (Minsky, 1986; Van Essen, Anderson, & Felleman, 1992). If such systems are modeled as distributed collections of voters with arbitrary preferences not subject to dictatorial rule, then there is a high probability that group choices will cycle among the alternatives (Arrow, 1963; Doyle & Wellman, 1989; McKelvey, 1979). Yet cyclic outcomes are not typically observed. Thus a puzzle in both social and cognitive science is how and when information from a large collection of agents can be aggregated in a democratic setting to reach noncyclic, or what we refer to as *stable*, collective decisions.

Reexamining Arrow's (1963) general possibility theorem that no democratic aggregation procedure can guarantee a stable collective ranking, we focus on the assumption that agents may rank-order choices in any way. Specifically, we assume that agents' information is not arbitrary but is constrained by the agents' mental model of the choice domain. Now a very positive result emerges: there is a high probability of stable collective outcomes provided that a model of the domain is shared among the agents (Richards, 2001; Richards, McKay, & Richards, 1998; Runkel, 1956). Examples of such models include the relations between animal species, kinship networks, the organization of taste choices, flow diagrams, library catalog systems, and maps (e.g., Borg & Lingoes, 1987; Chandrasekaran, Glasgow, & Narayanan, 1995; Denzau & North, 1994; Gunkel, 1999; Romney, Boyd, Moore, Batchelder, & Brazill, 1996; Schelling, 1960; Shepard, 1980). A *knowledge structure* is the representational form for such models. We refer to a knowledge structure as shared when agents use similar models to rank order their preferences.

KNOWLEDGE STRUCTURES AS GRAPHS

Let the term "agent" be used to designate an individual-level entity that is contributing to a collective decision over a finite set of alternatives $A = \{a_1, ..., a_n\}$, where A contains at least two elements. Agents organize the set of alternatives using a knowledge structure, which we assume is identical for all agents. This structure serves as a model of the choice domain and is represented as a labeled connected graph $\mathcal{M}(A, e)$ with a set of vertices A and a set of edges e. The vertices of the graph represent the objects, events, or behaviors—i.e. the alternatives—and the edges of the graph indicate the relationships between these alternatives.

Figure 1 shows a simple example of a graph $\mathcal{M}$ representing parameter changes between four alternatives, $a_1, ..., a_4$. Each edge in $\mathcal{M}$ connecting two alternatives indicates that those alternatives differ in a single attribute. To illustrate, if the alternatives are choices among soft drinks, a_1 and a_4 may be two brands that have similar taste but differ in that one is caffeinated. Alternatives a_1 and a_2 may both have no caffeine but differ in their citrus blend, such as comparing 7-Up to Sprite.

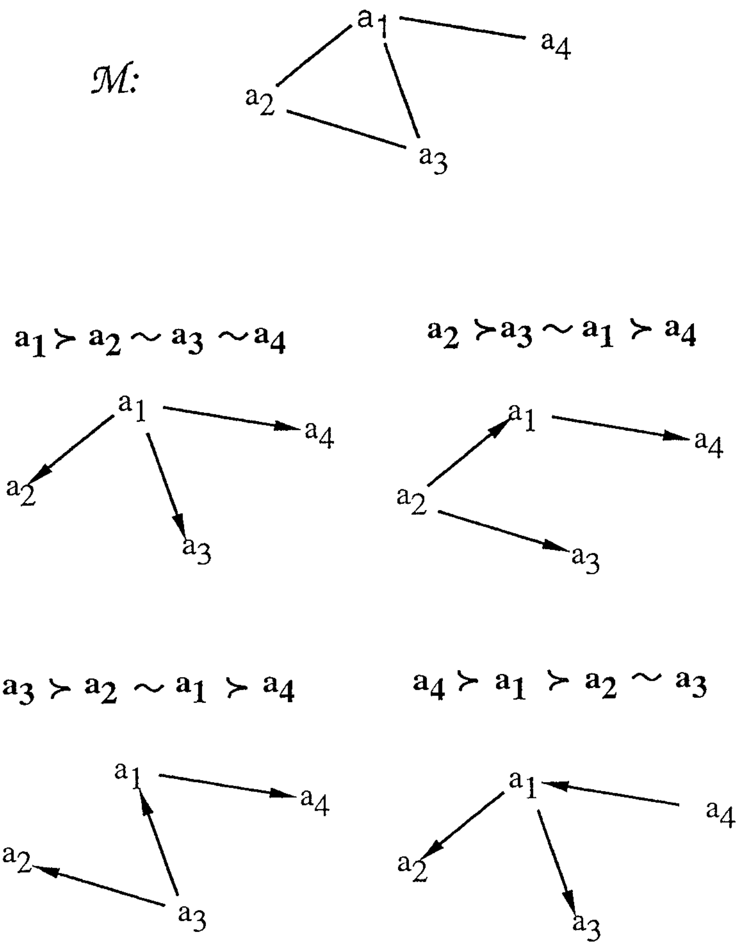

FIG. 1. Example of the set of four preference orders induced from a knowledge structure $\mathcal{M}$.

