6.045

Lecture 13:
Time Complexity

One more cool Computability Result

Define RE := {L | L is recognizable}

Problems like A_{TM} are in RE but not decidable

Thm: [Ji, Natarajan, Vidick, Wright, Yuen, January'20]
Every language in RE can be *decided* by an efficient verifier interacting with two all-powerful provers sharing quantum entanglement! "MIP* = RE"

You can be quickly convinced that an arbitrary program halts on an arbitrary input, using two all-powerful computers whose storage is *quantum entangled*!

Computational Complexity Theory

Computational Complexity Theory

What can and can't be computed with limited resources on computation, such as time, space, and so on

Captures many of the significant issues in practical problem solving

The field is rich with important open questions that no one has any idea how to begin answering!

We'll start with: Time complexity

Very Quick Review of Big-O

Let $f, g : \mathbb{N} \to \mathbb{N}$.
We say that $f(n) \leq O(g(n))$ if there are $c, n_0 \in \mathbb{N}$ so that for every integer $n \geq n_0$

$$f(n) \leq c\, g(n)$$

We say **g(n) is an upper bound on f(n)** if
$f(n) \leq O(g(n))$

$$5n^3 + 2n^2 + 22n + 6 \leq O(n^3)$$

Ex: If $c = 6$ and $n_0 = 10$, then $5n^3 + 2n^2 + 22n + 6 \leq cn^3$

$$2n^{4.1} + 200283n^4 + 2 \quad \leq O(n^{4.1})$$

$$3n \log_2 n + 5n \log_2 \log_2 n \quad \leq O(n \log_2 n)$$

$$n \log_{10} n^{78} \quad \leq O(n \log_{10} n)$$

$$\log_{10} n = \log_2 n \,/\, \log_2 10$$

$$O(n \log_2 n) \leq O(n \log_{10} n) \leq O(n \log n)$$

Big-O isolates the “dominant” term of a function

A Simpler Big-O Definition

Let $f, g : \mathbb{N} \to \mathbb{N}$ We say $f(n) \leq O(g(n))$ if there is a $c \in \mathbb{N}$ so that

for all $n \in \mathbb{N}$, $f(n) \leq c\, g(n) + c$

Exercise: Show this definition is equivalent to the other one!

Measuring Time Complexity of a TM

We measure time complexity by counting the steps taken for a Turing machine to halt on an input

Example: Let $A = \{ 0^k 1^k \mid k \geq 0 \}$

Here's a TM for A. On input x of length n:

O(n)	1. Scan across the tape and **reject** if x is not of the form $0^a 1^b$
O(n^2)	2. Repeat the following if both 0s and 1s remain on the tape: Scan across the tape, crossing off a single 0 and a single 1
O(n)	3. If 0s remain after all 1s have been crossed off, or vice-versa, **reject**. Otherwise **accept**.

Let M be a TM that halts on all inputs.

(We will only consider decidable languages now!)

Definition:

The **running time or time complexity of M** is the function $T : \mathbb{N} \to \mathbb{N}$ such that

$$T(n) = \text{maximum number of steps taken by M over all inputs of length } n$$

A "worst-case" measure of time complexity:

What's the longest time that a Turing machine could take over inputs of length n?

Time-Bounded Complexity Classes

Definition:

TIME(t(n)) = { L' | there is a Turing machine M with time complexity O(t(n)) so that L' = L(M) }

= { L' | L' is a language decided by a Turing machine with running time $\leq$ c t(n) + c, for some c $\geq$ 1 }

We just showed: A = { 0^k1^k | k $\geq$ 0 } $\in$ TIME(n^2)

Is there a faster Turing machine?

A = { 0^k1^k | k ≥ 0 } ∈ TIME(n log n)

