On Estimating the Size and Confidence of a Statistical Audit

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$$\boxed{u = \left\lceil \left(n - \frac{b-1}{2}\right) \cdot \left(1 - (1-c)^{1/b}\right) \right\rceil}$$

Outline

- Motivation
- Background
  - How Do We Audit?
  - The Problem
- Analysis
  - Model
  - Sample Size
  - Bounds
- Conclusions

Motivation

- There have been cases of electoral fraud

(Gumbel's *Steal This Vote*, Nation Books, 2005)

- Would like to ensure confidence in elections
- Auditing = comparing statistical sample of paper ballots to electronic tally
- Provides confidence in a software independent manner

How Do We Audit?

- Proposed Legislation: *Holt Bill (2007)*
  - Voter-verified paper ballots
  - Manual auditing
- Granularity: Machine, Precinct, County
- Procedure
  - Determine *u*, # precincts to audit, from margin of victory
  - Sample *u* precincts randomly
  - Compare hand count of paper ballots to electronic tally in sampled precincts
    - If all are sufficiently close, declare electronic result final
    - If any are significantly different, investigate!

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➡ Our formulas are independent of the auditing procedure

The Problem

- How **many** precincts should one audit to ensure high confidence in an election result?

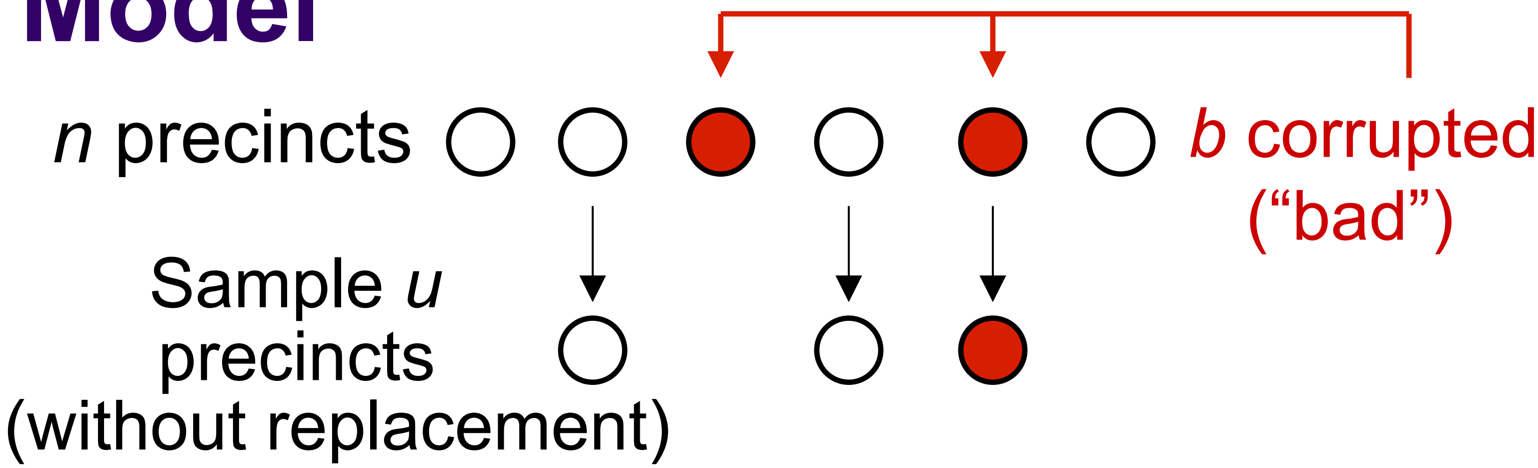
Previous Work

- *Saltman (1975):* The first to study auditing by sampling without replacement
- *Dopp and Stenger (2006):* Choosing appropriate audit sizes
- *Alvarez et al. (2005):* Study of real case auditing of punch-card machines

Hypothesis Testing

- Null hypothesis: The reported election outcome is **incorrect** (electronic tally indicates different winner than paper ballots)
- Want to reject the null hypothesis
  - Need to sample enough precincts to ensure that, if no fraud is detected, the election outcome is correct with high confidence