It is a central assumption of our framework that agents' preferences are consistent with a shared internal model of the relationships among alternatives. For example, in Black's (1958) or Coombs's (1964) constraint on linear orders, if $\mathcal{M}$ represents a shared cognitive ordering of musical instruments, such as flute–violin–cello–bass violin, then preferences that are inconsistent with the model, such as flute preferred to bass violin preferred to violin preferred to cello are assumed to be precluded. In our framework, the graph $\mathcal{M}$ is an abstraction of this shared model and serves as a constraint on agents' interpretations and preference orderings. Thus, given a most-preferred alternative, preference rankings over the remaining alternatives are assumed to be consistent with the structure of the graph $\mathcal{M}$. For example, for the model $\mathcal{M}$ shown in Fig. 1, an agent who most prefers alternative a_4 is assumed to rank a_1 as the second choice and to rank a_2 and a_3 as indifferent third choices. Specifically, we assume each agent has a unique most-preferred alternative, called an *ideal point*. Let a_o be the ideal point of agent o. If the number of edges on the shortest path from a_o to a_i is less than the number of edges on the shortest path from a_o to a_j then agent o prefers a_i to a_j, denoted $a_i \succ_o a_j$. If the number of edges on the shortest path from a_o to a_i is equal to the number of edges on the shortest path from a_o to a_j then we say that agent o is indifferent between a_i and a_j, denoted $a_i \sim a_j$. Thus, each agent has a (weak) preference ordering over the alternatives in A induced from the shared model $\mathscr{M}$.

We are interested in two possible outcomes: either there is a top cycle among alternatives or there is a top-ranked alternative. A *top cycle* exists if there is some set of alternatives such that a_i is preferred to a_j by a plurality of agents, a_j is preferred to a_k by a plurality of agents, and a_k is preferred to a_i by a plurality of agents, and every alternative not in the top cycle is beaten in a pairwise plurality vote by at least one alternative in the top cycle. (Note that there may be additional cycles that are not top cycles; we refer to these as *local cycles*.) For example, in the aggregation of information from several perceptual modules, an interpretation I_1 is unstable if there is another interpretation, I_2, that is favored over I_1 by some majority of agents and in turn I_2 is beaten in a pairwise plurality comparison by a set of agents favoring I_3, which in turn is favored by the original I_1 interpretation in a subsequent pairwise comparison. Such top cycles can be easily created in laboratory situations where perceptual information is controlled to be ambiguous, breaking the natural regularities and ordering relations normally encountered in the world (Gregory, 1970; Richards, Wilson, & Sommer, 1993; Rock 1983). If there is no top cycle then there must be a top-ranked alternative, which we call a *winner,* namely an alternative that cannot be overturned in a pairwise vote by any other alternative. Specifically, let $|a_j \succ a_k|$ denote the number of agents for whom a_j is preferred to a_k. If $|a_j \succ a_k| > |a_k \succ a_j|$ then a_j is preferred to a_k by a plurality of agents, denoted $a_j \succ a_k$. An alternative $a_j \in A$ is a winner if and only if for all $a_k \in A$, $a_k \neq a_j$, $|a_j \succ a_k| \geqslant |a_k \succ a_j|$. This definition corresponds to plurality rule, which is based on the same principle as the procedure that chooses the maximum likelihood ranking (Young, 1986).

METHODS AND TREATMENTS

Because the combinatorics quickly lead to very large numbers of preference profiles, most results on the probability of cycles are based on Monte Carlo simulations (e.g., Campbell & Tullock, 1965; Jones *et al.* 1995; Klahr, 1966). We follow suit. Our procedure is to construct a graph $\mathscr{M}_{n,p}$ with n vertices (corresponding to the n alternatives) and choose each of the $\frac{n(n-1)}{2}$ possible edges independently with probability p. Only one edge is allowed between any two vertices and graphs not connected were discarded. This random connected graph $\mathscr{M}_{n,p}$ determines the set of n feasible preference orders. Each preference order is assigned a weight w_i, $i = 1, ..., n$, drawn uniformly from the interval $[0, 1000]$. These weights create an n-tuple $\mathbf{w} = \{w_1, ..., w_n\}$ representing the distribution of agents over feasible preferences. We then evaluate all $\binom{n}{2}$ pairs of alternatives to determine whether a winner exists. The result gives the probability $Q(n)$ that the tournament induced from $\mathscr{M}_{n,p}$ with weights $\mathbf{w}$ has a winner. Following tradition, we present the results in terms of the probability of no winner, namely $1 - Q(n)$. The number of trials varied between 200 and 1000 depending on the number of vertices, the edge probabilities, and the probability $1 - Q(n)$. The results have estimated errors of 0.03 or less for values of $1-Q(n)$ between 0.2 and 1.0, errors of 0.02 or less for values of $1-Q(n)$ between 0.05 and 0.2, and errors of approximately 0.003 for $1-Q(n)$ less than 0.05.

RESULTS

Weakly Constrained Preferences

We begin by incorporating a small amount of constraint from the knowledge structure. Specifically, in this section it is assumed that only ideal points and alternatives adjacent to an agent's ideal point are constrained by the graph $\mathcal{M}_{n,p}$. The preference assignments for the remaining alternatives are chosen with equal probability from the set $\{\succ, \sim, \prec\}$. We restrict our attention to the case of $p = \frac{1}{2}$, which corresponds to sampling uniformly from all labeled graphs on n vertices and which generates the greatest variety of nonisomorphic graphs for any fixed n and p.