M(w) := If w is not of the form 0*1*, reject.
  Repeat until all bits of w are crossed out:
    If (parity of 0's) ≠ (parity of 1's), reject.
    Cross out every other 0. Cross out every other 1.
  Once all bits are crossed out, accept.

```
00000000000001111111111111
x0x0x0x0x0x0xx1x1x1x1x1x1x
xxx0xxx0xxx0xxxx1xxx1xxx1x
xxxxxxx0xxxxxxxxxxxxx1xxxx
xxxxxxxxxxxxxxxxxxxxxxxxxx
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A = { 0^k1^k | k ≥ 0 } ∈ TIME($n \log n$)

M(w) := If w is not of the form 0*1*, reject.
  Repeat until all bits of w are crossed out:
    If (parity of 0's) ≠ (parity of 1's), reject.
    Cross out every other 0. Cross out every other 1.
  Once all bits are crossed out, accept.

For a fixed w = 0^k1^k:

Let $zero_i$ be number of 0s left in w, after iteration i

Let $ones_i$ be number of 1s left in w, after iteration i

Start with $zero_0$ = k, $ones_0$ = k

Key Observation:

$zero_{i+1}$ = floor($zero_i/2$), $ones_{i+1}$ = floor($ones_i/2$)

Number of iterations $\leq O(\log n)$

It can be proved that *there is no* one-tape Turing Machine that can decide **A** in *less* than O(n log n) time!

(Hard) Puzzle:

Let f(n) = O $\left(\frac{n \log n}{\alpha(n)}\right)$ where $\alpha(n)$ is unbounded.

