6.045

Lecture 15: NP-Complete Problems and the Cook-Levin Theorem

Time-Bounded Complexity Classes

Turing machine M has **time complexity O(t(n))** if there is a $c > 0$ such that for all inputs x, M running on x halts within $c\,t(|x|) + c$ steps

Definition:
TIME(t(n)) = { L' | there is a Turing machine M with time complexity O(t(n)) so that L' = L(M) }

= { L' | L' is a language decided by a Turing machine with $\leq c\,t(n) + c$ running time, for some $c \geq 1$ }

$$\mathbf{P} = \bigcup_{k \in \mathbf{N}} \mathrm{TIME}(n^k)$$

Polynomial Time

The analogue of "decidability" in the world of complexity theory

Definition: NTIME(t(n)) =

{ L | L is decided by an O(t(n)) time *nondeterministic* Turing machine }

Note: TIME(t(n)) $\subseteq$ NTIME(t(n))

Is TIME(t(n)) = NTIME(t(n)) for all t(n)?

THIS IS AN OPEN QUESTION!

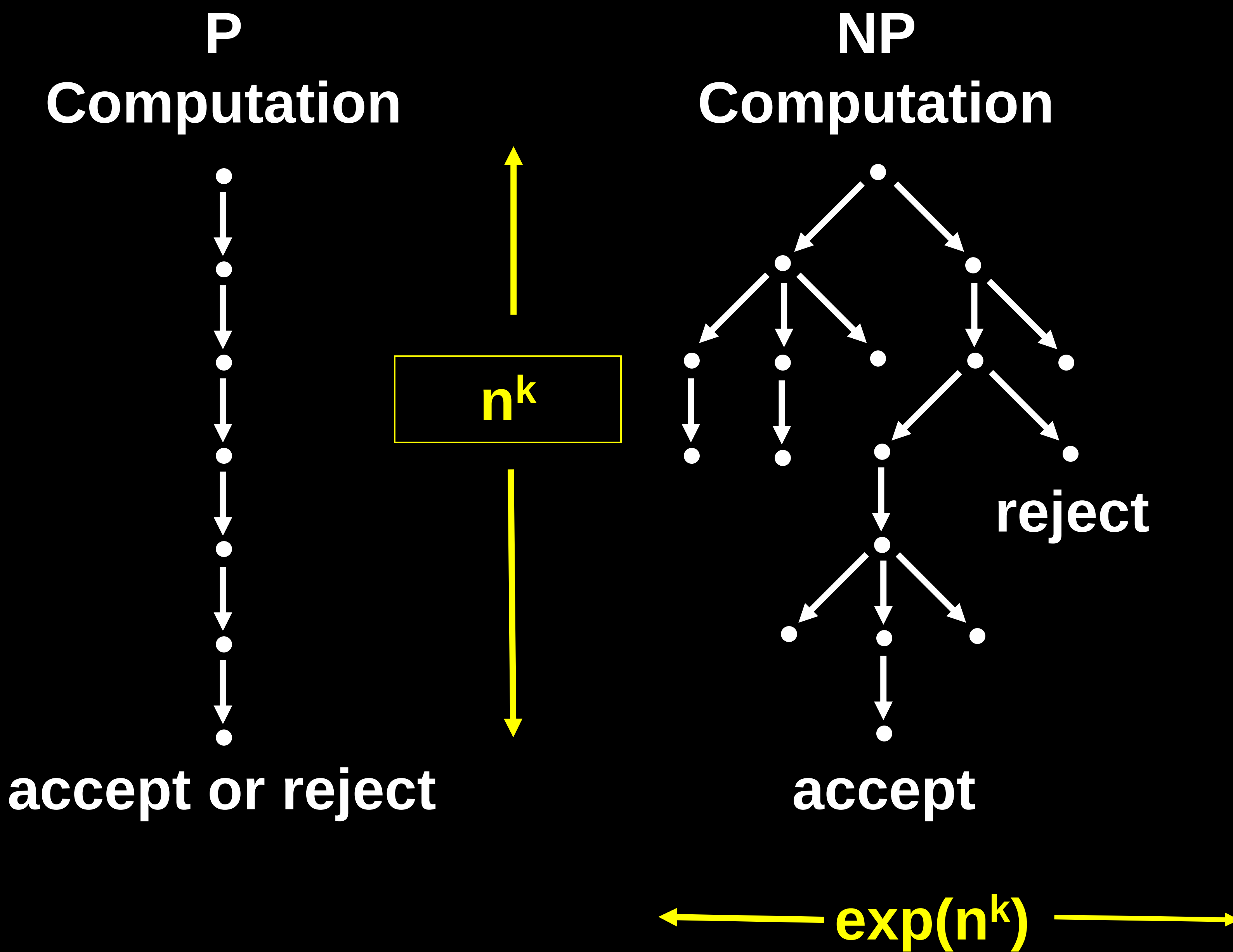
What can be done in "short" NTIME that cannot be done in "short" TIME?