Our first result is shown by the triangles in Fig. 2. To provide some context, the figure also includes results from Jones *et al.* (1995) for randomly distributed weak preference orders (dashed line). The figure shows that by constraining preferences over only the ideal point and the alternatives adjacent to the ideal point, the probability of no winner decreases by nearly one half over random weak orders, as shown by the triangles. However, if the knowledge structure constraint is eliminated entirely, by considering only an agent's ideal point, then the probability of no winner quickly asymptotes at one. This result is shown by the solid circles in Fig. 2. The comparison between the upper and the lower curves in this figure makes clear that, for small choice sets, incorporating even such a locally restricted constraint on feasible preferences dramatically reduces the probability of cycles. The following section shows how further improvements emerge as the scope of a shared model is increased.

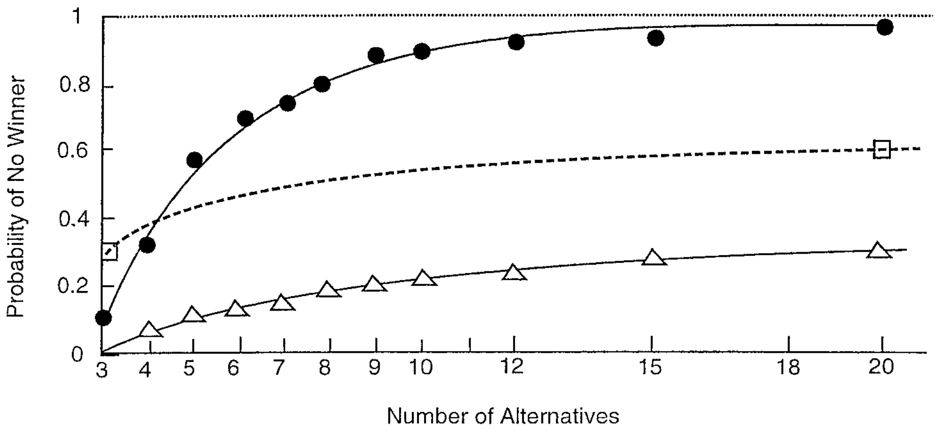

FIG. 2. Probability of No Winner. Upper curve (circles) shows results if the shared knowledge structure is ignored; middle curve (squares) shows results from Jones *et al.* (1995); lowest curve (triangles) shows the effect of a minimal knowledge constraint.

Strongly Constrained Preferences

Here we require that agents' preference orders be fully consistent with the shared model $\mathcal{M}_{n,p}$. The solid circles in Fig. 3 show the results for 3 to 100 alternatives for $p=\frac{1}{2}$. (Note the log scale for the ordinate.) For comparison, the triangles show our previous result when agents' preferences are constrained only by adjacent alternatives. We see that if agents use the full knowledge structure then the probability of no winner drops dramatically to less than 4% for $n \leqslant 100$. In other words, for less than 100 alternatives and for knowledge structures chosen uniformly over all labeled graphs, the probability of a stable outcome is at least 96% when agents' conflicting preferences are structured by a shared model. In nearly half the cases (47%), the winner is a vertex in the graph representation with maximum degree. In two thirds of the cases (67%), the winner is a vertex with either the maximum or one less than the maximum degree.

We now explore the full set of random graphs with edge probability p varying from 0 to 1. First consider the extremes. If $p=1$ then $\mathcal{M}_{n,1}$ is a fully connected graph and there is always a winner (Richards *et al.*, 1998). At the opposite end of the probability spectrum, edges will be sparse. Here, unrooted trees, T_n, are a limiting type of connected graph (see Harary, 1969, p. 13 for a formal definition of a tree). Trees also always have a winner. This new result is an important generalization of Black's theorem (1958) which demonstrates that a winner always exists when preferences are restricted to a simple linear ordering.

Definition 1. The *distance* between two vertices v, w of a graph is the least number of edges on a path from v to w. Note that weights on the vertices of the graph have no bearing on the distances in the graph. We will also find it convenient to define the distance between a vertex v and an edge e: it is $\frac{1}{2}$ more than the least of the two distances from v to the endpoints of e.

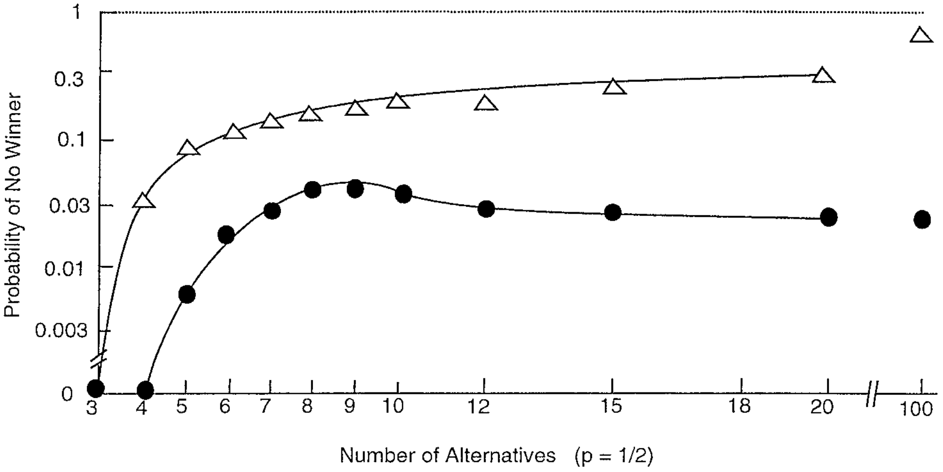

FIG. 3. Probability of No Winner with Mutual Knowledge Structures. Upper curve (triangles) repeats the minimal knowledge case from Fig. 2; lower curve (circles) shows results for full knowledge constraint and $p=\frac{1}{2}$.