Prove: TIME(f(n)) contains only regular languages(!)

For example, TIME(n log log n) contains only regular languages!

Two Tapes Can Be More Efficient

Theorem: A = { 0^k1^k | $k \geq 0$ } can be decided in O(n) time with a *two-tape* TM.

Proof Idea:
Sweep over all 0s, copy them over on the second tape.
Sweep over all 1s. For each 1, cross off a 0 from the second tape.

Different models of computation can yield different running times for the same language!

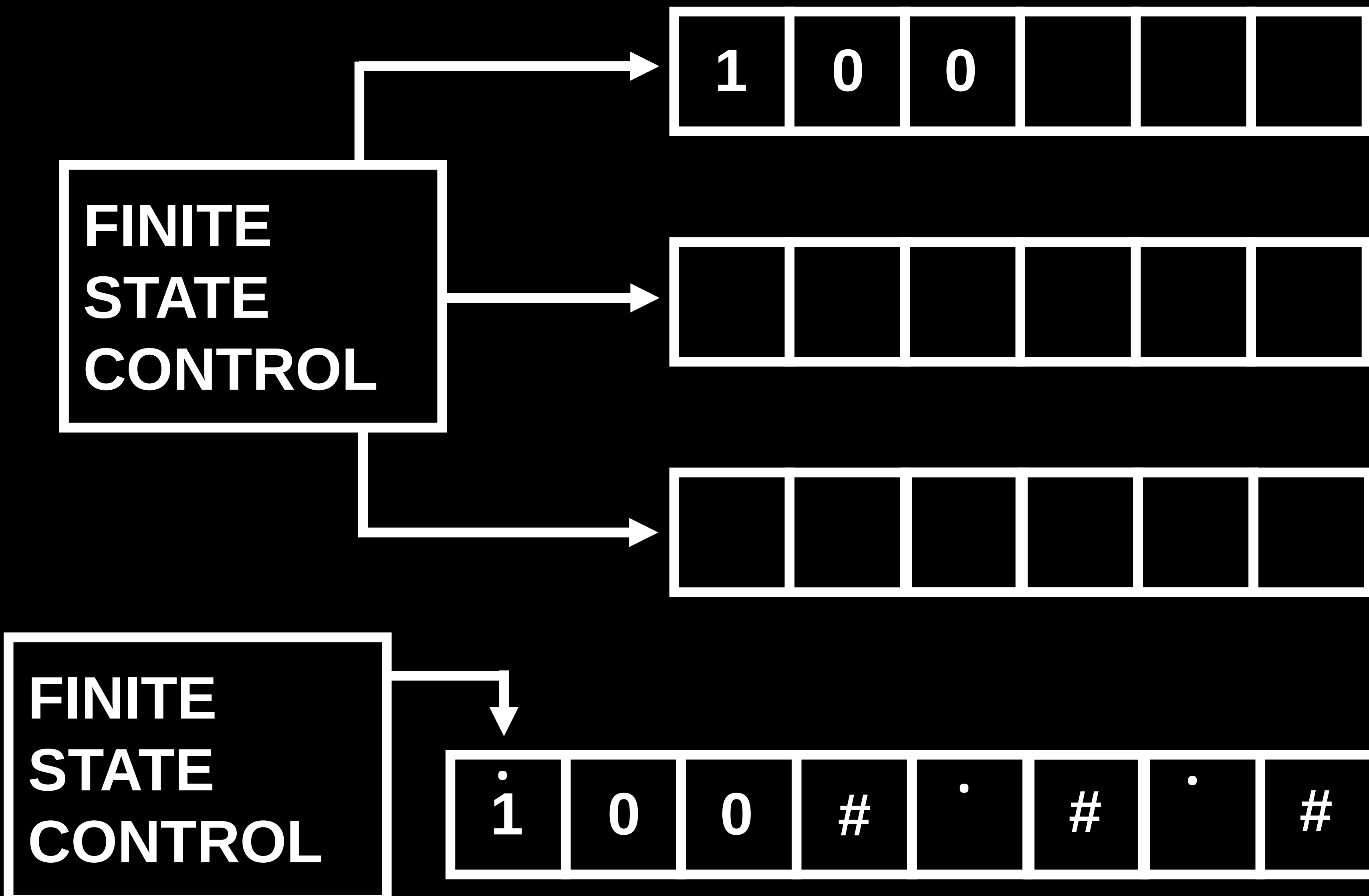
Let's revisit some of the key concepts from computability theory…

Theorem: Let t : $\mathbb{N} \rightarrow \mathbb{N}$ satisfy $t(n) \geq n$, for all n.
Then every $t(n)$ time multi-tape TM has an equivalent $O(t(n)^2)$ time one-tape TM

Our simulation of multitape TMs by one-tape TMs achieves this!

Corollary: Suppose language A can be decided by a multi-tape TM in p(n) time, for some polynomial p. Then A can also be decided by a one-tape TM in q(n) time, for some polynomial q.

Theorem: For every t(n) time multi-tape TM, there is an equivalent $O(t(n)^2)$ time one-tape TM



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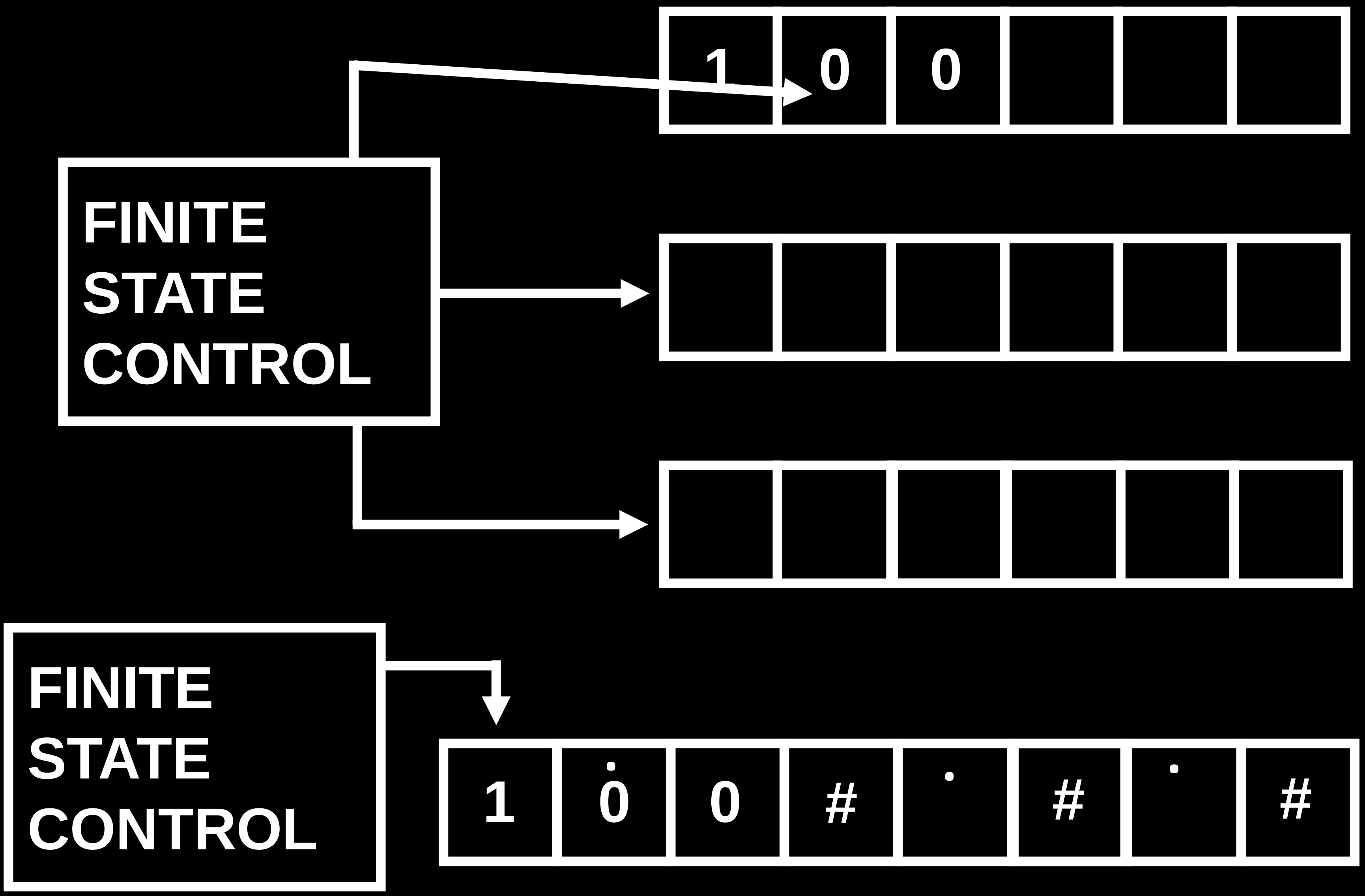