Last time we saw:
3SAT, CLIQUE, HAMPATH are in NP

$$\mathbf{NP} = \bigcup_{k \in \mathbb{N}} \mathrm{NTIME}(n^k)$$

Nondeterministic Polynomial Time

The analogue of "recognizability" in complexity


P
Computation
NP
Computation
n^k
reject
accept or reject
accept
$\exp(n^k)$

Theorem: $L \in NP \Leftrightarrow$ There is a constant k and polynomial-time TM V such that

$$L = \{ x \mid \exists\, y \in \Sigma^* \; [\, |y| \leq |x|^k \text{ and } V(x,y) \text{ accepts }] \}$$

A language L is in NP if and only if there are "polynomial-length proofs" for membership in the language L

P = the problems that can be *efficiently solved*

NP = the problems where *proposed solutions can be efficiently verified*

Is P = NP?

Can problem solving be automated?

Is SAT solvable in O(n) time on a multitape TM?

Logic circuits of $10n$ gates for SAT?

If yes, then there would be a "dream machine" that could crank out short proofs of theorems, quickly optimize all aspects of life…
recognizing **quality work is all you would need to** ***produce*** **quality work**

THIS IS AN OPEN QUESTION!

So how do we get a handle on a problem that we have no idea how to resolve?

Try to understand its consequences!
Understand its meaning!

Try to better understand NP problems!

In computability theory, we related problems by mapping reductions and oracle reductions….

Polynomial Time Reductions

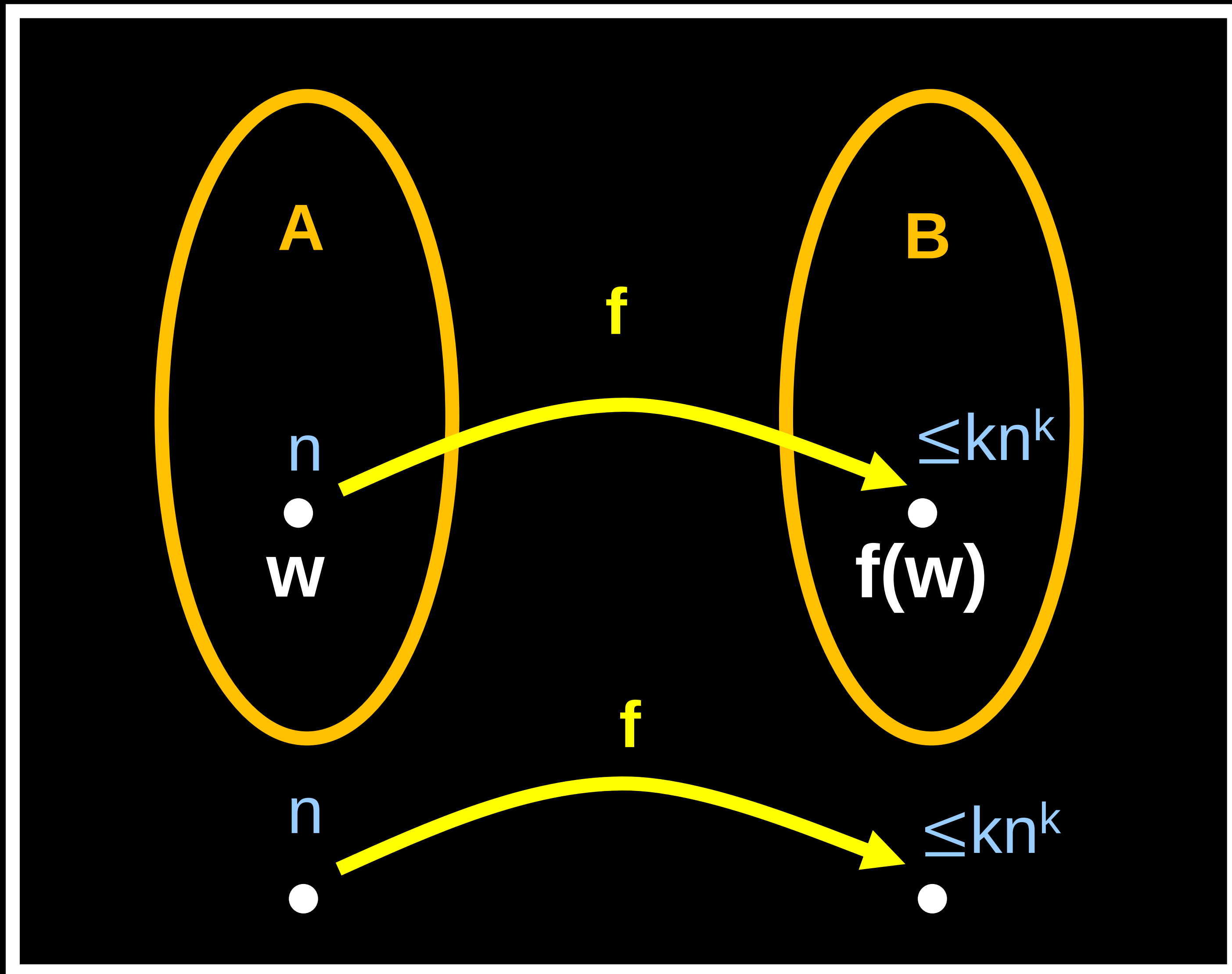
$f : \Sigma^* \rightarrow \Sigma^*$ is a **polynomial time computable function** if there is a poly-time Turing machine M that on every input w, halts with just f(w) on its tape

