New Lower Bounds for Channel Width

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ABSTRACT

We present here a simple yet effective technique for calculating a lower bound on the number of tracks required to solve a given channel-routing problem. The bound applies to the wiring model where horizontal wires run on one layer and vertical wires run on another layer. One of the major results is that at least $\sqrt{2n}$ tracks are necessary for <u>any</u> dense channel routing problem with n two-terminal nets that begin and end in different columns. For <u>example</u>, if each net i begins in column i and ends in column i+1, at least $\sqrt{2n}$ tracks are required, even though the channel "density" is only 2. This is the first technique which can give results which are significantly better than the naive channel density arguments. A modification results in the calculation of an improved bound, which we conjecture to be optimal to within a constant factor.

I. INTRODUCTION

The "channel-routing" problem has recently attracted a great amount of interest and is becoming increasingly important with the advent of VLSI. The results of this paper are of both practical and theoretical interest. On the practical side, the techniques allow a channel-routing algorithm to estimate more accurately a bound on the number of tracks required to solve a given problem, and thus to know when to stop looking for an impossibly good solution. From a theoretical point of view, this paper makes two points. The first is that channel "density" is not the only factor determining the limits of channel-routing performance in this wiring model; we must also consider how many nets must "switch columns" in order to be routed. The second point is closely related: the "traditional" wiring model - which we study here - seems to be in some significant sense provably worse than related wiring models where nets can overlap slightly (say at corners). In these models twice channel density is provably an upper bound on the number of tracks required [RBM81].

Related work has been done by, among others [HS71], [D76], [T80], and [DKSSU81].

II. DEFINITIONS AND THE WIRING MODEL

The (infinite) channel of width t consists of (1) the set V of grid points (x,y) such that x and y are integers and $0 \leq y \leq t+1$, $-\infty < x < \infty$, and (2) the set E of edges connecting points (x,y) and (x', y') whenever these points are at distance 1 from each other and y and y' are not both equal to 0 or t+1. Figure 1 shows a channel of width 4. If the width of the channel is t, we say that the channel has t <u>tracks</u>; track i (for $1 \leq i \leq t$) consists of all grid points with y=i and the (horizontal) edges connecting these points.

A (two-terminal) net N_i consists of a pair of integers (p_i,q_i). The intent is that a net specifies that a connection must be made between the point $(q_i,t+1)$ and the point $(p_i,0)$; these points are the terminals of the net.

A connection is made by a wire; a wire is defined to be a simple path $(v_0,\ell_0,v_1,\ell_1,\ldots,v_{k-1},\ell_{k-1},v_k)$ connecting v_0 to v_k. (Here $\ell_{i-1} \in E$ is the edge connecting grid points v_{i-1} and v_i.) A channel-routing problem is defined to be a set of nets (with $p_i \neq p_j$ and $q_i \neq q_j$ for $i \neq j$). A solution to a channel-routing problem is an integer t and a set of wires in the channel of width t, such that one wire connects the terminals of each net, and satisfying the restriction that two distinct wires can meet at a grid point only if one wire has only vertical edges touching that grid point and the other wire has only horizontal edges touching that grid point. (This grid point is then a crossover point.) This corresponds to the traditional model using one layer for horizontal wires and another for vertical wires.

Given a channel-routing problem, it is desired to find the least t permitting a solution. Szymanski [S81] has proved that this minimization problem is NP-hard if each net may require connection of an arbitrary number of terminals; it is natural to conjecture (but as yet unproven) that it is also NP-hard if each net connects only two terminals (as in our case).

An obvious lower bound on the minimum achievable channel width is the channel density: this is the maximum (over x) of the number of nets $N_i = (p_i,q_i)$ for which $p_i \leq x < q_i$ or $q_i \leq x < p_i$; i.e., the maximum number of nets whose wires must cross or touch some vertical line x in order to make the necessary connection. Previous to this paper, no better lower bound has been published.