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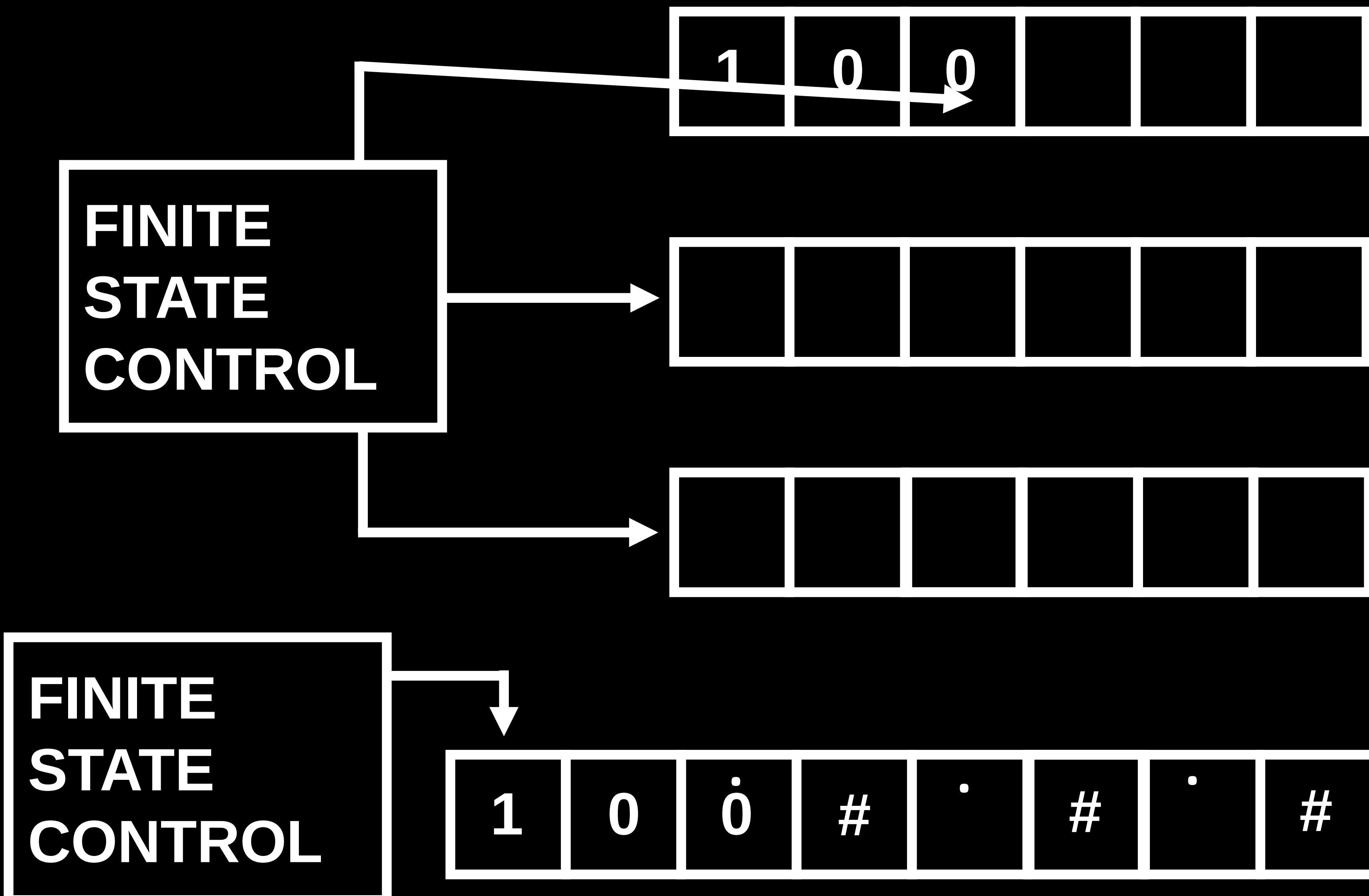
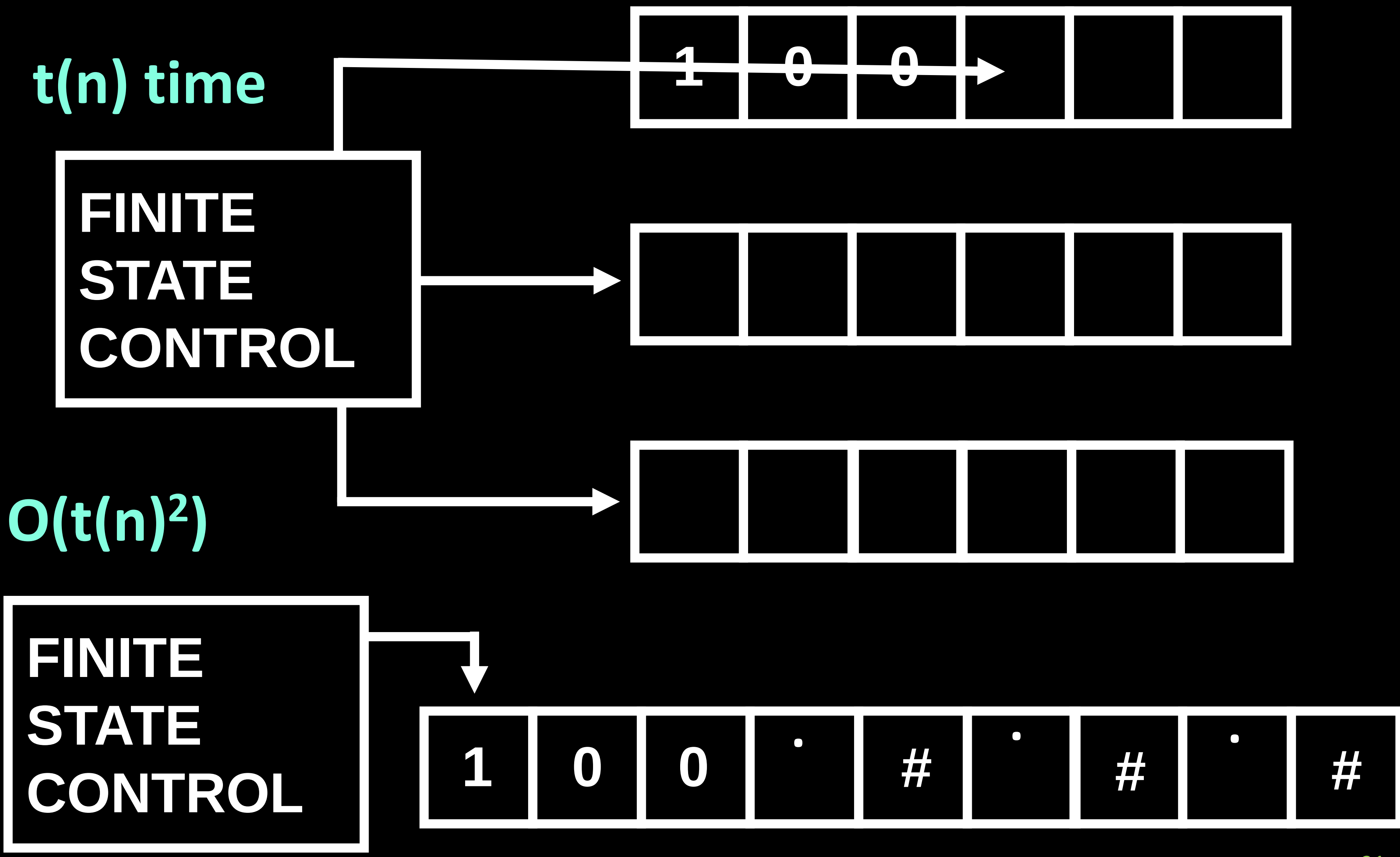


Theorem: For every $t(n)$ time multi-tape TM, there is an equivalent $O(t(n)^2)$ time one-tape TM



An Efficient Universal TM

Theorem: There is a (one-tape) Turing machine U which takes as input:

- the code of an arbitrary TM M
- an input string w
- and a string of t 1s, $t > |w|$

such that U on $\langle M, w, 1^t \rangle$ halts in $O(|M|^2\, t^2)$ steps and U accepts $\langle M, w, 1^t \rangle$ $\Leftrightarrow$ M accepts w in t steps

The Universal TM with a Clock

Idea: Make a multi-tape TM U' that does the above, and runs in $O(|M|\ t)$ steps.

Each step of M on w is $O(|M|)$ steps of U'

The Time Hierarchy Theorem

Intuition: If you get more time to compute, then you can solve **strictly more** problems.

Theorem: For all "reasonable" f, g : $\mathbb{N} \to \mathbb{N}$ where for all n, $g(n) > n^2 f(n)^2$, $\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(g(n))$

Proof Idea: Diagonalization with a clock

Make a TM N that on input $\langle M \rangle$ of length n, simulates the TM M on input $\langle M \rangle$ for $f(n)$ steps, *then* flips the answer.