Language A is **poly-time reducible** to language B, written as $A \leq_P B$, if there is a poly-time computable $f : \Sigma^* \rightarrow \Sigma^*$ so that:

$$w \in A \Leftrightarrow f(w) \in B$$

We say: f is a polynomial time reduction from A to B

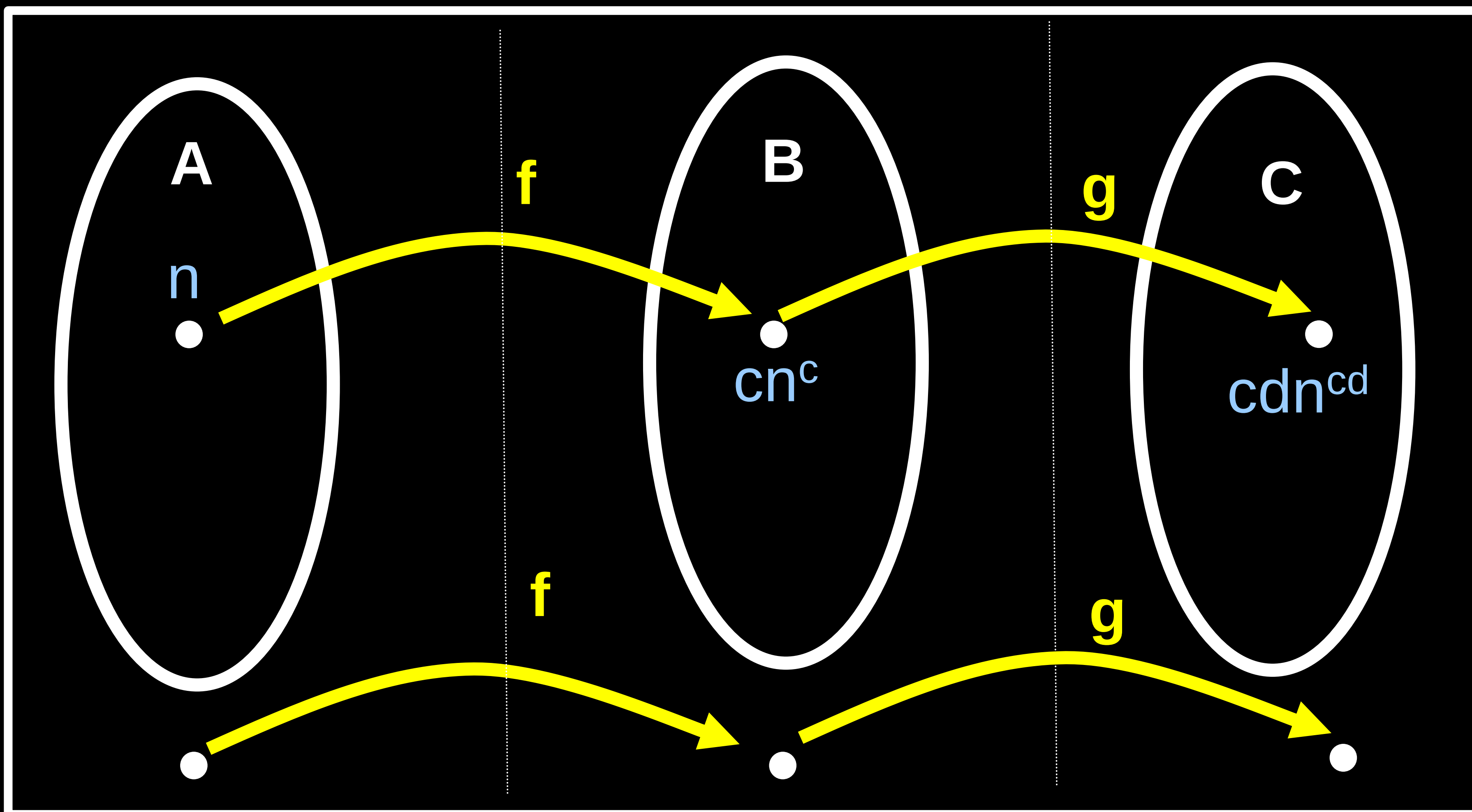
Note: there is a k such that for all w, $|f(w)| \leq k|w|^k$