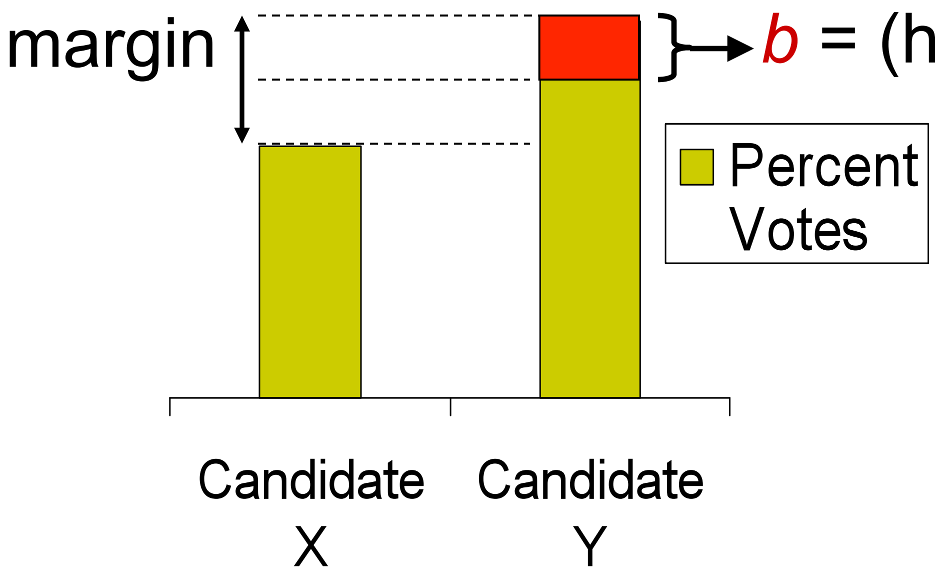
Model



- c = desired confidence
- Want: If there are ≥ b corrupted precincts, then sample contains at least one with probability ≥ c
- **Equivalently:** If the sample contains no corrupted precincts, then the election outcome is correct with probability ≥ c
- Typical values: n = 400, b = 50, c = 95%

What is *b*?

- Minimum # of precincts adversary must corrupt to change election outcome
- Derived from margin of victory



b = (half margin of victory) · *n*

[times 5 (Dopp and Stenger, 2006)]

- Our formulas are independent of *b*'s calculation

Rule of Three

- If we draw a sample of size ≥ $3n/b$ *with replacement*, then:
  - Expect to see at least **three corrupted** precincts
  - Will see at least **one corrupted** precinct with $c \geq 95\%$
- In practice, we sample *without replacement* (no repeated precincts)

Sample Size

- Probability that no corrupted precinct is detected:

$$Pr = \binom{n-b}{u} / \binom{n}{u}$$

- Optimal Sample Size: Minimum u such that $Pr \leq 1 - c$

⇨ Problem: Need a computer

⇨

- Goal: Derive a simple and accurate upper bound that an election official can compute on a hand-held calculator

Our Bounds

- Intuition: How many different precincts are sampled by the Rule of Three?
- Our *without replacement* upper bounds:

A
C
C
U
R
A
C
Y

$$u = \lceil n \cdot (1 - (1-c)^{1/b}) \rceil$$

$$\boxed{u = \left\lceil \left(n - \frac{b-1}{2}\right) \cdot (1 - (1-c)^{1/b}) \right\rceil}$$

$$u = \left\lceil \frac{b}{H_n - H_{n-b}} \cdot (1 - (1-c)^{1/b}) \right\rceil$$

Our Bounds

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- Our *without replacement* upper bounds:

$$u = \lceil n \cdot (1 - (1 - c)^{1/b}) \rceil$$

$$\boxed{u = \left\lceil \left(n - \frac{b-1}{2}\right) \cdot (1 - (1 - c)^{1/b}) \right\rceil}$$

- Example: n = 400, b = 50 (margin=5%), c = 95%

$$u = \left\lceil \left(400 - \frac{50-1}{2}\right) \cdot (1 - (1 - 0.95)^{1/50}) \right\rceil = \lceil 21.84 \rceil = 22$$