We will show L(N) cannot have time complexity $f(n)$

The Time Hierarchy Theorem

Theorem: For "reasonable" f, g where $g(n) > n^2 f(n)^2$,

$$\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(g(n))$$

Proof Sketch: Define a TM N as follows.

N on input $\langle M \rangle$: "Let $n = |\langle M \rangle|$. Simulate M on $\langle M \rangle$ for up to $f(n)$ steps. If the sim halts, output the opposite answer."

Claim: L(N) does not have time complexity $f(n)$.

Proof: Assume some D runs in $f(n)$ time, and L(D) = L(N). By assumption, D on $\langle D \rangle$ runs in $f(n)$ time and outputs the *opposite* answer of D on $\langle D \rangle$ after $f(n)$ steps!

This is a contradiction!

The Time Hierarchy Theorem

Theorem: For "reasonable" f, g where $g(n) > n^2 f(n)^2$,

$$\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(g(n))$$

Proof Sketch: Define a TM N as follows:

N on input $\langle M \rangle$: "Let $n = |\langle M \rangle|$. Simulate M on $\langle M \rangle$ for up to $f(n)$ steps. If the sim halts, output the opposite answer."

So, L(N) does *not* have time complexity f(n).

For what functions g(n) will N run in O(g(n)) time?

1. Compute $t = f(n)$ in $O(g(n))$ time ["reasonable"]