f converts any string w into a string f(w) such that

$w \in A \Leftrightarrow f(w) \in B$

Theorem: If $A \leq_P B$ and $B \leq_P C$, then $A \leq_P C$



Theorem: If $A \leq_P B$ and $B \in P$, then $A \in P$

Proof: Let M_B be a poly-time TM that decides B.
Let f be a poly-time reduction from A to B.

We build a machine M_A that decides A as follows:

M_A = On input w,

1. Compute f(w)
2. Run M_B on f(w), output its answer

$$w \in A \Leftrightarrow f(w) \in B$$

Theorem: If $A \leq_P B$ and $B \in NP$, then $A \in NP$

Proof: Analogous…

Theorem: If $A \leq_P B$ and $B \in P$, then $A \in P$

Theorem: If $A \leq_P B$ and $B \in NP$, then $A \in NP$

Corollary: If $A \leq_P B$ and $A \notin P$, then $B \notin P$

Question: What are the "hardest" NP problems under this partial ordering $\leq_P$?

Does there even *exist* a "hardest" NP problem??

Definition: A language B is NP-complete if:

1. $B \in NP$
2. **Every** $A \in NP$ is poly-time reducible to B

   That is, $A \leq_P B$

   When this is true, we say "B is NP-hard"

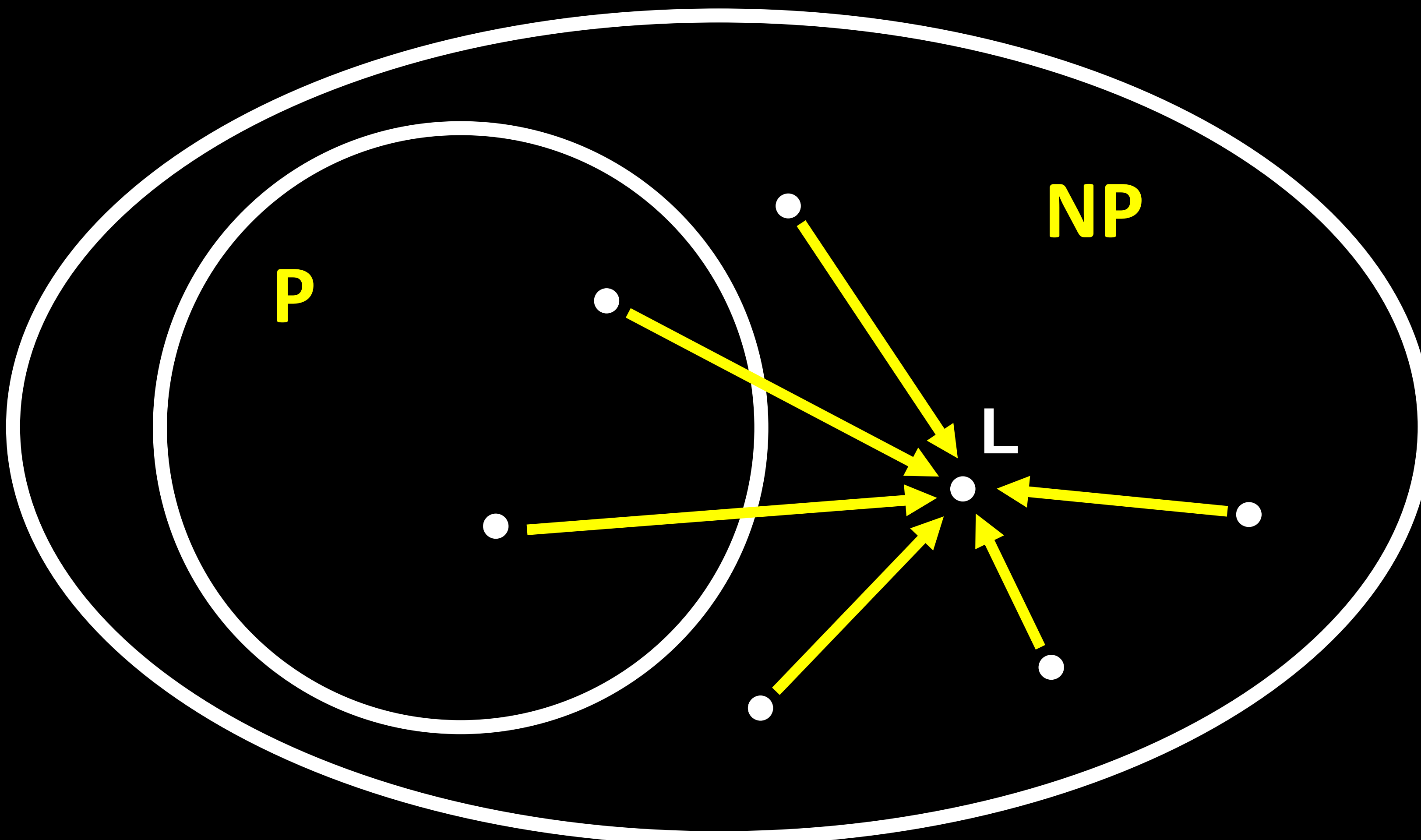
On homework, you showed (or will show!)

A language L is recognizable iff $L \leq_m A_{TM}$

A_{TM} is "*complete for recognizable languages*":

A_{TM} is recognizable, and for all recognizable L, $L \leq_m A_{TM}$

Suppose L is NP-Complete…



If $L \in P$, then $P = NP$!   If $L \notin P$, then $P \neq NP$!

