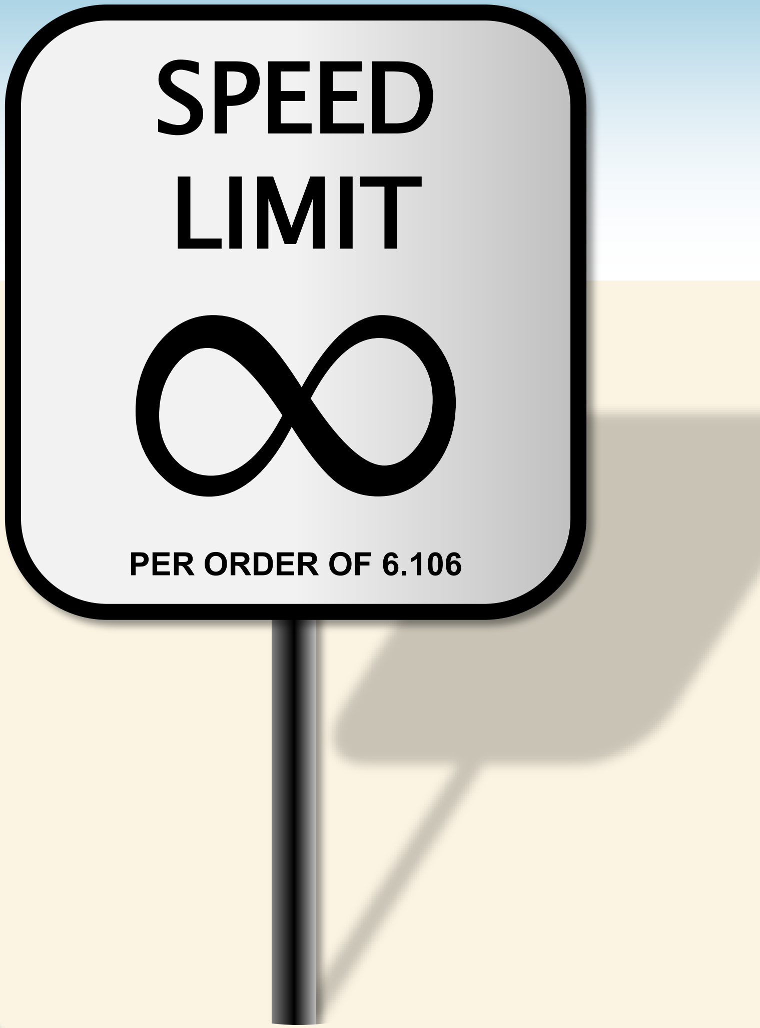
Performance
Engineering of
Software Systems




LECTURE 2

# Bentley Rules for Optimizing Work

Saman Amarasinghe

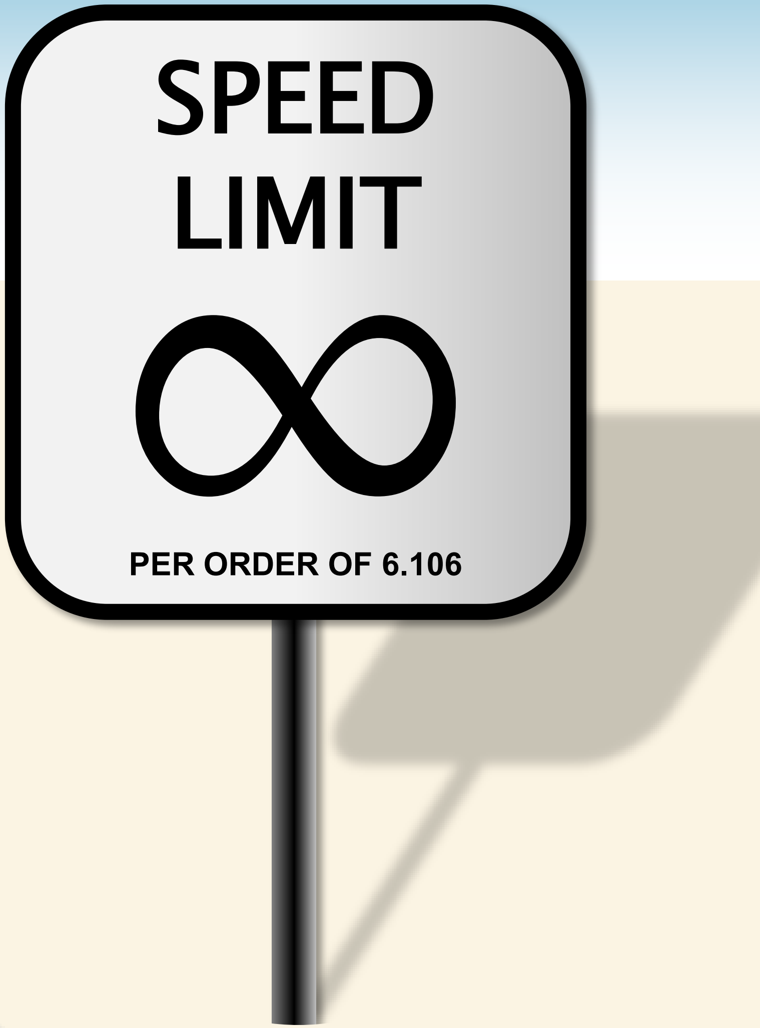
September 13, 2022

# Work

**Definition.**
The work of a program (on a given input) is the sum total of all the operations executed by the program.

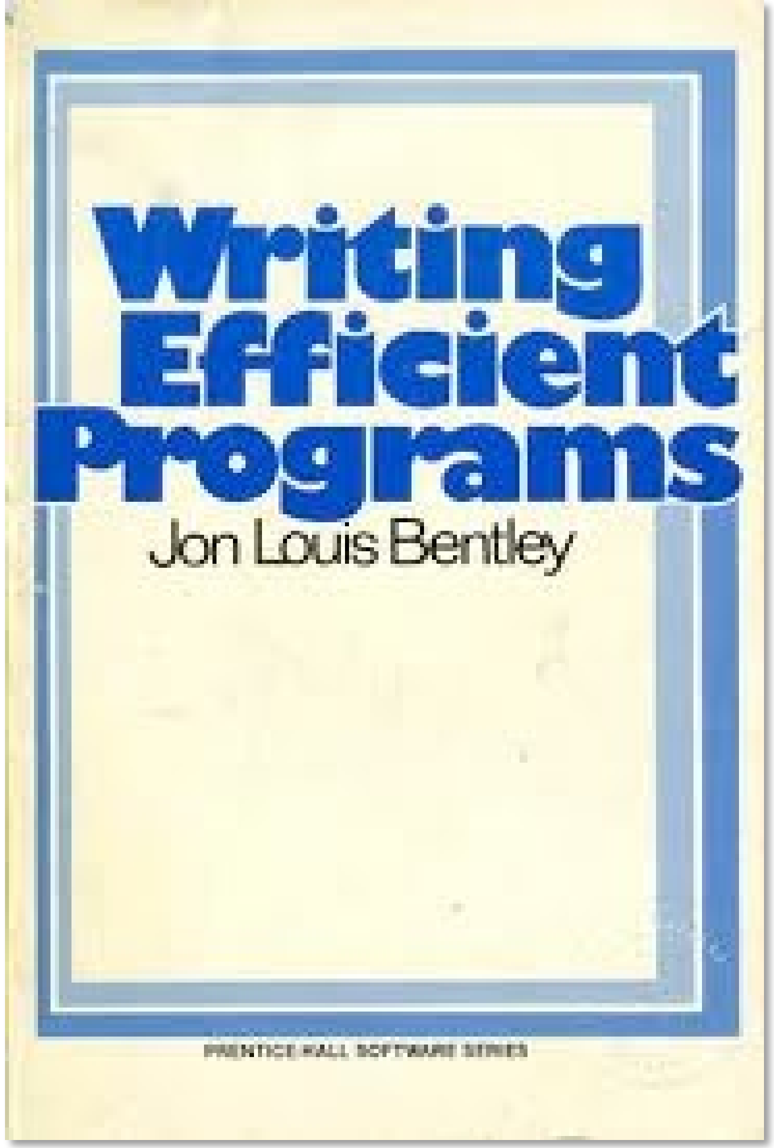
# Reducing Work

- Less work ≈ faster code.
- Reducing the work of a program does not automatically reduce its running time, however, due to the complex nature of computer hardware:
  - instruction-level parallelism (ILP),
  - caching,
  - vectorization,
  - speculation and branch prediction,
  - etc.
- Nevertheless, reducing the work serves as a good heuristic for reducing overall running time.
- Algorithm design can produce dramatic reductions in the work to solve a problem, as when a $\Theta(n \lg n)$-time sort replaces a $\Theta(n^2)$-time sort.