III. A SIMPLE LOWER BOUND

We begin with an investigation of the simple "shift-right-one" channel-routing problem with n nets. Here the top terminal of net i is in column $q_i=i$ and the bottom terminal of net i is in column $p_i = i+1$ for $i=1,\ldots,n$. We can show that at least $\sqrt{2n}$ tracks are required for this problem, even though the channel density is only 2. In fact, our simple argument does not depend upon the structure of the problem except that it contains n closely-packed nets with different starting and ending columns. Thus, an essentially same argument can be applied to any channel-routing problem with two-terminal nets.

Suppose that the leftmost terminal of any net occurs in column 1 (i.e. with p_i or q_i equal to 1), and that the rightmost terminal of any net occurs in column w. We consider the "window" of columns 1 to w, where obviously $w \geq n$. While wires do not need to lie entirely within the window, every wire must both start and end within the window. For our shift-right-one problem we have a window of size $w = n+1$.

Let m denote the number of nets which must be "moved" (i.e. which must switch columns because $p_i \neq q_i$). The structure of our argument is a track-by-track analysis of how many wires can be moved into their final columns on each track. Consider the first track (i.e. y=1). If below track 1 (i.e. connecting track 0 to track 1) we have $m_0 = m$ nets which must be moved, after track 1 (i.e. between tracks 1 and 2) we will have a number m_1 of nets to be moved, where $m_1 \leq m_0$. We continue in this manner for each track; when $m_i = 0$ we are done (with $t = i$).

How many nets can be moved into their target columns in one track? The fundamental but simple observation is that if net i moves from its current column to its target column q_i on the track, then column q_i must have been empty, (i.e. there were no wires in column q_i between this track and the previous one). Let e_i denote the number of empty columns between tracks i and i+1 in our window. Then clearly

$$m_i - m_{i+1} \leq e_i \quad \text{or} \quad m \leq \sum_{i=0}^{t-1} e_i \ .$$

The only way to change e_i from one track to the next is to route wires from a column inside the window to a column outside the window (which increases e_i by one) or vice versa (which decreases e_i by one). We also observe that $e_i - 2 \leq e_{i+1} \leq e_i + 2$, since at most two wires can cross the window boundary on any track.

Our initial conditions are $e_0 = e_t = w - n$ (w is the width of the window, n the number of nets), and we have the inequality

$$e_i \leq \min\{e_0 + 2i,\ e_t + 2(t-i)\} \ .$$

This implies that, for $t \geq 3$:

$$m \leq \sum_{i=0}^{\lfloor t/2 \rfloor} (e_0 + 2i) + \sum_{i=\lfloor t/2 \rfloor+1}^{t-1} (e_t + 2(t-i))$$

and so

$$m \leq t(w-n) + 2\sum_{i=0}^{\lfloor t/2 \rfloor} i + 2\sum_{i=1}^{\lfloor (t-1)/2 \rfloor} i$$

$$m \leq t(w-n) + \frac{t^2}{2}$$

$$\underline{t \geq -(w-n) + \left\lceil \sqrt{(w-n)^2 + 2m} \right\rceil} \ . \quad (*)$$

Thus, in our shift-right-one example we have $w-n = 1$ and $m = n$, yielding:

$$t \geq -1 + \left\lceil \sqrt{2n+1} \right\rceil \ .$$

Figure 2 illustrates a routing for this problem with $n = 13$; the lower bound of $t \geq -1 + \lceil \sqrt{27} \rceil = 5$ is achieved. It is not true, however, that the shift-right-one example can always achieve $-1 + \lceil \sqrt{2n+1} \rceil$ tracks. For instance, for $n = 12$, shift-right-one cannot be implemented using fewer than five tracks.