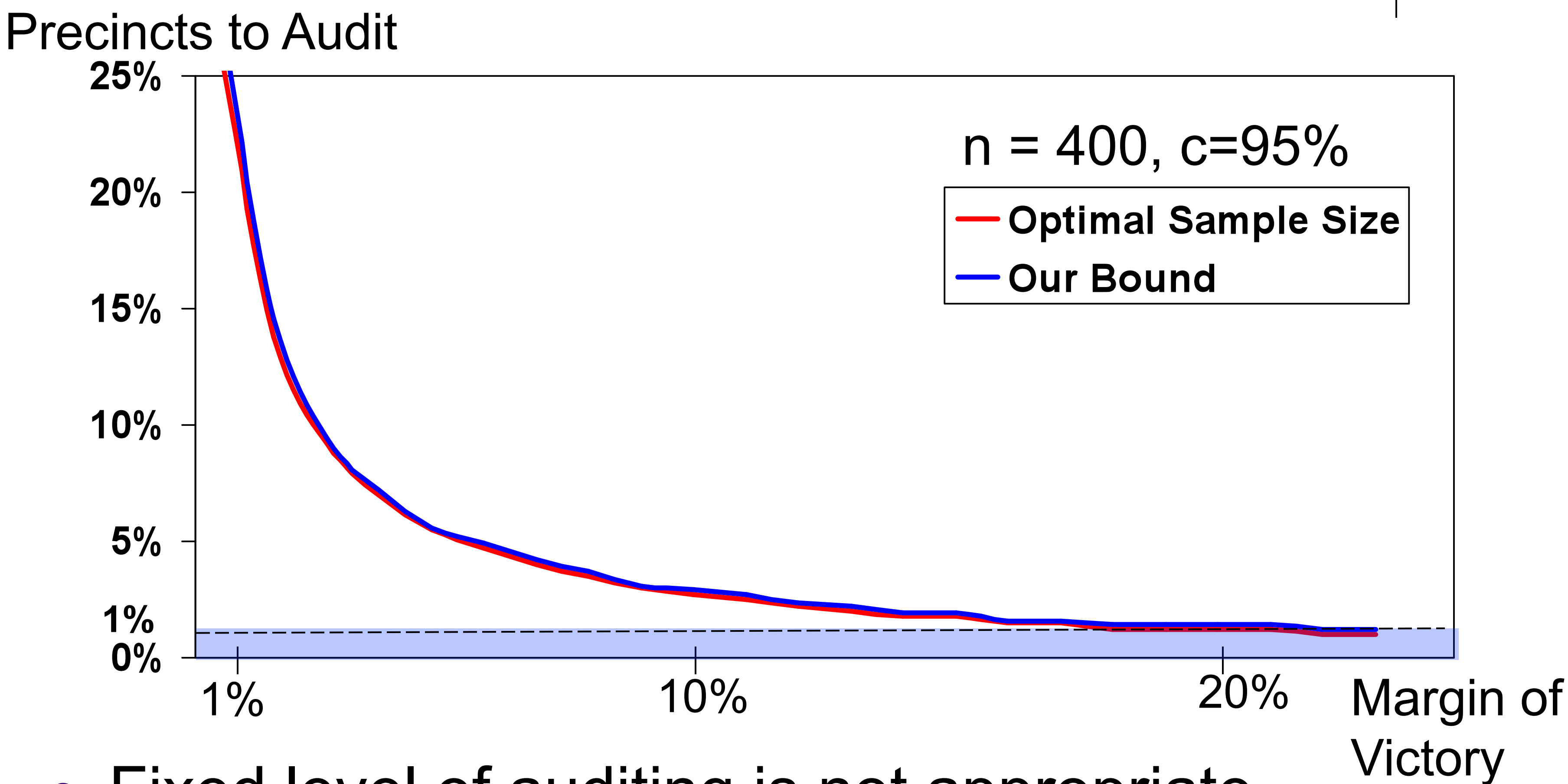
Our Bound

$$u = \left\lceil \left(n - \frac{b-1}{2}\right) \cdot \left(1 - (1-c)^{1/b}\right) \right\rceil$$