Suppose L is NP-Complete…

Then assuming the conjecture P $\neq$ NP,

L is not decidable in n^k time, for *every* k

Thm: There *exists* an NP-complete problem

NHALT = { ⟨N, x, 1^t⟩ | Nondeterministic TM N accepts input x in $\leq t$ steps }

Without $\mathbf{1^t}$, this is undecidable!

1. **NHALT ∈ NP**

Nondeterministically guess a sequence of t transitions of N, then check that N following these t transitions accepts x. Takes time polynomial in t, |x|, and |N|.

2. **Every A in NP is poly-time reducible to NHALT**

In other words, NHALT is NP-hard

For each A ∈ NP, there is an $k\,n^k$-time NTM N such that
A = { x | N(x) accepts }

Reduction: Map string x to the string ⟨N, x, $1^{p(|x|)}$⟩.

There are thousands of
natural NP-complete problems!

Your favorite topic certainly has an
NP-complete problem somewhere in it

Even the other sciences are not safe:
biology, chemistry, physics have
NP-complete problems too!

A 3cnf-formula has the form:

$(x_1 \vee \neg x_2 \vee x_3) \wedge (x_4 \vee x_2 \vee x_5) \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$

literals

clauses

where x_1 , x_2 , … are Boolean variables

A 3cnf-formula is satisfiable if there is a setting to the variables that makes the formula true.

3SAT = { ϕ | ϕ is a satisfiable 3cnf-formula }

The Cook-Levin Theorem:

3SAT is NP-complete

"Simple Logic can encode any NP problem!"