Notice that the argument outlined above does not apply solely to the shift-right-one example, and the bound (*) applies to all two-terminal channel-routing problems. It is in fact possible to show that the bound (*) is tight in the sense that, for any values of w-n and m, there is some channel-routing problem which achieves the minimum number of tracks given by (*). Figure 3 illustrates a particular routing problem with $m = 12$, $w - n = 1$ for which the lower bound of $t = 4$ can actually be achieved (even though it cannot be for the shift-right-one).

IV. AN IMPROVED LOWER BOUND

Let us examine the channel-routing problem specified by Figure 4a. The density is three, and since $w-n = 4$ and $m = 16$, (*) tells us that at least three tracks are required for a routing. But consider just the left side of this problem, using a window of size eight: $w-n = 0$, $m = 8$. The bound (*) guarantees that at least four tracks are required for this subproblem, and so it is certainly not possible to achieve a routing with fewer than four tracks for the entire problem. Therefore the routing illustrated in Figure 4a is indeed optimal. What is the problem with our (*) bound? The difficulty is that the bound nowhere takes into account the details of the particular routing or the locations of the initially empty columns. As shown by Figure 4b, there is <u>some</u> channel-routing problem with $w-n = 4$ and $m = 16$ which can be routed using only three tracks. But the empty columns are more spread out, so no subproblem is as "dense" as in Figure 4a.

We modify our Section III argument to consider not only the largest window but also all $O(n^2)$ "subwindows" it contains. A bound like (*) is computed for each subwindow, and the overall lower bound is then the maximum of the individual lower bounds.

The bound on a subwindow involves computing a solution to a quadratic formula as we did above. The formula is, however, more complicated because some nets may have only one terminal in the subwindow and some nets may have both terminals on opposite sides of (and outside of) the subwindow.

Consider a (sub)window of width w. Let D denote the number of nets whose top terminal is within the window and whose bottom terminal is outside of the window. Such a "departing" net may have its bottom terminal either to the "left" or to the "right" of the window; there are D_L and D_R of these, respectively ($D = D_L + D_R$). Similarly, let A denote the number of "arriving" nets, those with top terminal outside and bottom terminal inside the window; there are A_L (A_R) of these with top terminal to the left (right) of the window. There are T nets which have their terminals on opposite sides of the window and so must pass all the way "through". Finally, there are I nets which have both their terminals "inside" the window. (Note that $w = D + I + e_t = A + I + e_0$.)

In this extended abstract, we omit the (somewhat complicated) derivation of the modified bound for an arbitrary subwindow and consider only subwindows for which $D_L = D_R$ and $A_L = A_R$.

Clearly $\frac{D}{2}$ tracks are required for the D "departing" nets. But within these $\frac{D}{2}$ tracks, as many as

$$\sum_{i=0}^{\frac{D}{2}-1} (e_t + 2i)$$

"inside" nets might also be routed. Similarly, the A "arriving" nets require at least $\frac{A}{2}$ tracks, which could also be used to route

$$\sum_{i=0}^{\frac{A}{2}-1} (e_0 + 2i)$$

"inside" nets. This leaves $\max\{0, I'\}$, where

$$I' = I - \sum_{i=0}^{\frac{D}{2}-1} (e_t + 2i) - \sum_{i=0}^{\frac{A}{2}-1} (e_0 + 2i),$$

more "inside" nets to be routed. Bound (*), previously established, gives a minimum number of additional tracks required to route these. Recalling that T nets pass completely through the window, we obtain

$$t \geq T + \frac{D}{2} + \frac{A}{2} + \max\{0, -(e_t + D) + \sqrt{(e_t + D)^2 + 2I'}\}.$$

This formula is illustrated by the example in Figure 5 (where only the left half has been drawn; the right half is the mirror image of the left). For this problem, $T = 0$, $D = 2$, $A = 6$, $I = 42$, $e_0 = 0$, $e_t = 4$, and the above formula gives

$$t \geq 1 + 3 + \max\{0, -6 + \sqrt{36 + 64}\} = 8.$$

This minimal number of tracks is in fact achieved by the routing shown.