# Bentley Rules for Optimizing Work

# Jon Louis Bentley




1982

# New Bentley Rules

## Data structures

- Packing and encoding
- Augmentation
- Caching
- Precomputation
- Compile-time initialization
- Sparsity

Loops

## Loops

- Loop unrolling
- Hoisting
- Sentinels
- Loop fusion
- Eliminating wasted iterations

## Logic

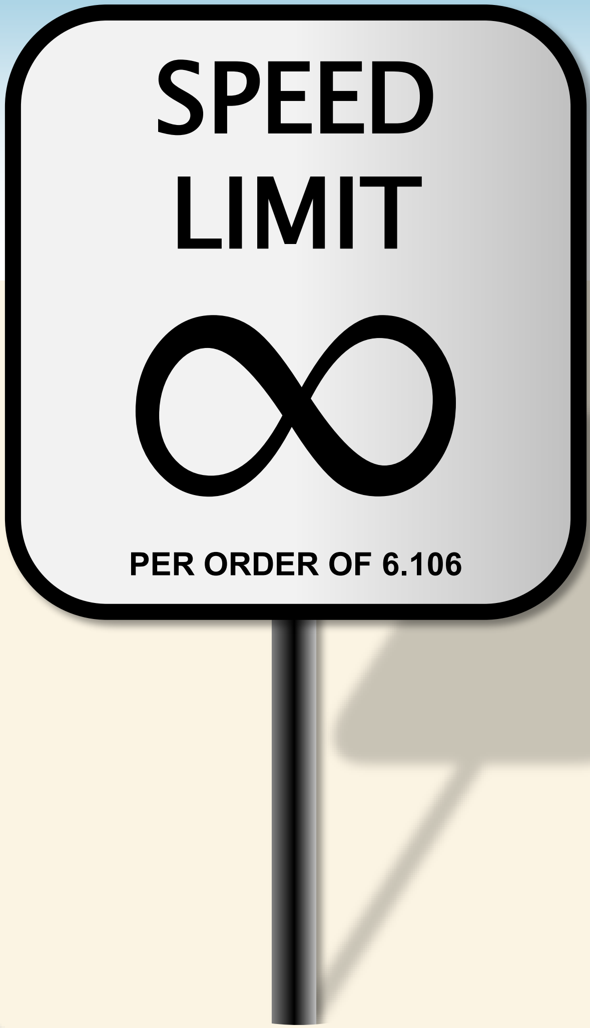
- Constant folding and propagation
- Common-subexpression elimination
- Algebraic identities
- Creating a fast path
- Short-circuiting
- Ordering tests
- Combining tests

Functions

## Functions

- Inlining
- Tail-recursion elimination
- Coarsening recursion

# DATA STRUCTURES



# Packing and Encoding

The idea of packing is to store more than one data value in a machine word. The related idea of encoding is to convert data values into a representation that requires fewer bits.

**Example:** Encoding dates

- The string "`September 3, 2020`" can be stored in 17 bytes — more than two 64-bit words — which must must move whenever the date is manipulated.

- Assuming that we only store dates between 4096 B.C.E. and 4096 C.E., there are about $365.25 \times 8192 \approx 3\,\mathrm{M}$ dates, which can be encoded in $\lceil \lg(3\times 10^6) \rceil = 22$ bits, easily fitting in a 32-bit word.

- **Problem:** How can we represent dates compactly so that determining the year, month, and day is fast?

# Packing and Encoding (2)

## Example: Packing dates

- Let us pack the three fields into a word:

```c
typedef struct {
  int year: 13;
  int month: 4;
  int day: 5;
} date_t;
```

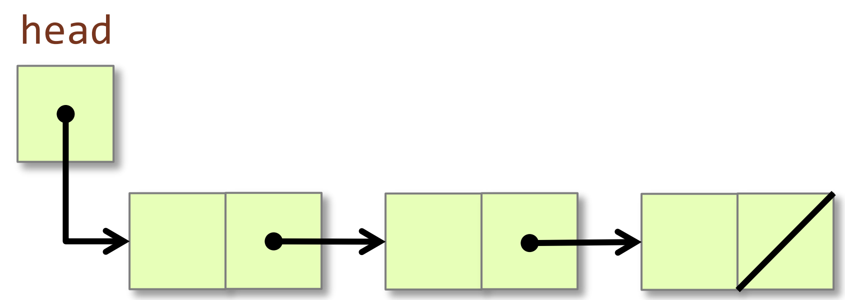
- This packed representation still only takes 22 bits, but the individual fields can be extracted much more quickly than if we had encoded the 3 M dates as sequential integers.

# Augmentation

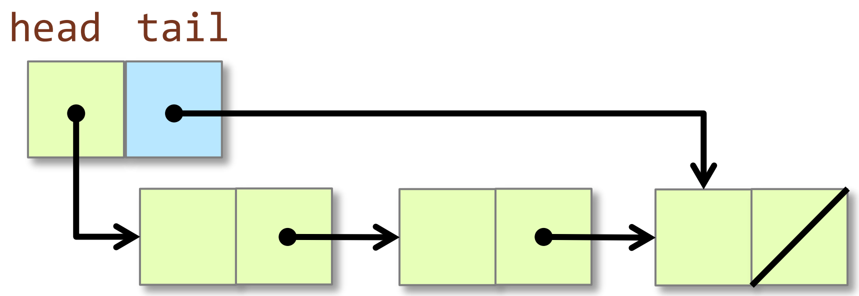
The idea of data-structure augmentation is to add information to a data structure to make common operations do less work.

**Example:** Appending singly linked lists.

- Appending one list to another requires walking the length of the first list to set its null pointer to the start of the second.




- Augmenting the list with a tail pointer allows appending to operate in constant time.



# Caching

The idea of caching is to store results that have been accessed recently so that the program need not compute them again.

*Before*

```
double hypotenuse(double A, double B) {
  return sqrt(A*A + B*B);
}
```

*After*

```
double cached_A = 0.0;
double cached_B = 0.0;
double cached_h = 0.0;

double hypotenuse(double A, double B) {
  if (A == cached_A && B == cached_B) {
    return cached_h;
  }
  cached_A = A;
  cached_B = B;
  cached_h = sqrt(A*A + B*B);
  return cached_h;
}
```

About 30% faster if cache is hit 2/3 of the time.

# Precomputation

The idea of precomputation is to perform calculations in advance so as to avoid doing them at “mission-critical” times.

Example: Binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Idea: Precompute the table of coefficients when initializing, and perform table look-up at runtime.

Note: Computing the “choose” function by implementing this formula can be expensive (lots of multiplications), and watch out for integer overflow for even modest values of $n$ and $k$.