DEFINITION 2. If T is a tree and v is a vertex of T, then the *limbs* at v are the components of the graph obtained by deleting v, and the edges incident with v, from T.

DEFINITION 3. Let the weights be normalized on a scale $[0, 1]$ such that their sum is 1. (Note that zero weights are allowed.) For a weighted graph with distinct vertices u and v, let $W(v, u)$ be the total weight of those vertices whose distance from v is less than their distance from u, including the weight of v itself. For a vertex v, define $e(v)$ to be the greatest total weight of a limb at v.

DEFINITION 4. The *weighted centroid* C of a tree is the set of all vertices v for which $e(v)$ is least.

LEMMA 5. *Let C be the weighted centroid of a tree and let $v \in C$. Then either $C = \{v\}$ and $e(v) < 1/2$ or C contains more than one vertex and $e(v) = 1/2$.*

Proof. If $e(v) < 1/2$, each vertex $u \neq v$ has a limb of weight greater than $1/2$, namely the limb containing v. Therefore, $C = \{v\}$ in that case.

If $e(v) = 1/2$, let u be a vertex adjacent to v that is in a limb of weight $1/2$ at v. Then $e(u) = 1/2$ and so $u \in C$. (Though not needed later, note that the centroid in this case consists of a single path whose internal vertices have weight 0.)

If $e(v) > 1/2$, choose that limb of weight greater than $1/2$ that has as few vertices as possible. Then the vertex u which lies in that limb and is adjacent to v has $e(u) < e(v)$, contradicting the minimality of $e(v)$. Therefore this case does not happen. ∎

THEOREM 6. *If v is in the weighted centroid of tree T, then $v \succ u$ for any u not in the weighted centroid of T.*

Proof. Let L be the limb of T at v that contains u. From Lemma 5 we know that $e(v) \leqslant 1/2$, so the weight of L is at most $1/2$. Therefore, the weight of all the other limbs at v plus the weight of v total at least $1/2$. Using Definition 1, these limbs (and v) are closer to v than to u, so $W(v, u) \geqslant 1/2$ and $W(u, v) \leqslant 1/2$. Thus $v \succ u$ unless $W(v, u) = W(u, v) = 1/2$. The latter can only happen if L has weight $1/2$ and $u \in C$, which is excluded by the conditions of the theorem. ∎

A more powerful result is that when the full knowledge constraint is invoked and the knowledge structure has the form of a tree; then there is a *complete* social ordering without cycles:

THEOREM 7. *For any set of weights and full knowledge depth, the social ordering induced from a tree knowledge structure has no cycles.*

Proof. See Appendix.

We next consider the case where $0 < p < 1$. Since the probability of top cycles is zero for $p = 0$ or 1, the probability of no winner must reach a maximum for some $0 < p < 1$. The effect of varying p for graphs with 5, 10, and 20 vertices is shown in Fig. 4. For these graphs, the greatest probability of no winner is approximately 0.12 and occurs when the probability of an edge is approximately 0.1 to 0.2 for a uniform distribution of weights on the alternatives. Since the effect of changing the probability of an edge is to change the distribution of classes of graphs, the results make clear that the structure of the graph $\mathcal{M}_{n,p}$ is a critical factor in collective stability. To further emphasize this point, a k-partite graph has an edge probability that is neither 0 nor 1, with $0 \ll p \ll 1$. Yet, as the following theorem shows, all complete k-partite graphs have no cycles.

THEOREM 8. *For any set of weights, the social ordering induced from a complete k-partite graph has no cycles.*

Proof. See Appendix.

Partial Knowledge Structures

A relevant variation on the general framework is to assume that the knowledge structure constrains preferences only over a subset of vertices near an agent's ideal point rather than over the entire choice set. Define the *knowledge depth*, k, as the maximum distance from the ideal point over which an agent's preference order is constrained by $\mathcal{M}$. A *knowledge depth with indifference* assumes that an agent is indifferent between all vertices more than k edges away from the agent's ideal point. Let $d(a_i, a_j)$ denote the distance between vertexes a_i and a_j. Then a knowledge depth of one implies that for any ideal point a_i and all a_j and $a_k \in \mathcal{M}$, if $d(a_i, a_j) \geqslant 2$ and $d(a_i, a_k) \geqslant 2$ then $a_j \sim_i a_k$.