1. 3SAT $\in$ NP
   A satisfying assignment is a "proof" that a 3cnf formula is satisfiable (already done!)

2. 3SAT is NP-hard
   Every language in NP can be polynomial-time reduced to 3SAT (complex logical formula)

Corollary: 3SAT $\in$ P if and only if P = NP

The Cook-Levin Theorem:

3SAT is NP-complete

"Simple Logic can encode any NP problem!"

This theorem is a cornerstone of complexity theory AND of modern (practical) system verification!

There are entire fields and conferences devoted solely to SAT solving!

Few theorems have had such an impact on *both* theory and practice!

Theorem (Cook-Levin): 3SAT is NP-complete

Proof Idea:

(1) 3SAT $\in$ NP (done)

(2) Every language A $\in$ NP is polynomial time reducible to 3SAT (this is the challenge)

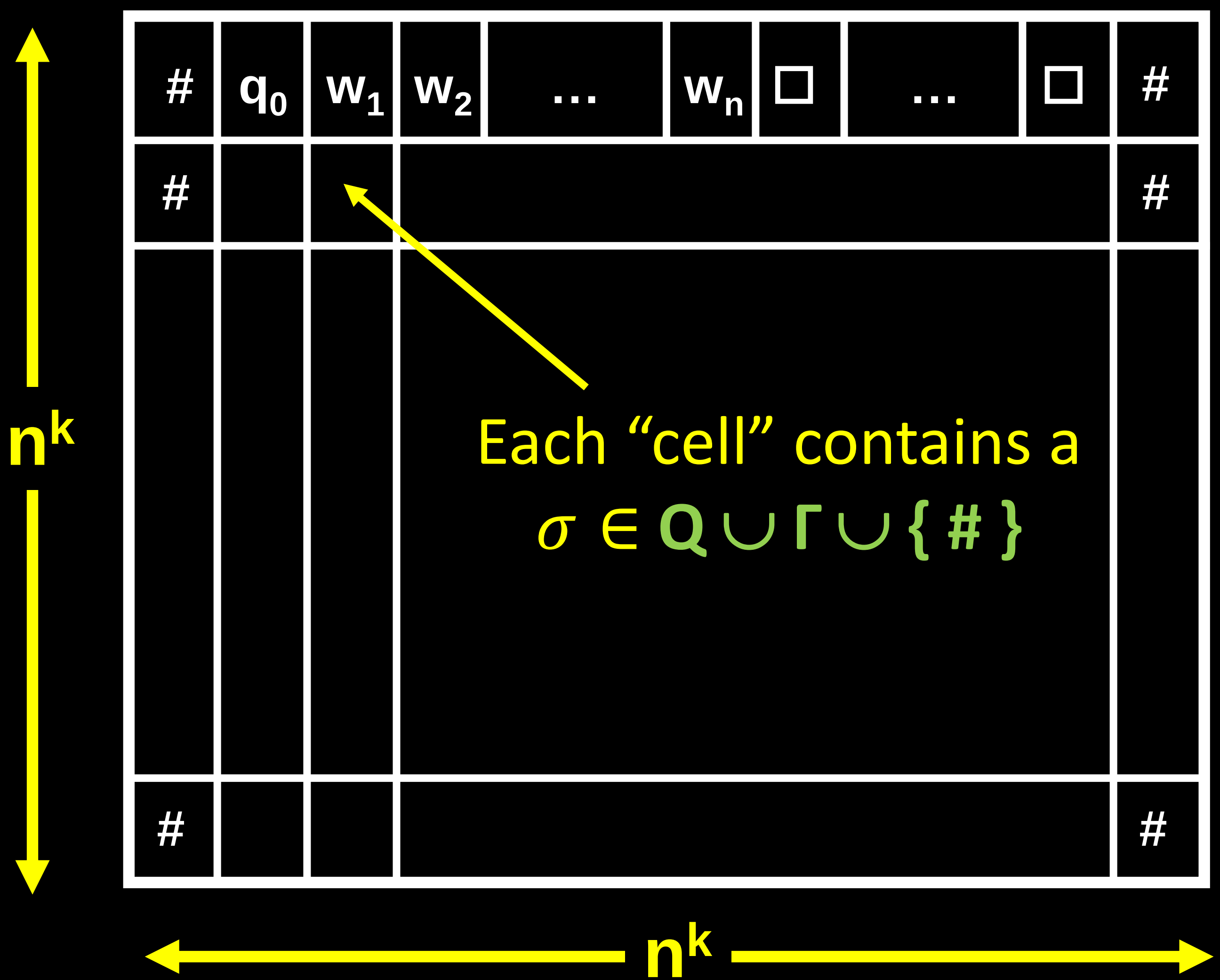
We give a poly-time reduction from A to 3SAT