# Step 1: Pascal's Triangle

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

| n \ k | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 3 | 3 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 4 | 6 | 4 | 1 | 0 | 0 | 0 | 0 |
| 1 | 5 | 10 | 10 | 5 | 1 | 0 | 0 | 0 |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | 0 | 0 |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | 0 |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 |

```c
int choose(int n, int k) {
  if (n < k)  return 0;
  if (k == 0) return 1;
  return choose(n-1, k-1) + choose(n-1, k);
}
```

# Step 2: Precomputing Pascal

```c
#define CHOOSE_SIZE 100
int choose[CHOOSE_SIZE][CHOOSE_SIZE];

void init_choose() {
  for (int n = 0; n < CHOOSE_SIZE; ++n) {
    choose[n][0] = 1;
    choose[n][n] = 1;
  }
  for (int n = 1; n < CHOOSE_SIZE; ++n) {
    choose[0][n] = 0;
    for (int k = 1; k < n; ++k) {
      choose[n][k] = choose[n-1][k-1] + choose[n-1][k];
      choose[k][n] = 0;
    }
  }
}
```

Now, whenever we need a binomial coefficient (less than 100), we can simply index the choose array.

# Compile-Time Initialization

The idea of compile-time initialization is to store the values of constants during compilation, saving work at execution time.

## Example

```
int choose[10][10] = {
  {  1,   0,   0,   0,   0,   0,   0,   0,   0,   0, },
  {  1,   1,   0,   0,   0,   0,   0,   0,   0,   0, },
  {  1,   2,   1,   0,   0,   0,   0,   0,   0,   0, },
  {  1,   3,   3,   1,   0,   0,   0,   0,   0,   0, },
  {  1,   4,   6,   4,   1,   0,   0,   0,   0,   0, },
  {  1,   5,  10,  10,   5,   1,   0,   0,   0,   0, },
  {  1,   6,  15,  20,  15,   6,   1,   0,   0,   0, },
  {  1,   7,  21,  35,  35,  21,   7,   1,   0,   0, },
  {  1,   8,  28,  56,  70,  56,  28,   8,   1,   0, },
  {  1,   9,  36,  84, 126, 126,  84,  36,   9,   1, },
};
```

# Compile-Time Initialization (2)

Idea: Create large static tables by metaprogramming.

```c
#define N 100
int main(int argc, const char *argv[]) {
  init_choose();
  printf("#define N %3d\n”, N);
  printf("int choose[N][N] = {\n");
  for (int a = 0; a < N; ++a) {
    printf("  {");
    for (int b = 0; b < N; ++b) {
      printf("%3d, ", choose[a][b]);
    }
    printf("},\n");
  }
  printf("};\n");
}
```

# Compile-Time Initialization (3)

Idea: Multi-stage Programming

```
static dyn_var<int> choose(dyn_var<int> n, dyn_var<int> k, const int MAX_N) {
	int comp[MAX_N][MAX_N];
	for (int i = 0; i < MAX_N; i++) {
		comp[i][0] = 1;
		comp[i][i] = 1;
	}
	for (int i = 1; i < MAX_N; ++i) {
		comp[0][i] = 0;
		for (int j = 1; j < i; ++j
			comp[i][j] = comp[
			comp[j][i] = 0;
		}
	}
	dyn_var<int[]> comp_r;
	resize(comp_r, MAX_N * MAX_N);
	for (static_var<int> i = 0; i < MA
		comp_r[i] = comp[i / MAX_N
	}
	return comp_r[n * MAX_N + k];
}
```

```
int choose (int arg0, int arg1) {
  int var0 = arg1;
  int var1 = arg0;
  int var2[100];
  var2[0] = 1;
  var2[1] = 0;
  var2[2] = 0;
  …
  var2[94] = 126;
  var2[95] = 126;
  var2[96] = 84;
  var2[97] = 36;
  var2[98] = 9;
  var2[99] = 1;
  int var3 = var2[(var1*10)+ var0];
  return var3;
}
```

See the **BuildIt** research project if you are interested (https://buildit.so/)

# Sparsity

The idea of exploiting sparsity is to avoid storing and computing on zeroes. "The fastest way to compute is not to compute at all."

Example: Matrix-vector multiplication

$$y = \begin{pmatrix} 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 5 & 9 \\ 0 & 0 & 0 & 2 & 0 & 6 \\ 5 & 0 & 0 & 3 & 0 & 0 \\ 5 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 9 & 7 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 2 \\ 8 \\ 5 \\ 7 \end{pmatrix}$$

Dense matrix-vector multiplication performs $n^2 = 36$ scalar multiplies, but only 14 entries are nonzero.

# Sparsity

The idea of exploiting sparsity is to avoid storing and computing on zeroes. "The fastest way to compute is not to compute at all."

## Example: Matrix-vector multiplication

$$
y = \begin{pmatrix}
3 & & & & 1 & \\
 & 4 & 1 & & 5 & 9 \\
 & & & 2 & & 6 \\
5 & & & 3 & & \\
5 & & & & 8 & \\
 & & & 9 & 7 &
\end{pmatrix}
\begin{pmatrix} 1 \\ 4 \\ 2 \\ 8 \\ 5 \\ 7 \end{pmatrix}
$$

Dense matrix-vector multiplication performs $n^2 = 36$ scalar multiplies, but only 14 entries are nonzero.

# Sparsity (2)

## Compressed Sparse Rows (CSR)

```
        0   1   2   3   4   5   6   7   8   9   10  11  12  13
rows: [0   2   6   8   10  11  14]

cols: [0   4   1   2   4   5   3   5   0   3   0   4   3   4]
vals: [3   1   4   1   5   9   2   6   5   3   5   8   9   7]
```

$$
\begin{array}{c}
0\\1\\2\\3\\4\\5\\ \\
\end{array}
\begin{array}{c}
\begin{pmatrix}
3 & 0 & 0 & 0 & 1 & 0\\
0 & 4 & 1 & 0 & 5 & 9\\
0 & 0 & 0 & 2 & 0 & 6\\
5 & 0 & 0 & 3 & 0 & 0\\
0 & 0 & 0 & 0 & 5 & 0\\
0 & 0 & 0 & 8 & 9 & 7
\end{pmatrix}\\
\begin{array}{cccccc}0 & 1 & 2 & 3 & 4 & 5\end{array}
\end{array}
$$

