Balanced Dense Polynomial Multiplication on Multicores

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Overview

In symbolic computation, polynomial multiplication is a fundamental operation akin to matrix multiplication in numerical computation. We present efficient implementation strategies for FFT-based dense polynomial multiplication targeting multi-cores. We show that balanced input data can maximize parallel speed-up and minimize cache complexity for bivariate multiplication. However, unbalanced input data, which are common in symbolic computation, are challenging. We provide efficient techniques that we call *contraction* and *extension* to reduce multivariate (and univariate) multiplication to balanced bivariate multiplication. Our implementation in `Cilk++` demonstrates good speed-up on multi-cores.

FFT-based Multivariate Multiplication

Let $\mathbb{K}$ be a field and $f, g \in \mathbb{K}[x_1 < \cdots < x_n]$ be polynomials. Define $d_i = \deg(f, x_i)$ and $d'_i = \deg(g, x_i)$, for all i. Assume there exists a primitive s_i-th root $\omega_i \in \mathbb{K}$, for all i, where s_i is a power of 2 satisfying $s_i \geq d_i + d'_i + 1$. Then fg can be computed as follows.

***Step* 1.** Evaluate f and g at each point of the n-dimensional grid $((\omega_1^{e_1}, \ldots, \omega_n^{e_n}), 0 \leq e_1 < s_1, \ldots, 0 \leq e_n < s_n)$ via n-D FFT.

***Step* 2.** Evaluate fg at each point P of the grid, simply by computing $f(P)g(P)$,

***Step* 3.** Interpolate fg (from its values on the grid) via n-D FFT.

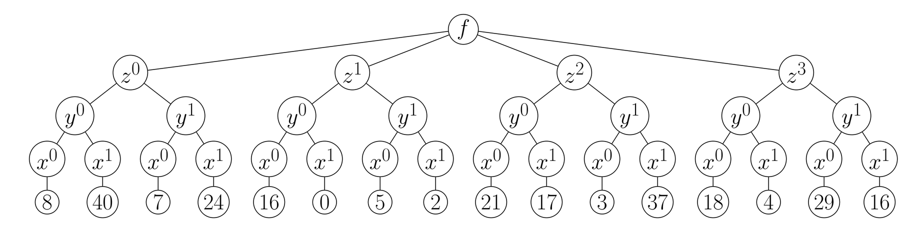
Complexity Estimates

- Let $s = s_1 \cdots s_n$. The number of operations in $\mathbb{K}$ for computing fg based on FFTs is
$$\frac{9}{2}\sum_{i=1}^{n}(\prod_{j\neq i} s_j) s_i \lg(s_i) + (n+1)s = \frac{9}{2} s \lg(s) + (n+1)s.$$
- Under our serial 1-D FFT assumption, the span of *Step* 1 is $\frac{9}{2}(s_1\lg(s_1) + \cdots + s_n \lg(s_n))$, and the parallelism of *Step* 1 is lower bounded by
$$s/\max(s_1, \ldots, s_n). \qquad (1)$$
- Let L be the size of a cache line. For some constant $c > 0$, the number of cache misses of *Step* 1 is upper bounded by
$$n\frac{cs}{L} + cs(\frac{1}{s_1} + \cdots + \frac{1}{s_n}). \qquad (2)$$

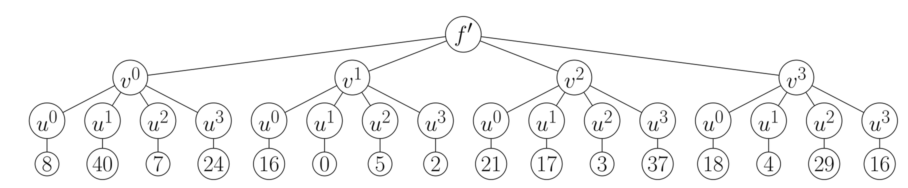
- **Remark**: For $n \geq 2$, Expr. (2) is minimized at $n = 2$ and $s_1 = s_2 = \sqrt{s}$. Moreover, when $n = 2$, under a fixed $s = s_1 s_2$, Expr. (1) is maximized at $s_1 = s_2 = \sqrt{s}$.