$$(1-c)^{1/b} = e^{\ln(1-c)/b}$$

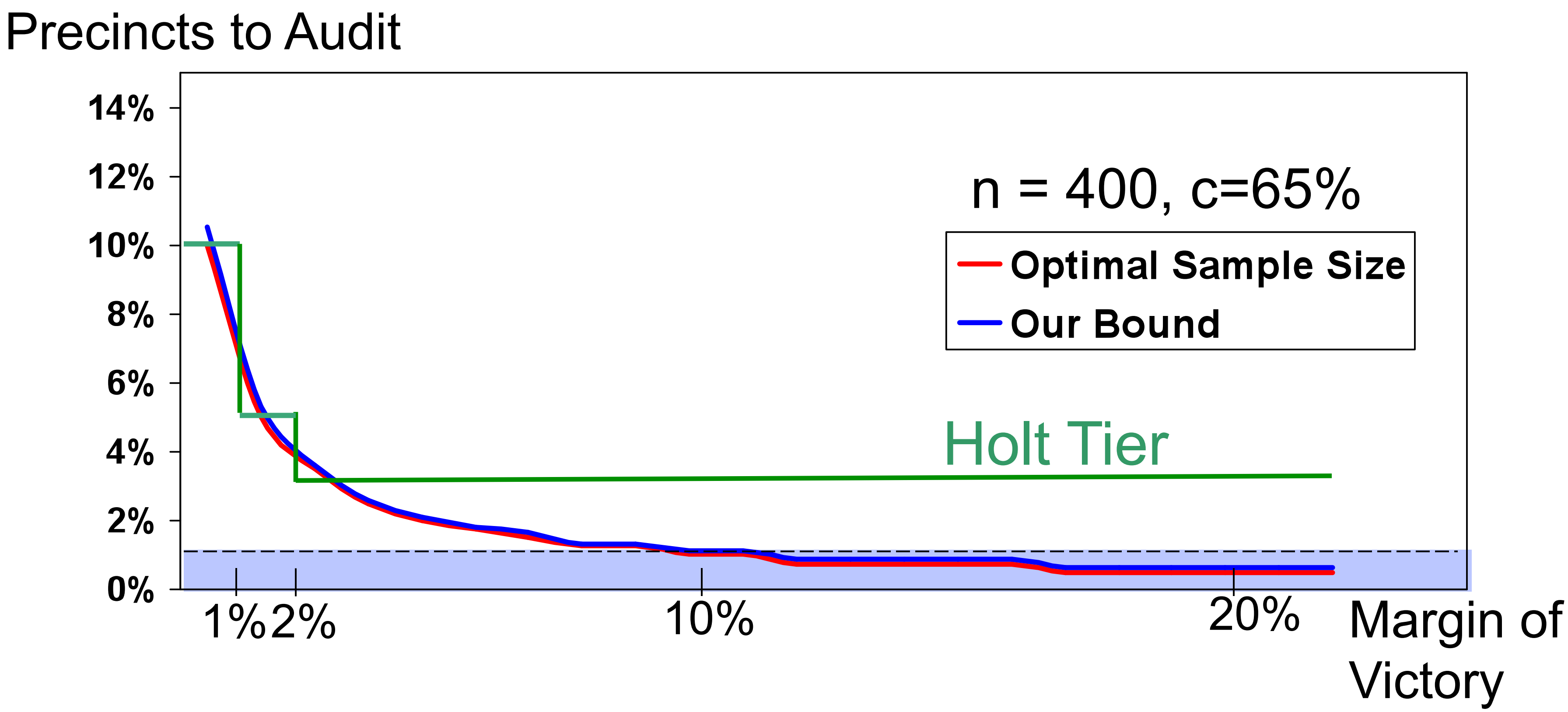
- Conservative: provably an upper bound
- Accurate:
  - For $n \leq 10{,}000$, $b \leq n/2$, $c \leq 0.99$ (steps of 0.01):
    - 99% is exact, 1% overestimates by 1 precinct
  - Analytically, it overestimates by at most $-\ln(1-c)/2$, e.g. three precincts for $c < 0.9975$
- Can be computed on a hand-held calculator

Observations



- Fixed level of auditing is not appropriate

Observations (cont'd)



- *Holt Bill (2007):* Tiered auditing

Related Problems

- Inverse questions
  - Estimate confidence level *c* from *u*, *b*, and *n*
  - Estimate detectable fraud level *b* from *u*, *c*, and *n*
- Auditing with constraints
  - *Holt Bill (2007):* Audit at least one precinct in each county
- Future work
  - Handling precincts of variable sizes (*Stanislevic, 2006*)

Conclusions

- We develop a formula for the sample size:

$$\boxed{u = \left\lceil \left(n - \frac{b-1}{2}\right) \cdot \left(1 - (1-c)^{1/b}\right) \right\rceil}$$

that is:

- Conservative (an upper bound)
- Accurate
- Simple, easy to compute on a pocket calculator
- Applicable to different other settings

Thank you!

- Questions?