$n = 6$

$nnz = 14$

Storage is $O(n+nnz)$ instead of $n^2$

# Sparsity (3)

## CSR matrix-vector multiplication

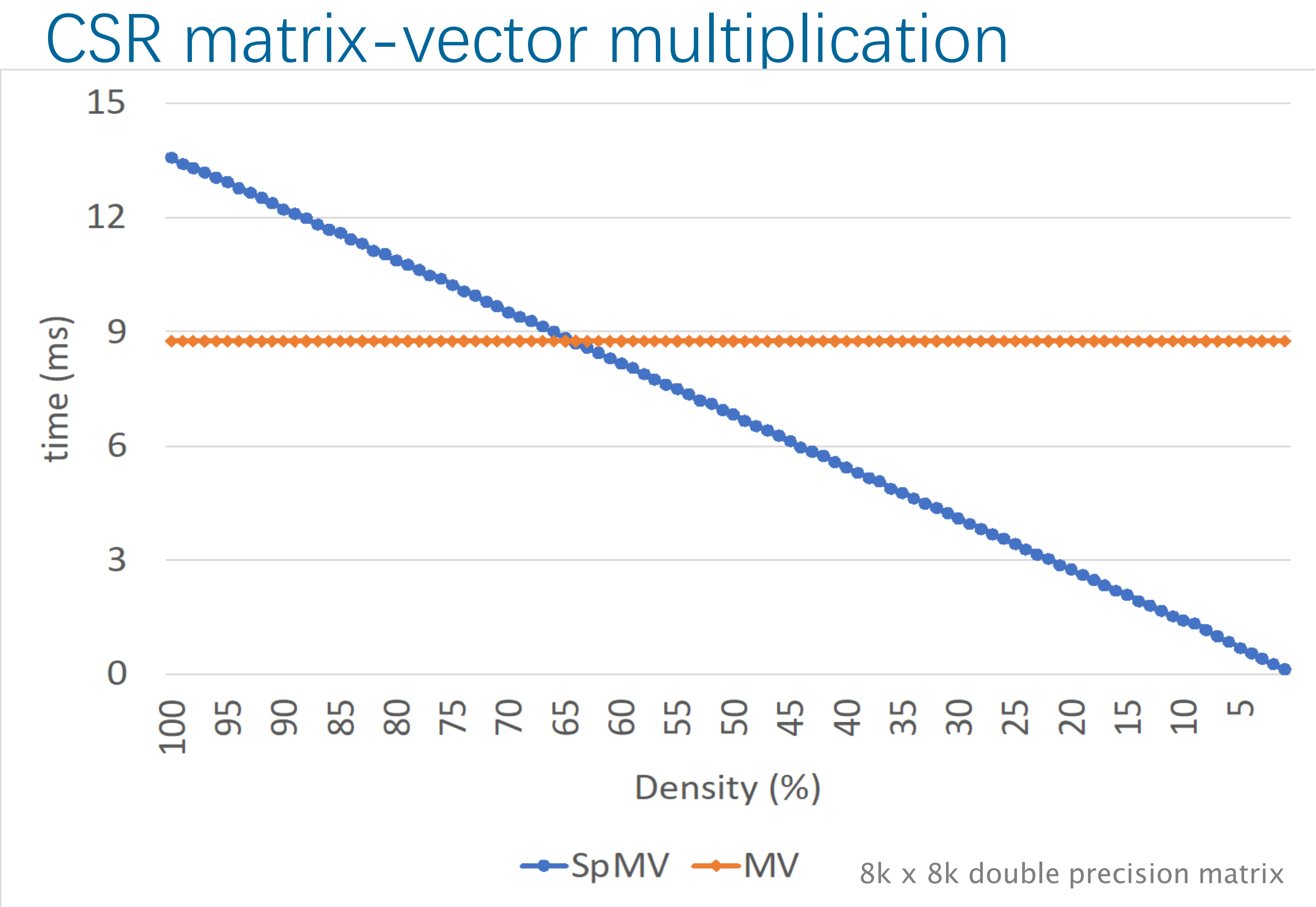
```
typedef struct {
  int n, nnz;
  int *rows;      // length n
  int *cols;      // length nnz
  double *vals;   // length nnz
} sparse_matrix_t;

void spmv(sparse_matrix_t *A, double *x, double *y) {
  for (int i = 0; i < A->n; i++) {
    y[i] = 0;
    for (int k = A->rows[i]; k < A->rows[i+1]; k++) {
      int j = A->cols[k];
      y[i] += A->vals[k] * x[j];
    }
  }
}
```

Number of scalar multiplications = nnz, which is potentially much less than $n^2$.

See the **TACO** research project if you are interested (https://tensor-compiler.org/)

# Sparsity (3)



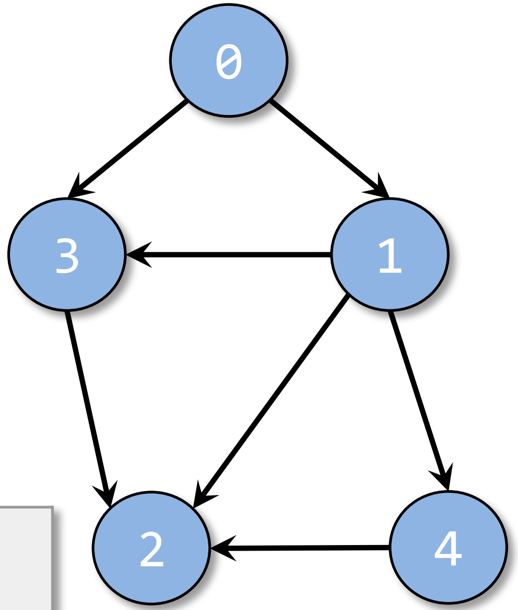

Number of scalar multiplications = nnz, which is potentially much less than $n^2$.

See the **TACO** research project if you are interested (https://tensor-compiler.org/)

# Sparsity (4)

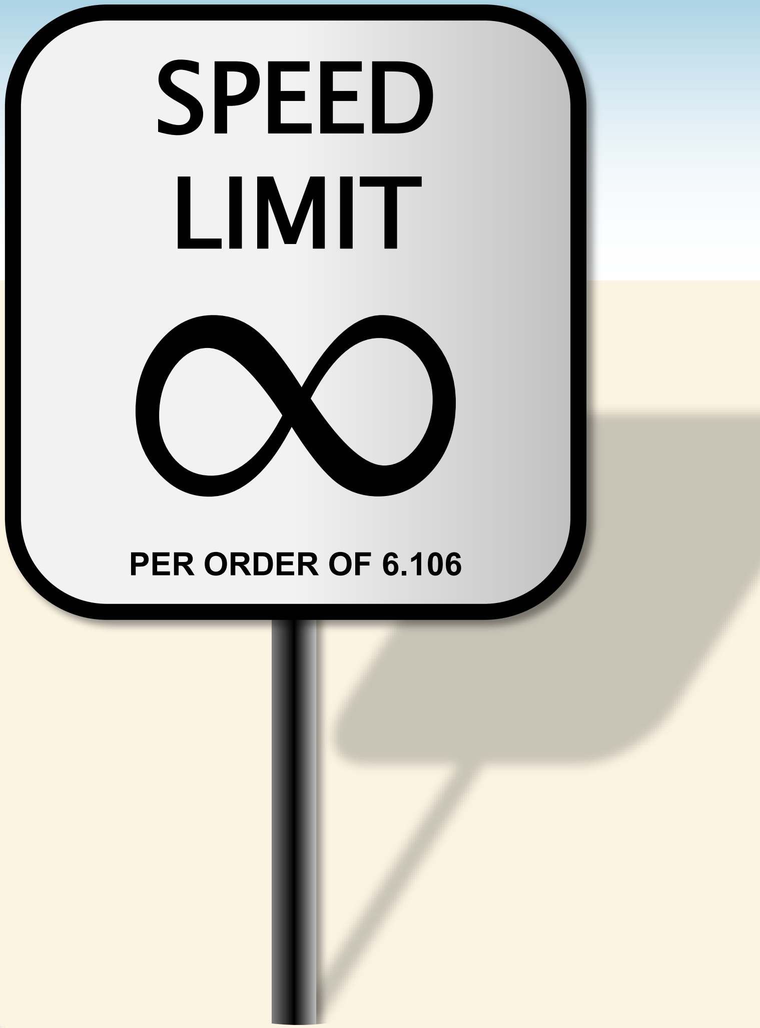
## Storing a static sparse graph

| Vertex ID | 0 | 1 | 2 | 3 | 4 | | |
|---|---|---|---|---|---|---|---|
| offsets | 0 | 2 | 5 | 5 | 6 | 7 | |
| edges | 1 | 3 | 2 | 3 | 4 | 2 | 2 |




- Many graph algorithms run efficiently on this representation, e.g., breadth-first search, PageRank.
- Edge weights can be stored in an additional array or by making each edges element a record containing the both the edge index and the edge weight.

See the **GraphIt** research project if you are interested (https://graphit-lang.org/)



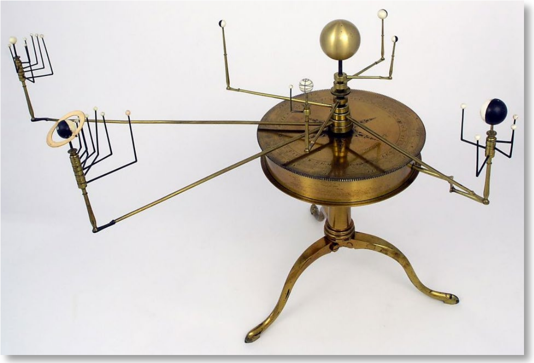

# Logic

# Constant Folding and Propagation

The idea of constant folding and propagation is to evaluate constant expressions and substitute the result into further expressions, all during compilation.

```c
#include <math.h>

void orrery() {
  const double radius = 6371000.0;
  const double diameter = 2 * radius;
  const double circumference = M_PI * diameter;
  const double cross_area = M_PI * radius * radius;
  const double surface_area =
      circumference * diameter;
  const double volume =
      4 * M_PI * radius * radius * radius / 3;
  // ...
}
```



*mechanical orrery*[1]

With a sufficiently high optimization level, all the expressions are evaluated at compile-time.