The above formula can, of course, be extended to subwindows where $A_L \neq A_R$ and $D_L \neq D_R$. In addition, small improvements can easily be made by considering relative positions of, say, the D nets and the e_t columns.

Finally, it should be noted that channel density is, in fact, a subcase of what we are here computing. If the (maximum) density d is in column i, then the subwindow of size one which includes i will require at least d tracks. So the maximum over all subwindows can be no less than d.

V. CONCLUSIONS

We have presented a new simple but powerful technique for deriving a lower bound on the number of tracks required to solve a traditional channel-routing problem for two-terminal nets. We have as of yet found no example for which this bound is more than a constant factor from optimal.

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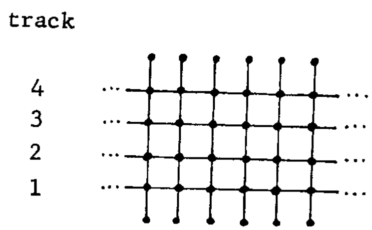


Figure 1. (Infinite) channel of width 4.

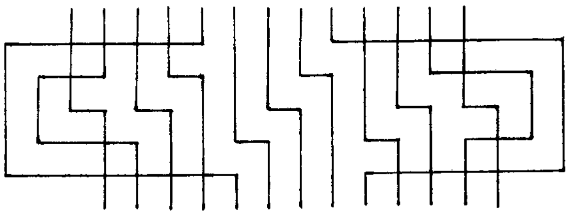


Figure 2. Shift-right-one example for n = 13.

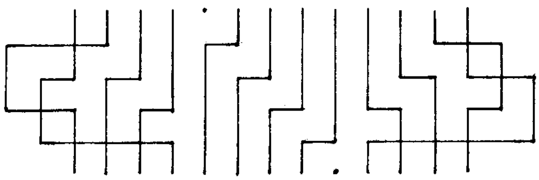
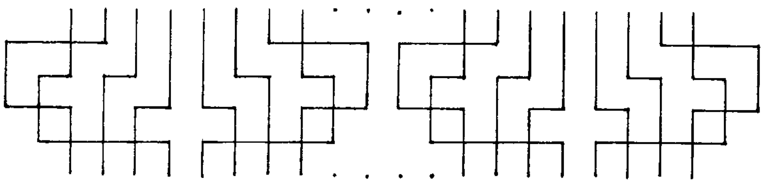
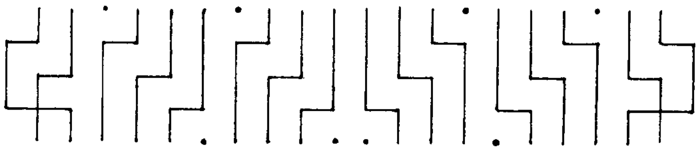


Figure 3. Example achieving bound (*) for w-n = 1, m = 12.



(a) Optimal but does not achieve bound (*).



(b) Does achieve bound (*).

Figure 4. Examples for $w - n = 4$, $m = 16$.

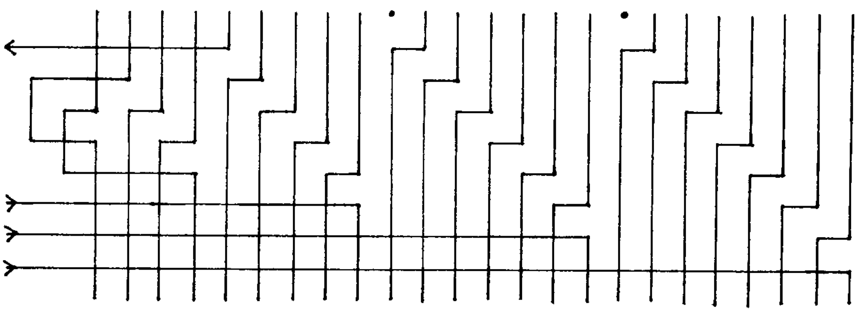


Figure 5. Illustration for improved lower bound.