2. To sim M on $\langle M \rangle$: run $U(M, M, 1^t)$ in $O(g(n))$ time

Recall: $U(M, w, 1^t)$ halts in $O(|M|^2 t^2)$ steps

So, set $g(n)$ so that $g(|M|) > |M|^2 f(|M|)^2$ for all n. QED

Remark: Time hierarchy also holds for multitape TMs!

A Better Time Hierarchy Theorem

Theorem: For "reasonable" f, g where
g(n) > f(n) $\log^2$ f(n), TIME(f(n)) $\subsetneq$ TIME(g(n))

Corollary: TIME(n) $\subsetneq$ TIME(n^2) $\subsetneq$ TIME(n^3) $\subsetneq$ …

There is an infinite hierarchy of
increasingly more time-consuming problems

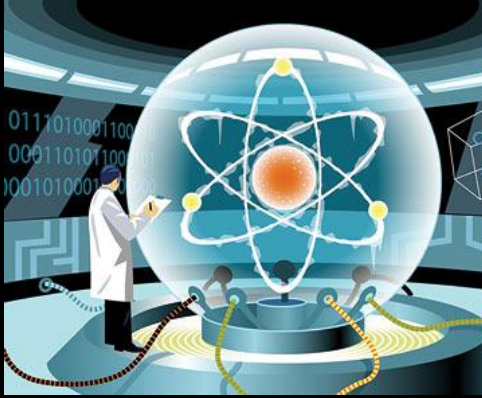
Question: Are there important everyday problems
that are high up in this time hierarchy?
A *natural* problem that *needs precisely* n^{10} time?

THIS IS AN OPEN QUESTION!

$$P = \bigcup_{k \in N} \text{TIME}(n^k)$$

Polynomial Time

The analogue of “decidability”
in the world of complexity theory



The EXTENDED Church-Turing Thesis

Everyone's Intuitive Notion of Efficient Algorithms = **Polynomial-Time Turing Machines**

A controversial (dead?) thesis!

Counterexamples include n^{100} time algorithms, randomized algorithms, quantum algorithms, …

Nondeterminism and NP

The analogue of “recognizability” in complexity theory