Figure 5 illustrates the effect of reducing the knowledge depth to $k = 1$. As in the previous figure, the probability of no winner is plotted against the probability of

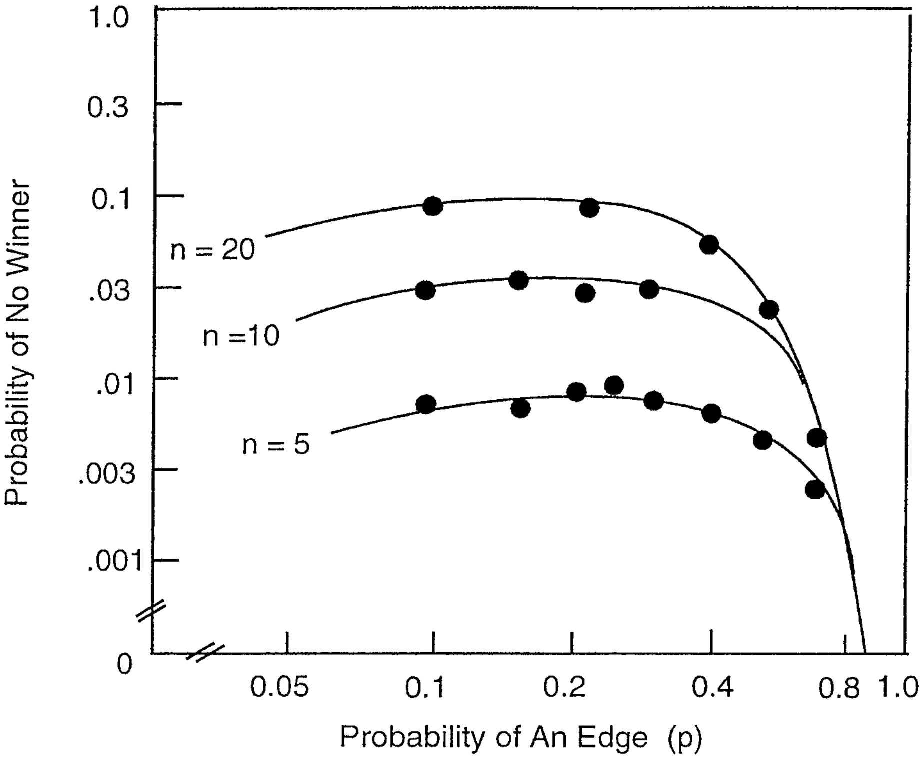

FIG. 4. Probability of No Winner for Full Knowledge Depth. Preference orders constrained by the full knowledge structure $\mathcal{M}_{n,p}$.

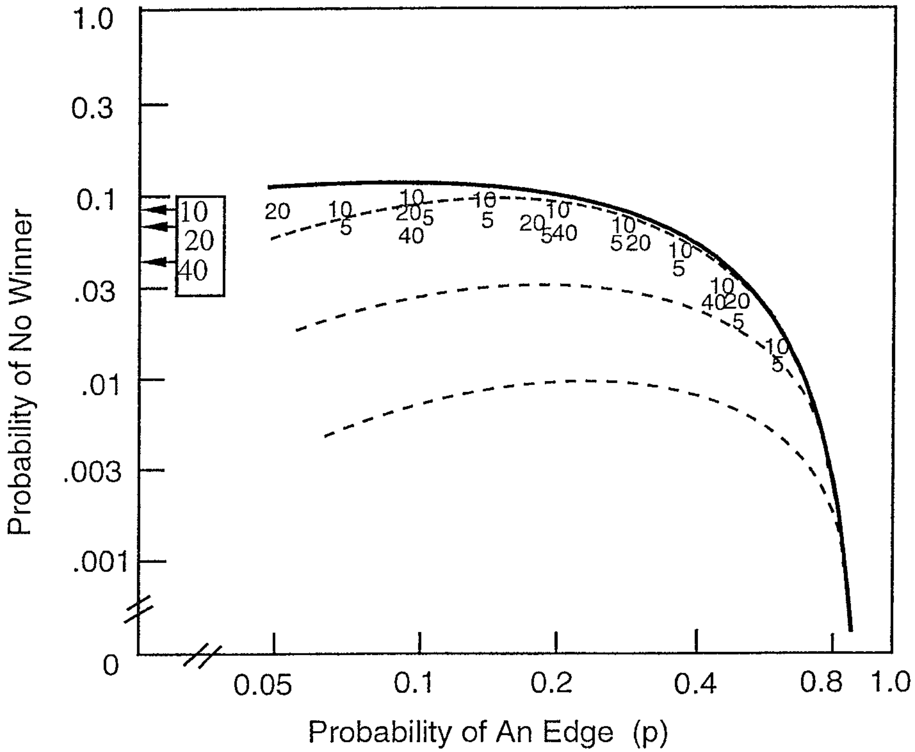

FIG. 5. Probability of No Winner for Partial Knowledge Depth. Preference orders only partially constrained by the knowledge structure $\mathscr{M}_{n,p}$. The small numbers show results for 5-, 10-, 20-, and 40-vertex graphs. The dashed curve repeats results from Fig. 4. The solid curve is a suggested upper bound on the probability of no winner. The box at the left shows results for trees.

choosing an edge from the set of $\binom{n}{2}$ edges. For comparison, the dashed curves repeat the result from Fig. 4 when agents use the full structure of the graph. The small numbers show the new results for 5-, 10-, 20-, and 40-vertex graphs. The solid curve is drawn to suggest the upper bound for the probability of no-winner for all graphs with less than 40 vertices. Note that this curve asymptotes near 0.11 as $p \to 0$. To keep these probabilities in perspective, recall that the lowest probability of cycles found by Jones *et al.* (1995) is approximately 40%—well above the upper bounds of our results at approximately 11%.