Contraction to Bivariate

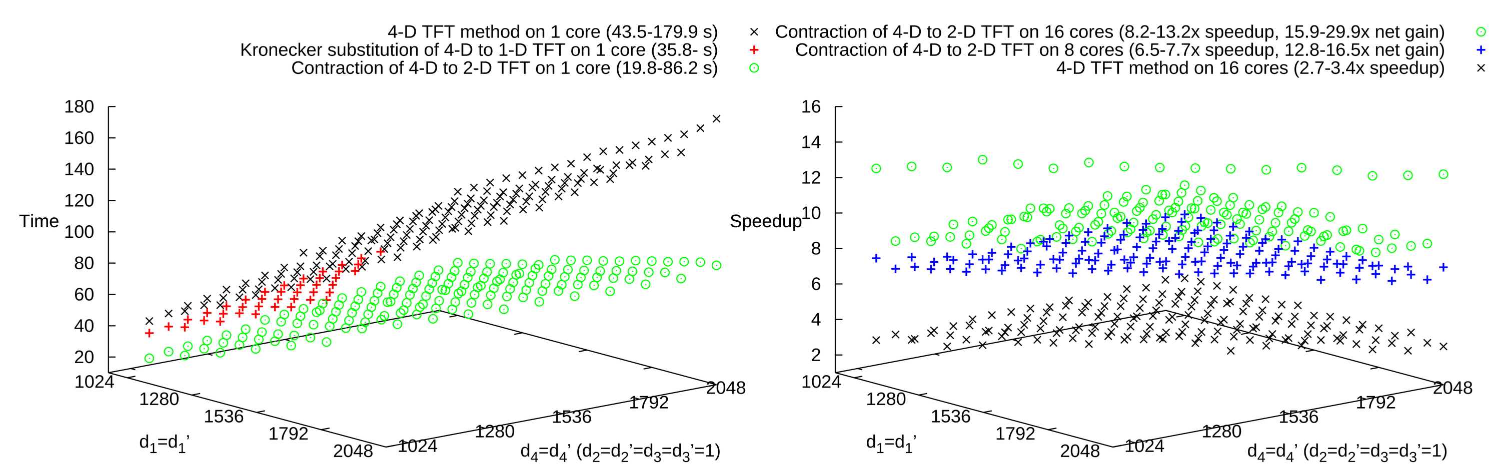
- **Example**. Let $f \in \mathbb{K}[x, y, z]$ where $\mathbb{K} = \mathbb{Z}/41\mathbb{Z}$, with $\deg(f, x) = \deg(f, y) = 1$, $\deg(f, z) = 3$ and recursive dense representation:



Contracting $f(x, y, z)$ to $f'(u, v)$ by $x^{e_1}y^{e_2} \mapsto u^{e_1+2e_2}$, $z^{e_3} \mapsto v^{e_3}$:

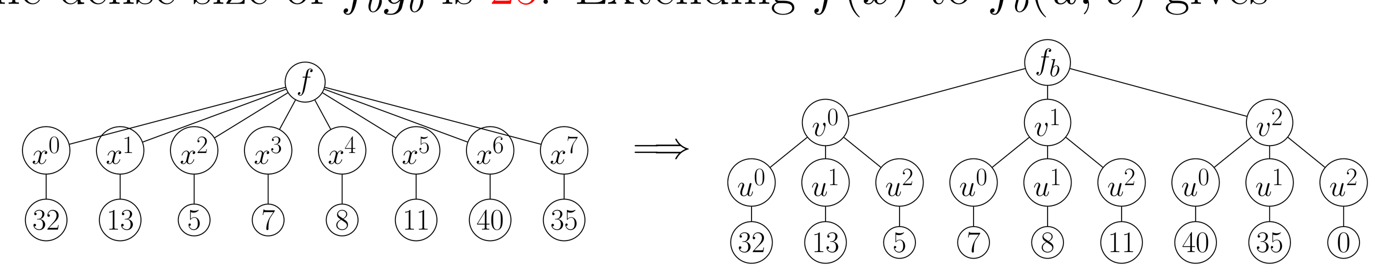


- **Remark**. The data is "essentially" unchanged by contraction, which is a property of recursive dense representation.
- Below, the left figure displays the timing of 4-variate multiplication via 4-D TFT, 1-D TFT by Kronecker substitution and contraction to balanced 2-D TFT on 1 core; The right figure shows the speedups of 4-variate multiplication using 4-D TFT and contraction to balanced 2-D TFT on 8 and 16 cores.

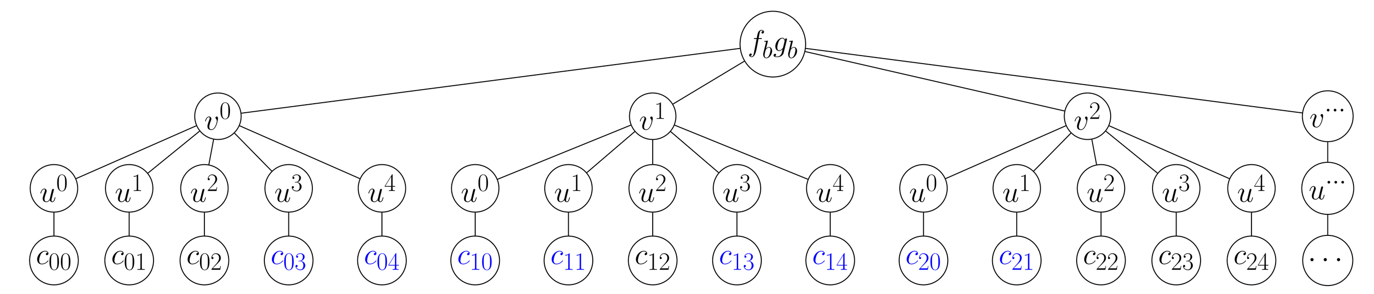


Extension from Univariate to Bivariate

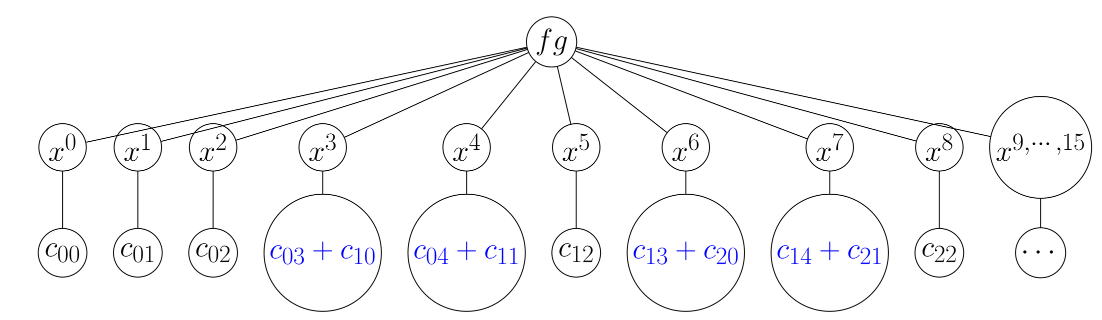
- **Example**: Consider $f, g \in \mathbb{K}[x]$ univariate, with $\deg(f) = 7$ and $\deg(g) = 8$; fg has "dense size" 16. We obtain an integer b, such that fg can be performed via $f_b g_b$ using "nearly square" 2-D FFTs, where $f_b := \Phi_b(f)$, $g_b := \Phi_b(g)$ and
$$\Phi_b : \ x^e \longmapsto u^{e \text{ rem } b}\, v^{e \text{ quo } b}.$$
Here $b = 3$ works since $\deg(f_b g_b, u) = \deg(f_b g_b, v) = 4$; moreover the dense size of $f_b g_b$ is 25. Extending $f(x)$ to $f_b(u, v)$ gives



- **Proposition**: For any non-constant $f, g \in \mathbb{K}[x]$, one can always compute b such that $|deg(f_b g_b, u) - deg(f_b g_b, v)| \leq 2$ and the dense size of $f_b g_b$ is at most twice that of fg.
- **Example (ctnd)**: Computing the bivariate product $f_b g_b$:



Converting back to fg from $f_b g_b$ requires only to traverse the coefficient array once and perform at most $\deg(fg, x)$ additions.