[1]https://en.wikipedia.org/wiki/Orrery#/media/File:Thinktank_Birmingham_-_object_1956S00682.00001(1).jpg

# Common-Subexpression Elimination

The idea of common-subexpression elimination is to avoid computing the same expression multiple times by evaluating the expression once and reusing the result when you later need it.

```
a = b + c;
b = a - d;
c = b + c;
d = a - d;
```

```
a = b + c;
b = a - d;
c = b + c;
d = b;
```

# Common-Subexpression Elimination

The idea of common-subexpression elimination is to avoid computing the same expression multiple times by evaluating the expression once and storing the result for later use.

```
a = b + c;
b = a - d;
c = b + c;
d = a - d;
```

```
a = b + c;
b = a - d;
c = b + c;
d = b;
```

The third line cannot be replaced by `c = a`, because the value of `b` changes in the second line.

# Algebraic Identities

The idea of exploiting algebraic identities is to replace expensive algebraic expressions with algebraic equivalents that require less work.

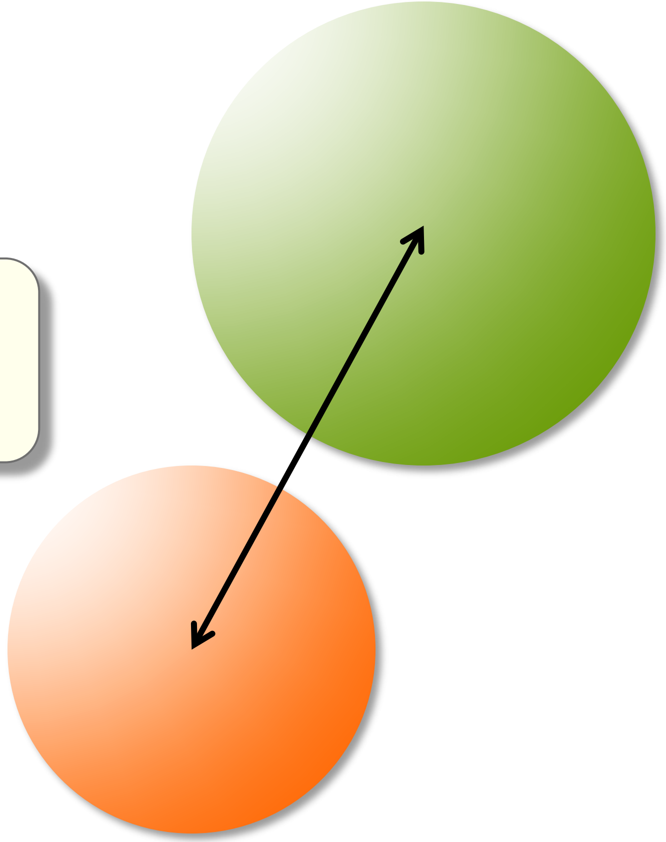
```
#include <stdbool.h>
#include <math.h>

typedef struct {
  double x, y, z;  // spatial coordinates
  double r;        // radius of ball
} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  double d = sqrt(square(b1->x - b2->x)
                + square(b1->y - b2->y)
                + square(b1->z - b2->z));
  return d <= b1->r + b2->r;
}
```

Expensive routine!



# Algebraic Identities

The idea of exploiting algebraic identities is to replace expensive algebraic expressions with algebraic equivalents that require less work.

```c
#include <stdbool.h>
#include <math.h>

typedef struct {
  double x, y, z;  // spatial coordinates
  double r;        // radius
} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  double d = sqrt(square(b1->x - b2->x)
                + square(b1->y - b2->y)
                + square(b1->z - b2->z));
  return d <= b1->r + b2->r;
}
```

$\sqrt{u} \le v$ exactly when $u \le v^2$.

```c
bool collides(ball_t *b1, ball_t *b2) {
  double dsquared = square(b1->x - b2->x)
                  + square(b1->y - b2->y)
                  + square(b1->z - b2->z);
  return dsquared <= square(b1->r + b2->r);
}
```

**Caution:** Be careful with floating point!

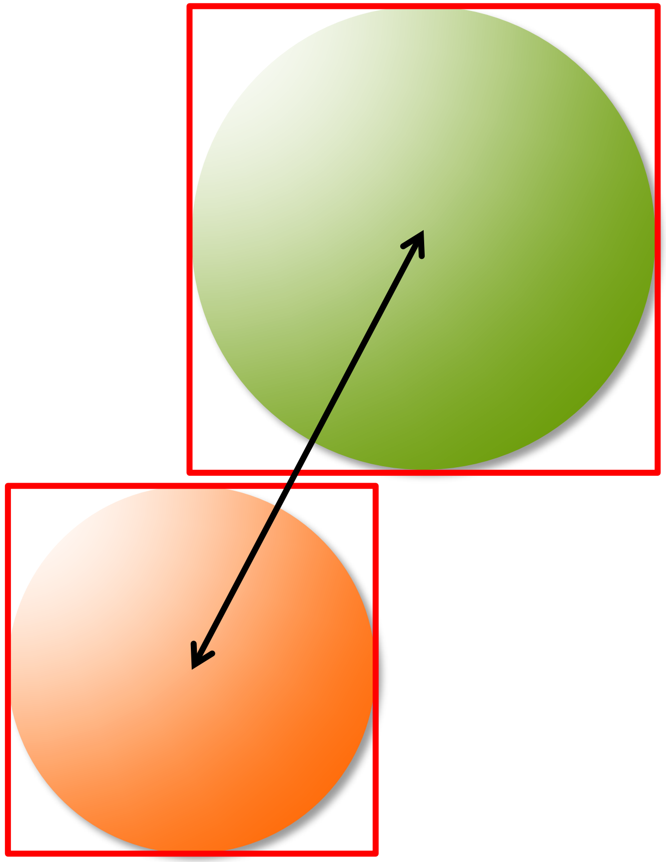
# Creating a Fast Path

```c
#include <stdbool.h>
#include <math.h>

typedef struct {
  double x, y, z;  // spatial coordinates
  double r;        // radius of ball
} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  double dsquared = square(b1->x - b2->x)
                  + square(b1->y - b2->y)
                  + square(b1->z - b2->z);
  return dsquared <= square(b1->r + b2->r);
}
```



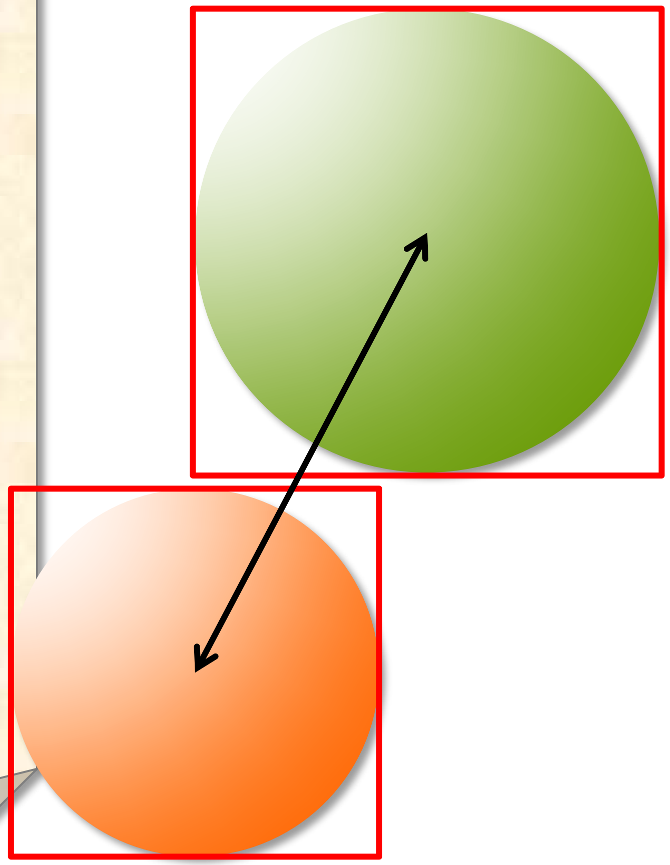
# Creating a Fast Path

```c
#include <stdbool.h>
#include <math.h>

typedef struct {
  double x, y, z;  // spatial coordinates
  double r;        // radius of ball
} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  double dsquared = square(b1->x - b2->x)
                  + square(b1->y - b2->y)
                  + square(b1->z - b2->z);
  return dsquared <= square(b1->r + b2->r);
}
```