As in the case of a full knowledge depth, unrooted trees T_n give some insight into asymptotes for sparse connected graphs in the partial knowledge case. The labeled tics at the left margin of Fig. 5 show some probabilities of no-winner for the case of trees with a knowledge depth of one. Unlike the case of a full knowledge depth (where a winner was guaranteed), a partial knowledge constraint does admit the possibility of top cycles.

DISCUSSION

Shortly after Arrow's general possibility theorem showed that a majority winner could not be guaranteed for aggregation procedures satisfying a minimal set of conditions, Tullock (1967) raised the question of the significance of this result in terms of the probability of aggregation failures. Over the past three decades, a host of papers have appeared showing that unconstrained preference orders lead to a significant likelihood of no majority winner (e.g., Campbell & Tullock, 1965; DeMeyer & Plott, 1970; Garman & Kamien, 1968; Jones *et al.*, 1995; Kelly, 1986; Klahr, 1966; Niemi & Weisberg, 1968). The current consensus is that cycles in the collective aggregation of individual-level information are not a rare theoretical occurrence, with the probability of cycles approaching one as either the number of alternatives or the number of agents increases.

In contrast, when preferences are structured by a model shared by the agents, the probability of cycles is dramatically reduced. This knowledge structure constraint allows agents to have different preferences but places structure on relationships within the choice set. The effect of a shared knowledge structure is to markedly reduce the probability of cycles: even with a shallow knowledge depth, the probability of no winner is never greater than 12% for choice sets up to 20 alternatives. If agents' preferences are fully constrained by a shared knowledge structure then the probability of no winner is less than 5%.

An important limitation to this result is that all preference orders must be consistent with a shared model. However, in practice differences among the agents' models would be expected because similarity relationships may be asymmetric or intransitive (Coombs and Avrunin, 1977; Tversky 1969, 1977). Nevertheless, preliminary studies indicate that minor inconsistencies among the individual models of a domain have little effect on the probability of cycles. A much more important factor in increasing the probability of no winner is if agents vote probabilistically when they cannot discern differences between alternatives. This result is contrary to previous findings that emphasize the stability-inducing properties of aggregating probabilistic individual choices, such as the Condorcet jury theorem. We find that stable collective outcomes are more likely if agents refrain from voting over alternatives outside their knowledge depth.

Although a stable winner occurred roughly 90% of the time when the shared model is respected, an important lesson is that the *form* of the knowledge structure—namely the characteristics of the model $\mathcal{M}$, are an important variable in collective stability. In a previous paper we showed that certain knowledge relationships lead to top cycles whenever they appear as induced graphs (Richards *et al.*, 1998). The main effect of varying the probability of an edge in $\mathcal{M}$ is to change the distribution of the subgraphs that have a high probability of no winner. One may speculate as to the frequency of these troublesome graph forms as models in natural contexts of the social and physical world (McMahon, 1976; Richards & Bobick, 1988; Thompson, 1968). For example, tree-like structures, radial graphs, and "small world" graphs (Kasturirangan, 1999; Watts & Strogatz, 1998) are known to be empirically prevalent representational forms. An important issue is to identify which kinds of knowledge structures depicted as graphs *always* insure stable outcomes. Classes of graphs presently known to guarantee stable outcomes with a full knowledge depth include (1) all covered graphs (Richards *et al.*, 1998), (2) trees, and (3) complete k-partite graphs.

APPENDIX

Tree Knowledge Structures

DEFINITION 9. The *midpoint* between two vertices v, w of a tree is the vertex or edge x, on the unique path from v to w, which is equidistant from v and w. (Recall that our definition of *distance* covers both vertices and edges.)

DEFINITION 10. If p and q are two positions on a tree (possibly vertices or midpoints of edges), $w(p, q)$ denotes the total weight of all the vertices strictly between p and q along the (unique) path from p to q. If $p = q$, then $w(p, q) = 0$.

(Note that $w(p, q)$ is distinct from $W(v, u)$ as in Definition 3. Also, as before, the weight of p itself is $w(p)$, sometimes abbreviated w_p.)

Proof of Theorem 7. We first prove two special cases and then show that the general case follows.

Case (i). Suppose the tree T is a path and distinct vertices x, y, z are on the path in that order with x and z at the ends of the path. Let a be the midpoint between x and y, and let b be the midpoint between x and z, where b may be on either side of y or equal to y. Then

$$W(x, y) = w(x) + w(x, a), \tag{1}$$

$$W(y, x) = w(a, z) + w(z), \tag{2}$$

$$W(x, z) = w(x) + w(x, b), \tag{3}$$

$$W(z, x) = w(b, z) + w(z). \tag{4}$$

Thus, $W(x, y) \leqslant W(x, z)$ and $W(z, x) \leqslant W(y, x)$. These inequalities give implication (a): $x \succ y \Rightarrow x \succ z$. By the same reasoning with x and z exchanged, we have implication (b): $z \succ y \Rightarrow z \succ x$. Since each of the two possible cycles $x \succ y \succ z \succ x$ and $x \succ z \succ y \succ x$ violate one of (a) and (b), they do not exist.