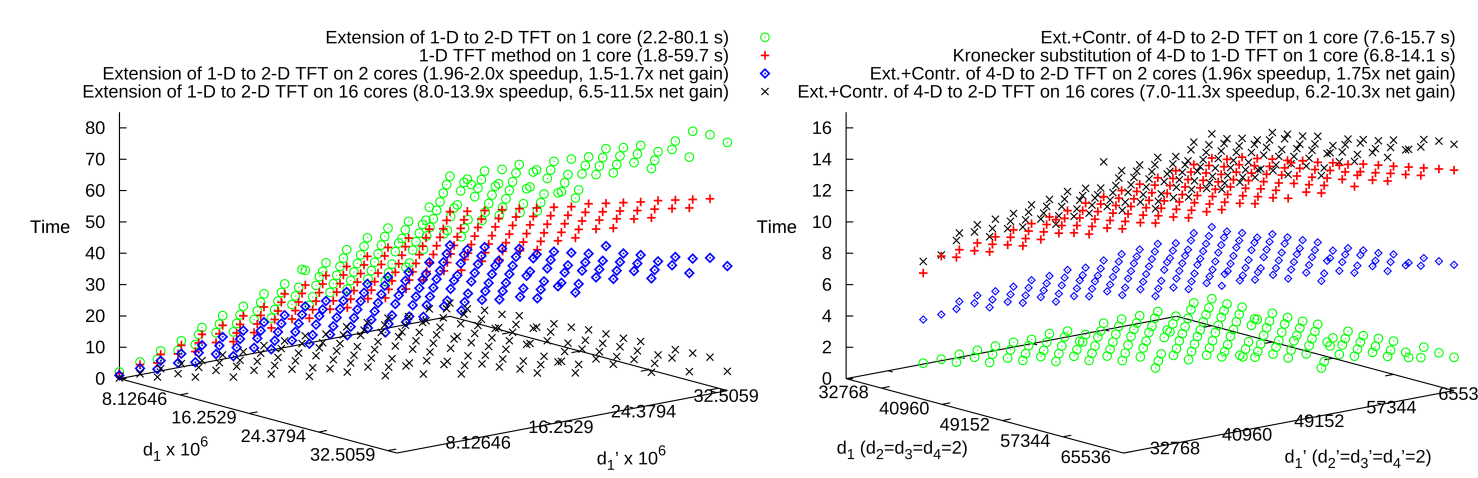
Balanced Multiplication

- **Definition**. A pair of bivariate polynomials $p, q \in \mathbb{K}[u, v]$ is *balanced* if $\deg(p, u) + \deg(q, u) = \deg(p, v) + \deg(q, v)$.
- **Algorithm**. Let $f, g \in \mathbb{K}[x_1 < \ldots < x_n]$. W.l.o.g. one can assume $d_1 >> d_i$ and $d_1{}' >> d_i$ for $2 \leq i \leq n$ (up to variable re-ordering and contraction). We obtain fg by

  ***Step* 1.** Extending x_1 to $\{u, v\}$.

  ***Step* 2.** Contracting $\{v, x_2, \ldots, x_n\}$ to v.

  Determine the above extension Φ_b such that f_b, g_b is (nearly) a balanced pair and $f_b g_b$ has dense size at most twice that of fg.
- The left figure shows the timing of univariate multiplication via 1-D TFT and extension to balanced 2-D TFT on 1, 2, 16 cores; The right one shows the timing of our balanced multiplication for an unbalanced 4-variate case on 1, 2, 16 cores vs the method based on 1-D TFT via Kronecker substitution.



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