The reduction converts a string w into a 3cnf formula ϕ such that w $\in$ A iff $\phi \in$ 3SAT

For A $\in$ NP, let N be a nondeterministic TM deciding A in n^k time

Idea: ϕ will "simulate" N on w

Let L(N) $\in$ NTIME(n^k). A tableau for N on w is an $n^k \times n^k$ matrix whose rows are the configurations of *some* computation history of N on w



A tableau is accepting if the last row of the tableau has an accept state

Therefore, N accepts string w if and only if there is an accepting tableau for N on w

Given w, we will construct a 3cnf formula ϕ with $O(|w|^{2k})$ clauses, describing logical constraints that any accepting tableau for N on w must satisfy

The 3cnf formula ϕ will be satisfiable

if and only if

there is an accepting tableau for N on w

Programming with Boolean logic!

Variables of formula ϕ will *encode* a tableau

Let $C = Q \cup \Gamma \cup \{ \# \}$ *(constant-sized set!)*

Each cell of a tableau contains a symbol from C

cell[i,j] = symbol in the cell at row i and column j
= the jth symbol in the ith configuration

For every i and j ($1 \leq i, j \leq n^k$) and for every $s \in C$
we make a Boolean variable $x_{i,j,s}$ in ϕ

Total number of variables = $|C|n^{2k}$, which is $O(n^{2k})$

The $x_{i,j,s}$ variables represent the cells of a tableau

We will enforce the condition: for all i, j, s,

$$x_{i,j,s} = 1 \Leftrightarrow \text{cell}[i,j] = s$$

Idea: Make ϕ so that every *satisfying assignment* to the variables $x_{i,j,s}$ corresponds to an *accepting tableau* for **N** on **w** (an assignment to all cell[i,j]'s of the tableau)

The formula ϕ will be the AND of four CNF formulas:

$$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$

ϕ_{cell} : for all i, j, there is a *unique* $s \in C$ with $x_{i,j,s} = 1$

ϕ_{start} : the first row of the table equals the *start* configuration of N on w

ϕ_{accept} : the last row of the table has an accept state

ϕ_{move} : every row is a configuration that yields the configuration on the next row

ϕ_{start} : the first row of the table equals the *start* configuration of N on w