Case (ii). Suppose the tree T consists of three paths joined at a common vertex v and that x, y, z are the other ends of each path. Let $d(x, y)$ denote the distance between vertex x and y. Without loss of generality, suppose $d(x, v) \leqslant d(y, v) \leqslant d(z, v)$. Let a be the midpoint between y and z (at v or on the path from v to z), b be the midpoint between x and z (at v or on the path from v to z), and c be the midpoint between x and y (at v or on the path from v to y). Let p be a position on a tree. Define

$$\omega(p, a) = \begin{cases} w(v) + w(v, x) + w(x) & \text{if} \quad p \neq v \\ 0 & \text{else;} \end{cases} \tag{5}$$

$$\omega(p, b) = \begin{cases} w(v) + w(v, y) + w(y) & \text{if} \quad p \neq v \\ 0 & \text{else;} \end{cases} \tag{6}$$

$$\omega(p, c) = \begin{cases} w(v) + w(v, z) + w(z) & \text{if} \quad p \neq v \\ 0 & \text{else.} \end{cases} \tag{7}$$

Then

$$W(x, y) = w(x) + w(x, v) + w(v, c) + \omega(p, c), \tag{8}$$

$$W(y, x) = w(y) + w(y, c), \tag{9}$$

$$W(x, z) = w(x) + w(x, v) + w(v, b) + \omega(p, b), \tag{10}$$

$$W(z, x) = w(z) + w(z, b), \tag{11}$$

$$W(y, z) = w(y) + w(y, v) + w(v, a) + \omega(p, a), \tag{12}$$

$$W(z, y) = w(z) + w(z, a). \tag{13}$$

Now we consider two possibilities. If $b = v$, then also $a = c = v$, so we have $W(x, y) = W(x, z)$, $W(y, x) = W(y, z)$, and $W(z, x) = W(z, y)$. It is easy to see that this prevents $\{x, y, z\}$ from forming a cycle. The remaining case is that $b \neq v$. Then we can see from above that $W(y, x) \leqslant W(y, z)$, $W(z, x) \leqslant W(z, y)$, and $W(y, z) \leqslant W(x, z)$. This gives the two implications $y \succ x \Rightarrow y \succ z$ and $z \succ x \Rightarrow z \succ y$, which again prevents $\{x, y, z\}$ from forming a cycle.

Now we can consider the general case. Let x, y, z be arbitrary distinct points of T. The smallest subtree S of T that contains all of $\{x, y, z\}$ is either a path or a star of three paths joined at some other vertex v. Consider any vertex u on S and any limb L at u that is disjoint from S. Then Eqs. (8)–(13) are unchanged if L is removed from T and its weight added to u. Continuing in this manner, we finally reach a weighting for S such that a cycle is formed by $\{x, y, z\}$ in S if and only if a cycle is formed by $\{x, y, z\}$ in T. However, S is one of the first two cases and so such a cycle cannot exist. ▮

k-Partite Knowledge Structures

DEFINITION 11. A graph G is a *complete k-partite* graph if its point set V can be partitioned into subsets $V_1, ..., V_k$ such that the edges of G join every member of V_i to every member of V_j for all $i \neq j$ and no edges join one member of V_i to another member of V_i for any i.

Proof of Theorem 8. Let a, b, and c be three vertices in a complete k-partite graph and assume a cycle with $a \succ b$, $b \succ c$, and $c \succ a$. There are three distinct cases: (i) $a, b, c \in V_i$, (ii) $a, b \in V_i$ and $c \in V_j$, or (iii) $a \in V_i$, $b \in V_j$, and $c \in V_k$. Let w_a, w_b, and w_c denote the weight on vertices a, b, and c, respectively. Let W_i^{-x} denote the weights of all vertices in V_i except for vertex x. The weights on all vertices not in V_i, V_j, or V_k can be ignored since these voters are indifferent between a, b, and c by the construction. Each case results in a contradiction: case (i) requires that $w_a > w_b$, $w_b > w_c$, and $w_c > w_a$; case (ii) requires that $w_a > w_b$, $w_b + W_j^{-c} > W_i^{-a,-b} + w_a + w_c$, and $W_i^{-a,-b} + w_b + w_c > w_a + W_j^{-c}$, which implies that $w_a > w_b$ and $w_b > w_a$; case (iii) requires that $w_a + W_j^{-b} > W_i^{-a} + w_b$, $w_b + W_k^{-c} > W_j^{-b} + w_c$, and $W_i^{-a} + w_c > w_a + W_k^{-c}$. This leads immediately to a contradiction. The same contradictions result for the reverse *abc* cycle. ▮

REFERENCES

Arrow, K. J. (1963). *Social choice and individual values*. New Haven: Yale Univ. Press.

Black, D. (1958). *The theory of committees and elections*. London: Cambridge Univ. Press.

Borg, I., & Lingoes, J. (1987). *Multidimensional similarity structure analysis*. New York: Springer-Verlag.

Campbell, C., & Tullock, G. (1965). A measure of the importance of cyclical majorities. *The Economic Journal*, **75**, 853–857.

Chandrasekaran, B., Glasgow, J., & Narayanan, N. H. (Eds.) (1995). *Diagrammatic reasoning*. Cambridge, MA: MIT Press.

Clarke, J. J., & Yuille, A. L. (1990). *Data fusion for sensory information processing systems*. London: Kluwer Academic.