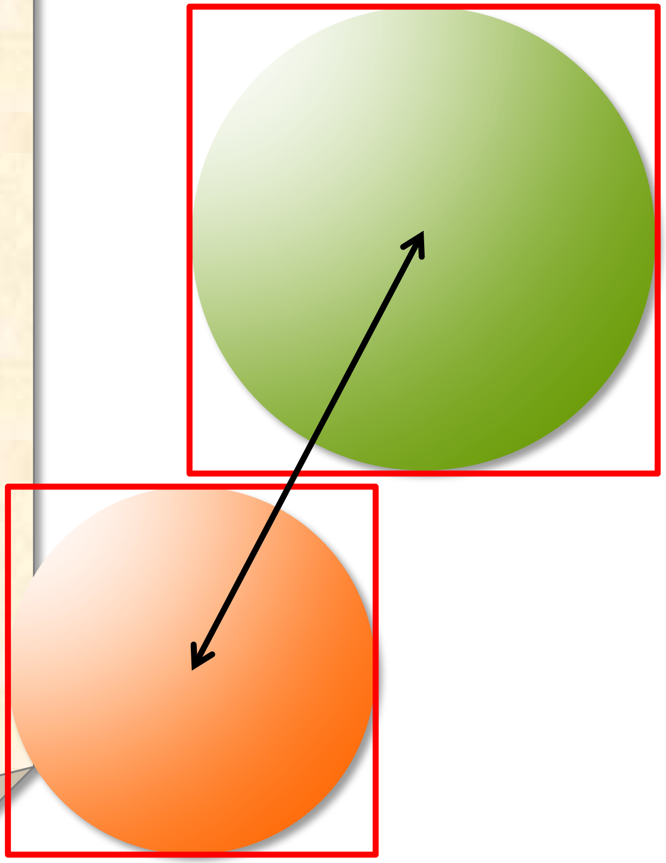
# Creating a Fast Path

```c
#include <stdbool.h>
#include <math.h>

typedef struct {
  double x, y, z;  // spatial coordinates
  double r;        // radius of ball
} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  if ((abs(b1->x - b2->x) > (b1->r + b2->r)) ||
      (abs(b1->y - b2->y) > (b1->r + b2->r)) ||
      (abs(b1->z - b2->z) > (b1->r + b2->r)))
    return false;
  double dsquared = square(b1->x - b2->x)
                  + square(b1->y - b2->y)
                  + square(b1->z - b2->z);
  return dsquared <= square(b1->r + b2->r);
}
```



# Short-Circuiting

When performing a series of tests, the idea of short-circuiting is to stop evaluating as soon as you know the answer.

```c
#include <stdbool.h>
// All elements of A are nonnegative
bool sum_exceeds(int *A, int n, int limit) {
  int sum = 0;
  for (int i = 0; i < n; i++) {
    sum += A[i];
  }
  return sum > limit;
}
```

# Short-Circuiting

When performing a series of tests, the idea of short-circuiting is to stop evaluating as soon as you know the answer.

*Before*

```c
#include <stdbool.h>
// All elements of A are nonnegative
bool sum_exceeds(int *A, int n, int limit) {
  int sum = 0;
  for (int i = 0; i < n; i++) {
    sum += A[i];
  }
  return sum > limit;
}
```

*After*

```c
#include <stdbool.h>
// All elements of A are nonnegative
bool sum_exceeds(int *A, int n, int limit) {
  int sum = 0;
  for (int i = 0; i < n; i++) {
    sum += A[i];
    if (sum > limit) {
      return true;
    }
  }
  return false;
}
```

# Ordering Tests

Consider code that executes a sequence of logical tests. The idea of ordering tests is to perform those that are more often "successful" — a particular alternative is selected by the test — before tests that are rarely successful.

*Before*

```
#include <stdbool.h>
bool is_whitespace(char c) {
  return (c == '\r' || c == '\t' || c == ' ' || c == '\n');
}
```

*After*

```
#include <stdbool.h>
bool is_whitespace(char c) {
  return (c == ' ' || c == '\n' || c == '\t' || c == '\r');
}
```

Note that && and || are short-circuiting logical operators, whereas & and | are not.

# Combining Tests

The idea of combining tests is to replace a sequence of tests with one test or switch.

## Full adder

| a | b | c | carry | sum |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

```
void full_add(int a,
              int b,
              int c,
              int *sum,
              int *carry) {
  if (a == 0) {
    if (b == 0) {
      if (c == 0) {
        *sum = 0;
        *carry = 0;
      } else {
        *sum = 1;
        *carry = 0;
      }
    } else {
      if (c == 0) {
        *sum = 1;
        *carry = 0;
      } else {
        *sum = 0;
        *carry = 1;
      }
    }
```

```
  } else {
    if (b == 0) {
      if (c == 0) {
        *sum = 1;
        *carry = 0;
      } else {
        *sum = 0;
        *carry = 1;
      }
    } else {
      if (c == 0) {
        *sum = 0;
        *carry = 1;
      } else {
        *sum = 1;
        *carry = 1;
      }
    }
  }
}
```

# Combining Tests (2)

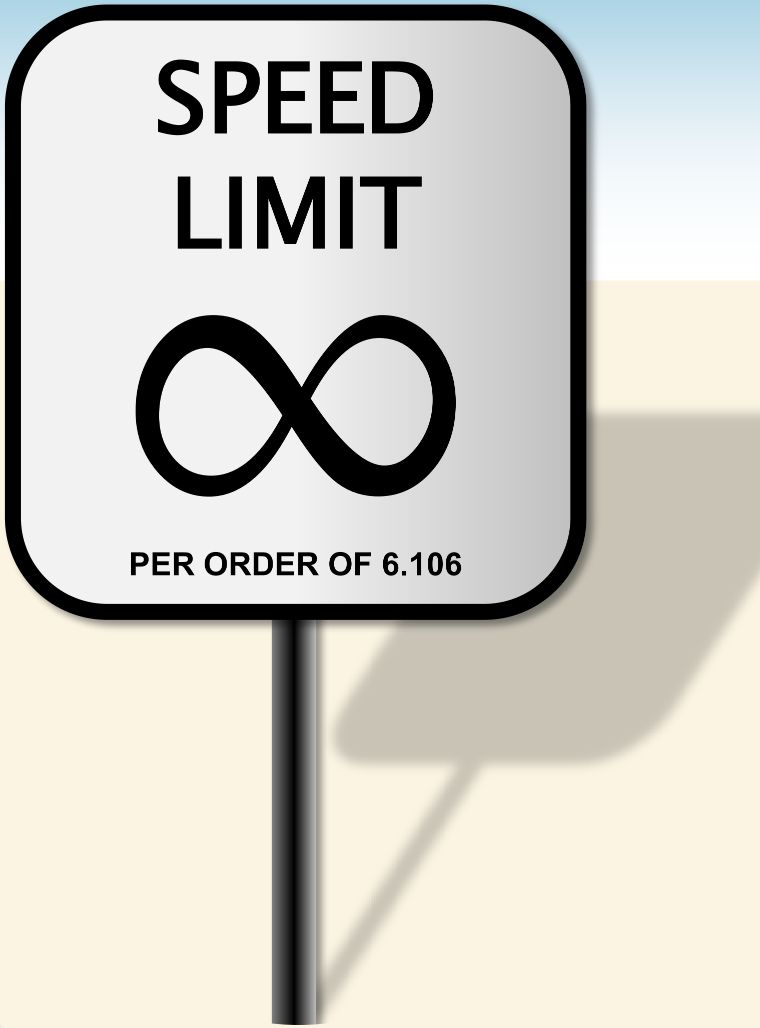
The idea of combining tests is to replace a sequence of tests with one test or switch.