Coombs, C. H. (1964). *A theory of data*. New York: Wiley.

Coombs, C. H., & Avrunin, G. S. (1977) Single-peaked functions and the theory of preferences. *Psychological Review*, **84**, 216–230.

DeMeyer, F., & Plott, C. (1970). The probability of a cyclical majority. *Econometrica*, **38**, 345–354.

Denzau, A. T., & North, D. C. (1994). Shared mental models: ideologies and institutions. *Kyklos*, **47**, 3–31.

Doyle, J., & Wellman, M. P. (1989). Impediments to universal preference-based default theories. In R. J. Brachman, H. J. Levesque, & R. Reiter (Eds.), *Proceedings of the first international conference on principles of knowledge representation and reasoning* (pp. 94–102). San Mateo, CA: Morgan Kaufmann.

Garman, M., & Kamien, M. (1968). The paradox of voting: probability calculations. *Behavioral Science*, **13**, 306–323.

Gregory, R. (1970). *The intelligent eye*. New York: McGraw–Hill.

Gunkel, P. (1999). *Ideonomy*. (http://www.ideonomy.edu)

Harary, F. (1969). *Graph theory*. Reading, MA: Addison–Wesley.

Jacobs, R. A., Jordan, M. I., Nowlan, S. J., & Hinton, G. E. (1991). Adaptive mixtures of local experts. *Neural Computation* **3**, 79–87.

Jones, B., Radcliff, B., Taber, C., & Timpone, R. (1995). Condorcet winners and the paradox of voting: probability calculations for weak preference orders. *American Political Science Review*, **89**, 137–144.

Kasturirangan, R. (1999). Multiple scales in small-world networks. Los Alamos Preprint Archive (http://www.lanl.gov/abs/cond-mat/9904055) and M.I.T. A.I. Memo No. 1663.

Kelly, J. S. (1986). Condorcet winner proportions. *Social Choice and Welfare*, **3**, 311–314.

Klahr, D. (1966). A computer simulation of the paradox of voting. *American Political Science Review*, **60**, 384–390.

Knill, D. C., & Richards, W. (Eds.) (1996). *Perception as Bayesian inference*. New York: Cambridge Univ. Press.

Luce, R. D., & Raiffa, H. (1957). *Games and decisions*. New York: Wiley.

McKelvey, R. D. (1979). General conditions for global intransitivities in formal voting models. *Econometrica*, **47**, 1085–1112.

McMahon, T. A. (1976). Allometry. In *Yearbook of science and technology* (pp. 48–57). New York: McGraw–Hill.

Minsky, M. (1986). *Society of mind*. New York: Simon and Schuster.

Niemi, R., & Weisberg, H. (1968). A mathematical solution for the probability of the paradox of voting. *Behavioral Science*, **13**, 317–323.

Richards, D. (2001). Coordination and shared mental models. *American Journal of Political Science*, **45**, 259–276.

Richards, D., McKay, B., & Richards, W. (1998). Collective choice and mutual knowledge structures. *Advances in Complex Systems*, **1**, 221–236 (http://www.polisci. umn.edu/faculty/richards/index.html).

Richards, W., & Bobick, A. (1988). Playing twenty questions with nature. In Z. Pylyshyn (Ed.), *Computational processes in human vision*. Norwood, NJ: Ablex.

Richards, W., Wilson, H., & Sommer, M. (1993). Chaos in percepts? *Biological Cybernetics*, **70**, 345–349.

Rock, I. (1983). *The logic of perception*. Cambridge, MA: M.I.T. Press.

Romney, A. K., Boyd, J. P., Moore, C. C., Batchelder, W. H., & Brazill, T. J. (1996). Culture as shared cognitive representations. *Proceedings of the National Academy of Sciences*, **93**, 1–7.

Runkel, P. J. (1956). Cognitive similarity in facilitating communication. *Sociometry*, **19**, 178–191.

Saari, D., & Haunsperger, D. (1991). The lack of consistency for statistical decision procedures. *The American Statistician* **45**, 252–255.

Schelling, T. C. (1960). *The strategy of conflict*. Cambridge, MA: Harvard Univ. Press.

Shepard, R. (1980). Multidimensional scaling, tree-fitting and clustering. *Science*, **210**, 390–398.

Thompson, D. (1968). *On growth and form*. New York: Cambridge Univ. Press.

Tullock, G. (1967). The general irrelevance of the general impossibility theorem. *Quarterly Journal of Economics*, **81**, 256–270.

Tversky, A. (1969). Intransitivity of preferences. *Psychological Review*, **76**, 31–48.

Tversky, A. (1977). Features of similarity. *Psychological Review*, **84**, 327–352.

Tversky, A., & Gati, I. (1982). Similarity, separability, and the triangle inequality. *Psychological Review*, **89**, 123–154.

Van Essen, D. C., Anderson, C., & Felleman, D. J. (1992). Information processing in the primate visual system: an integrated systems perspective. *Science*, **255**, 419–423.

Watts, D. J., & Strogatz, S. H. (1998). Collective dynamics of small-world networks. *Nature*, **393**, 440–442.

Young, H. P. (1986). Optimal ranking and choice from pairwise comparisons. In B. Grofman & G. Owen (Eds.), *Information pooling and group decision making*. Greenwich, CT: JAI Press.

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