## Full adder

| a | b | c | carry | sum |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

In this case, the outputs can be computed mathematically.

```
void full_add(int a,
              int b,
              int c,
              int *sum,
              int *carry) {
  int test = ((a == 1) << 2)
            | ((b == 1) << 1)
            | (c == 1);
  switch(test) {
    case 0:
      *sum = 0;
      *carry = 0;
      break;
    case 1:
      *sum = 1;
      *carry = 0;
      break;
    case 2:
      *sum = 1;
      *carry = 0;
      break;
    case 3:
      *sum = 0;
      *carry = 1;
      break;
    case 4:
      *sum = 1;
      *carry = 0;
      break;
    case 5:
      *sum = 0;
      *carry = 1;
      break;
    case 6:
      *sum = 0;
      *carry = 1;
      break;
    case 7:
      *sum = 1;
      *carry = 1;
      break;
  }
}
```




# Loops

# Why Loops?

**Loops** are often the focus of performance optimization. Why?

Loops account for a lot of work!

Consider this thought experiment:

- Suppose that a 2 GHz processor can execute 1 instruction per clock cycle.
- Suppose that a program contains 16 GB of instructions, but it is all simple straight-line code, i.e., no backwards branches.
- **Question:** How long does the code take to run?

Answer: at most 8 seconds!

# What Happens When a Loop Runs?

## A simple loop

```
int sum = 0;
for (int i = 0; i < N; i++) {
  sum += A[i];
}
```

Loop control

## Pseudocode for loop execution

```
int sum = 0;
int i = 0;
if (i >= N)
  goto loop_exit;
sum += A[i];
i++;
if (i >= N)
  goto loop_exit;
sum += A[i];
i++;
if (i >= N)
  goto loop_exit;
sum += A[i];
i++;
if (i >= N)
  goto loop_exit;
sum += A[i];
i++;
if (i >= N)
  goto loop_exit;
// ...
```

# Loop Unrolling

Loop unrolling attempts to save work by combining several consecutive iterations of a loop into a single iteration, thereby reducing the total number of iterations of the loop and, consequently, the number of times that the instructions that control the loop must be executed.

- Full loop unrolling: All iterations are unrolled.
- Partial loop unrolling: Several, but not all, of the iterations are unrolled.

# Full Loop Unrolling

*Before*

```c
int sum = 0;
for (int i = 0; i < 10; i++) {
  sum += A[i];
}
```

*After*

```c
int sum = 0;
sum += A[0];
sum += A[1];
sum += A[2];
sum += A[3];
sum += A[4];
sum += A[5];
sum += A[6];
sum += A[7];
sum += A[8];
sum += A[9];
```

# Partial Loop Unrolling

*Before*

```c
int sum = 0;
for (int i = 0; i < n; ++i) {
  sum += A[i];
}
```

*After*

```c
int sum = 0;
int j;
for (j = 0; j < n-3; j += 4) {
  sum += A[j];
  sum += A[j+1];
  sum += A[j+2];
  sum += A[j+3];
}
for (int i = j; i < n; ++i) {
  sum += A[i];
}
```

## Benefits of loop unrolling

- Fewer instructions devoted to loop control.
- Enables more compiler optimizations.

**Caution:** Unrolling too much can cause poor use of the instruction cache, because the code is bigger.

# Hoisting

The goal of hoisting — also called loop-invariant code motion — is to avoid recomputing loop-invariant code each time through the body of a loop.

*Before*

```c
#include <math.h>

void scale(double *X, double *Y, int N) {
  for (int i = 0; i < N; i++) {
    Y[i] = X[i] * exp(sqrt(M_PI/2));
  }
}
```

*After*

```c
#include <math.h>

void scale(double *X, double *Y, int N) {
  double factor = exp(sqrt(M_PI/2));
  for (int i = 0; i < N; i++) {
    Y[i] = X[i] * factor;
  }
}
```

# Hoisting

The goal of hoisting — also called loop-invariant code motion — is to avoid recomputing loop-invariant code each time through the body of a loop.

*Before*

```
#include <math.h>

void scale(double *X, double *Y, int N) {
  for (int i = 0; i < N; i++) {
    Y[i] = X[i] * exp(sqrt(M_PI/N));
  }
}
```

# Sentinels

Sentinels are special dummy values placed in a data structure to simplify the logic of boundary conditions, and in particular, the handling of loop-exit tests.

```
#include <stdint.h>
#include <stdbool.h>

bool overflow(uint64_t *A, size_t n) {
  // All elements of A are nonnegative
  uint64_t sum = 0;
  for (size_t i = 0; i < n; ++i) {
    sum += A[i];
    if (sum < A[i]) return true;
  }
  return false;
}
```

# Sentinels

Sentinels are special dummy values placed in a data structure to simplify the logic of boundary conditions, and in particular, the handling of loop-exit tests.

*Before*

```c
#include <stdint.h>
#include <stdbool.h>

bool overflow(uint64_t *A, si
  // All elements of A are no
  uint64_t sum = 0;
  for (size_t i = 0; i < n; +
    sum += A[i];
    if (sum < A[i]) return tr
  }
  return false;
}
```

*After*

```c
#include <stdint.h>
#include <stdbool.h>

// Assumes that A[n] and A[n+1] exist and
// can be clobbered
bool overflow(uint64_t *A, size_t n) {
  // All elements of A are nonnegative
  A[n] = UINT64_MAX;
  A[n+1] = 1;    // or any positive number
  size_t i = 0;
  uint64_t sum = A[0];
  while (sum >= A[i]) {
    sum += A[++i];
  }
  return (i < n);
}
```

Sentinel

# Loop Fusion

The idea of loop fusion — also called jamming — is to combine multiple loops over the same index range into a single loop body, thereby saving the overhead of loop control.

*Before*

```
for (int i = 0; i < n; ++i) {
  C[i] = (A[i] <= B[i]) ? A[i] : B[i];
}

for (int i = 0; i < n; ++i) {
  D[i] = (A[i] <= B[i]) ? B[i] : A[i];
}
```

Ternary operator for `if-else`.

*After*

```
for (int i = 0; i < n; ++i) {
  C[i] = (A[i] <= B[i]) ? A[i] : B[i];
  D[i] = (A[i] <= B[i]) ? B[i] : A[i];
}
```

# Eliminating Wasted Iterations

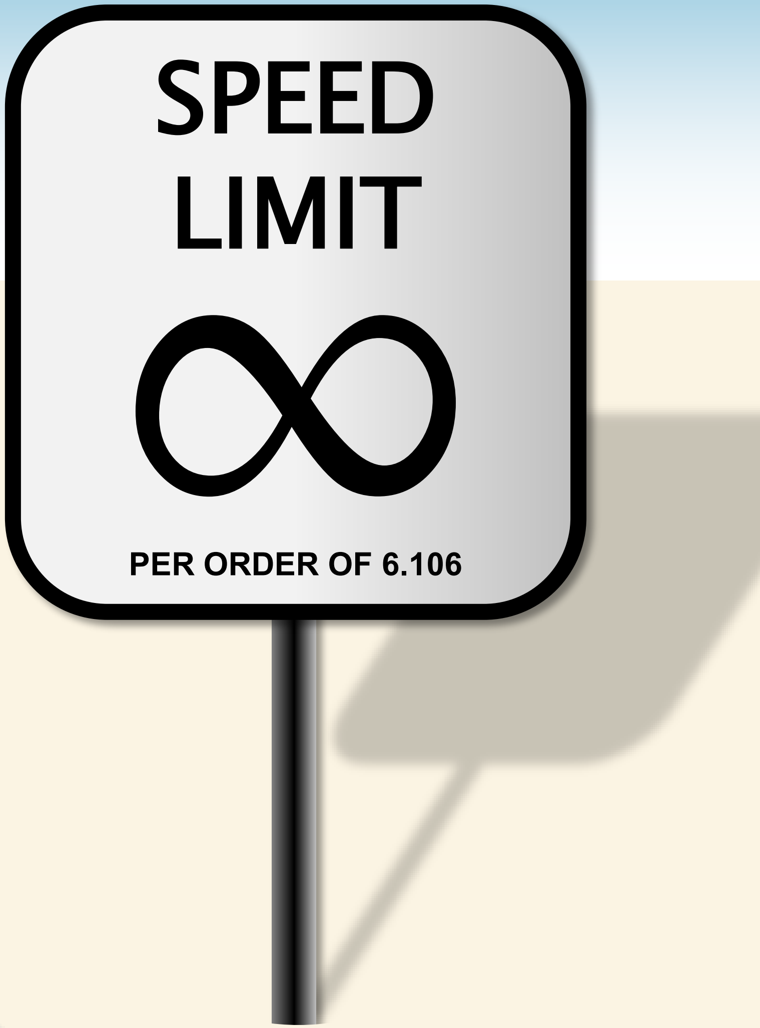
The idea of eliminating wasted iterations is to modify loop bounds to avoid executing loop iterations over essentially empty loop bodies.

*Before*

```
for (int i = 0; i < n; ++i) {
  for (int j = 0; j < n; ++j) {
    if (i > j) {
      int temp = A[i][j];
      A[i][j] = A[j][i];
      A[j][i] = temp;
    }
  }
}
```

*After*

```
for (int i = 1; i < n; ++i) {
  for (int j = 0; j < i; ++j) {
      int temp = A[i][j];
      A[i][j] = A[j][i];
      A[j][i] = temp;
  }
}
```




# Functions

# Inlining

The idea of inlining is to avoid the overhead of a function call by replacing a call to the function with the body of the function itself.

*Before*

```
double square(double x) {
  return x*x;
}

double sum_of_squares(double *A, int n) {
  double sum = 0.0;
  for (int i = 0; i < n; ++i) {
    sum += square(A[i]);
  }
  return sum;
}
```

*After*

```
double sum_of_squares(double *A, int n) {
  double sum = 0.0;
  for (int i = 0; i < n; ++i) {
    double temp = A[i];
    sum += temp*temp;
  }
  return sum;
}
```

# Inlining (2)

The idea of inlining is to avoid the overhead of a function call by replacing a call to the function with the body of the function itself.

Ask the compiler to inline for you.

```
inline double square(double x) {
  return x*x;
}

double sum_of_squares(double *A, int n) {
  double sum = 0.0;
  for (int i = 0; i < n; ++i)
    sum += square(A[i]);
  return sum;
}
```

Inlined functions can be just as efficient as macros, and they are safer to use and better structured.

# Tail-Recursion Elimination

Tail-recursion elimination removes the overhead of a recursive call that occurs as the last step of a function. The call is replaced with a branch to the top of the function, and the storage for the local variables of the function is reused by the erstwhile recursive call.

*Before*

```
void quicksort(int *A, int n) {
  if (n > 1) {
    int r = partition(A, n);
    quicksort (A, r);
    quicksort (A + r + 1, n - r - 1);
  }
}
```

*After*

```
void quicksort(int *A, int n) {
  while (n > 1) {
    int r = partition(A, n);
    quicksort (A, r);
    A += r + 1;
    n -= r + 1;
  }
}
```

# Coarsening Recursion

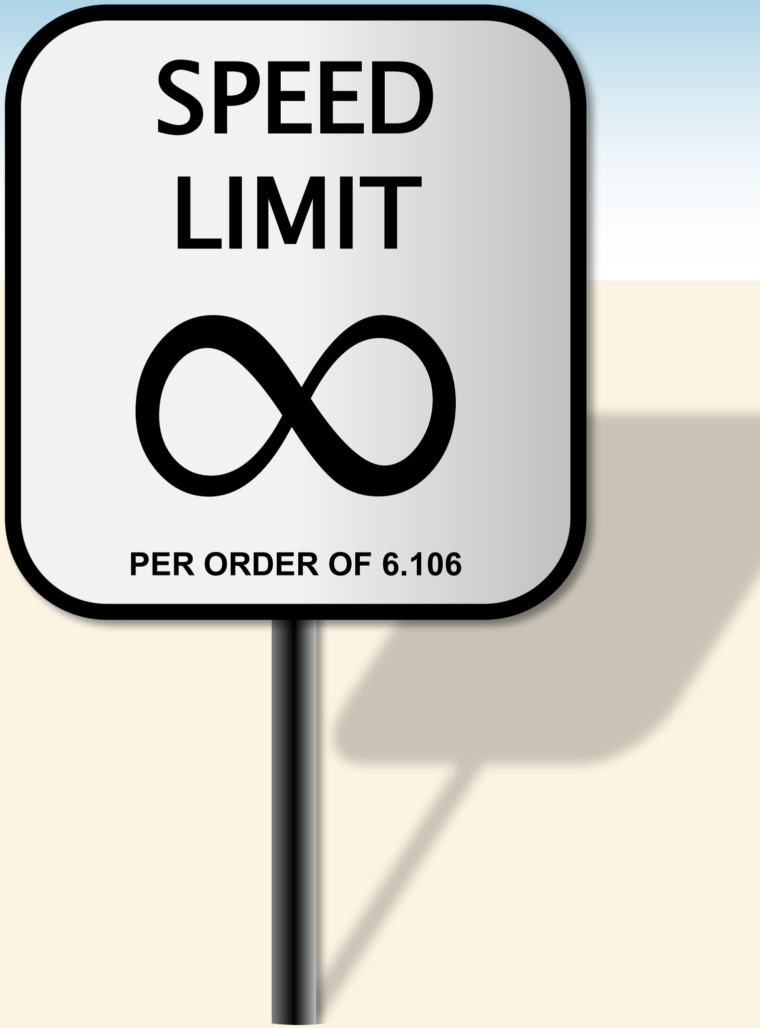
The idea of coarsening recursion is to increase the size of the base case and handle it with more efficient code that avoids function-call overhead.

*Before*

```
void quicksort(int *A, int n) {
  while (n > 1) {
    int r = partition(A, n);
    quicksort (A, r);
    A += r + 1;
    n -= r + 1;
  }
}
```

*After*

```
#define THRESHOLD 64
void quicksort(int *A, int n) {
  while (n > THRESHOLD) {
    int r = partition(A, n);
    quicksort (A, r);
    A += r + 1;
    n -= r + 1;
  }
  // insertion sort for small arrays
  for (int j = 1; j < n; ++j) {
    int key = A[j];
    int i = j - 1;
    while (i >= 0 && A[i] > key) {
      A[i+1] = A[i];
      --i;
    }
    A[i+1] = key;
  }
}
```




# SUMMARY

# New Bentley Rules

## Data structures
- Packing and encoding
- Augmentation
- Caching
- Precomputation
- Compile-time initialization
- Sparsity

## Loops
- Loop unrolling
- Hoisting
- Sentinels
- Loop fusion
- Eliminating wasted iterations

## Logic
- Constant folding and propagation
- Common-subexpression elimination
- Algebraic identities
- Creating a fast path
- Short-circuiting
- Ordering tests
- Combining tests

## Functions
- Inlining
- Tail-recursion elimination
- Coarsening recursion

# Closing Advice

- Avoid premature optimization. First, get correct working code. Then optimize, preserving correctness by regression testing.
- Reducing the work of a program does not necessarily decrease its running time, but it is a good heuristic.
- Many optimizations involve tradeoffs. Use a profiler to see what code needs to be optimized. (See Homework 2.)
- The compiler automates many low-level optimizations, but not all. We will see how to look at the compiler output in upcoming lectures.

If you find interesting examples of work optimization, please let us know!