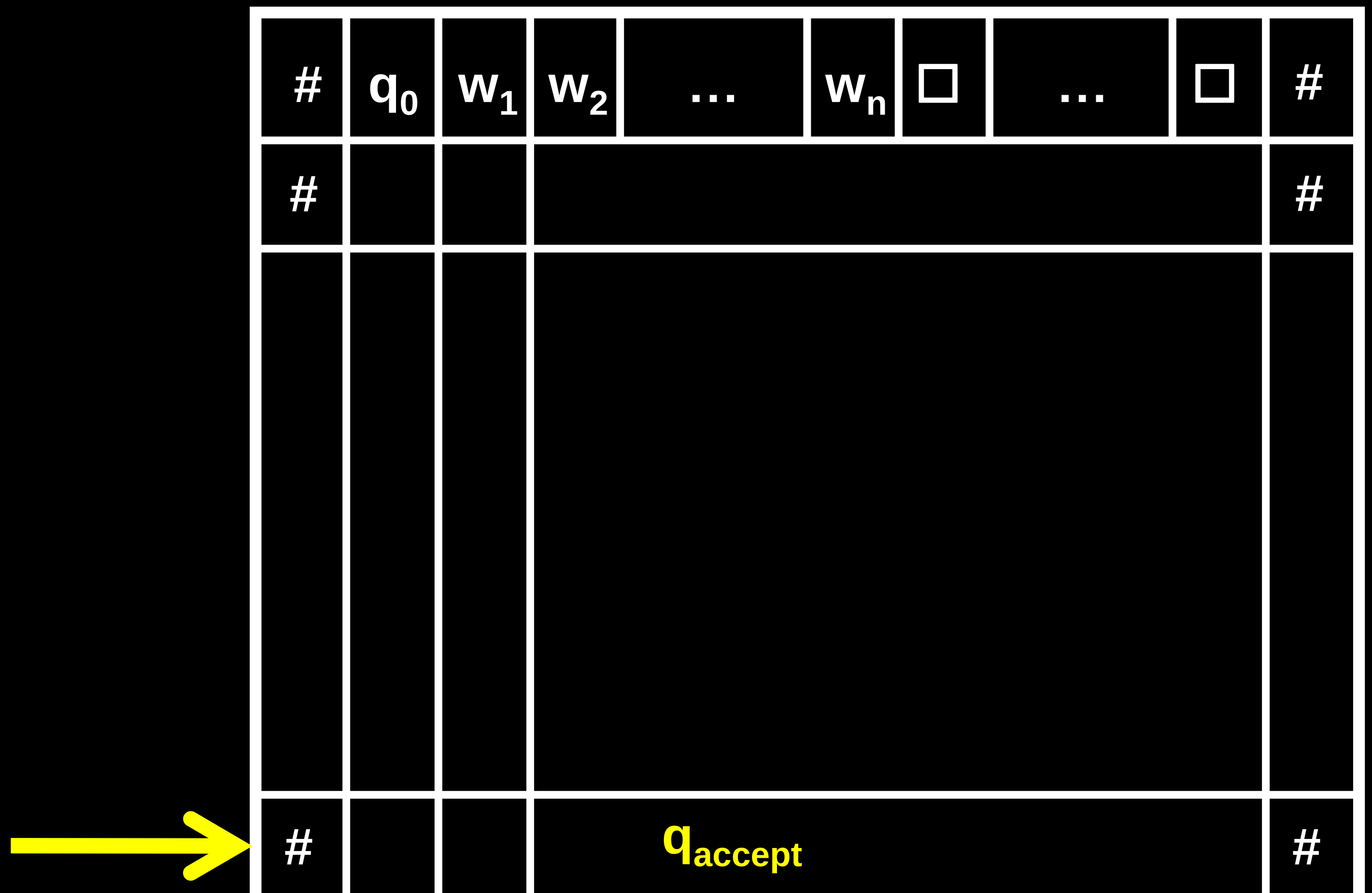
$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \cdots \wedge x_{1,n+2,w_n} \wedge x_{1,n+3,\square} \wedge \cdots \wedge x_{1,n^k-1,\square} \wedge x_{1,n^k,\#}$$

#	q_0	w_1	w_2	…	w_n	$\square$	…	$\square$	#
#									#

$O(n^k)$ clauses

ϕ_{accept} : the last row of the table has an accept state

$$\phi_{accept} = \bigvee_{1 \leq j \leq n^k} x_{n^k, j, q_{accept}}$$



ϕ_{accept} : the last row of the table has an accept state

$$\phi_{accept} = \bigvee_{1 \leq j \leq n^k} x_{n^k, j, q_{accept}}$$

How can we convert ϕ_{accept} into a 3-cnf formula?

Can write the clause $(a_1 \vee a_2 \vee \ldots \vee a_t)$ as

$$(a_1 \vee a_2 \vee z_1) \wedge (\neg z_1 \vee a_3 \vee z_2) \wedge \ldots \wedge (\neg z_{t-3} \vee a_{t-1} \vee a_t)$$

where z_i are brand new variables.

This produces $O(t)$ new 3cnf clauses, and the new formula is SAT iff the old one is SAT.

$O(n^k)$ 3cnf clauses

ϕ_{cell} : for all i, j, there is a *unique* $s \in C$ with $x_{i,j,s} = 1$

$$\phi_{cell} = \bigwedge_{1 \le i,\, j \le n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} (\neg x_{i,j,s} \vee \neg x_{i,j,t}) \right) \right]$$

for all i,j

at least one $x_{i,j,s}$ is set to 1

at most one $x_{i,j,s}$ is set to 1

$O(n^{2k})$ 3cnf clauses

ϕ_{move} : every row is a configuration that yields the configuration on the next row

Key Question: If one row yields the next row, how many cells can be different between the two rows?

Answer: AT MOST THREE CELLS!

#	b	a	a	q_1	b	c	b	#
#	b	a	q_2	a	c	c	b	#

ϕ_{move} : every row is a configuration that yields the configuration on the next row

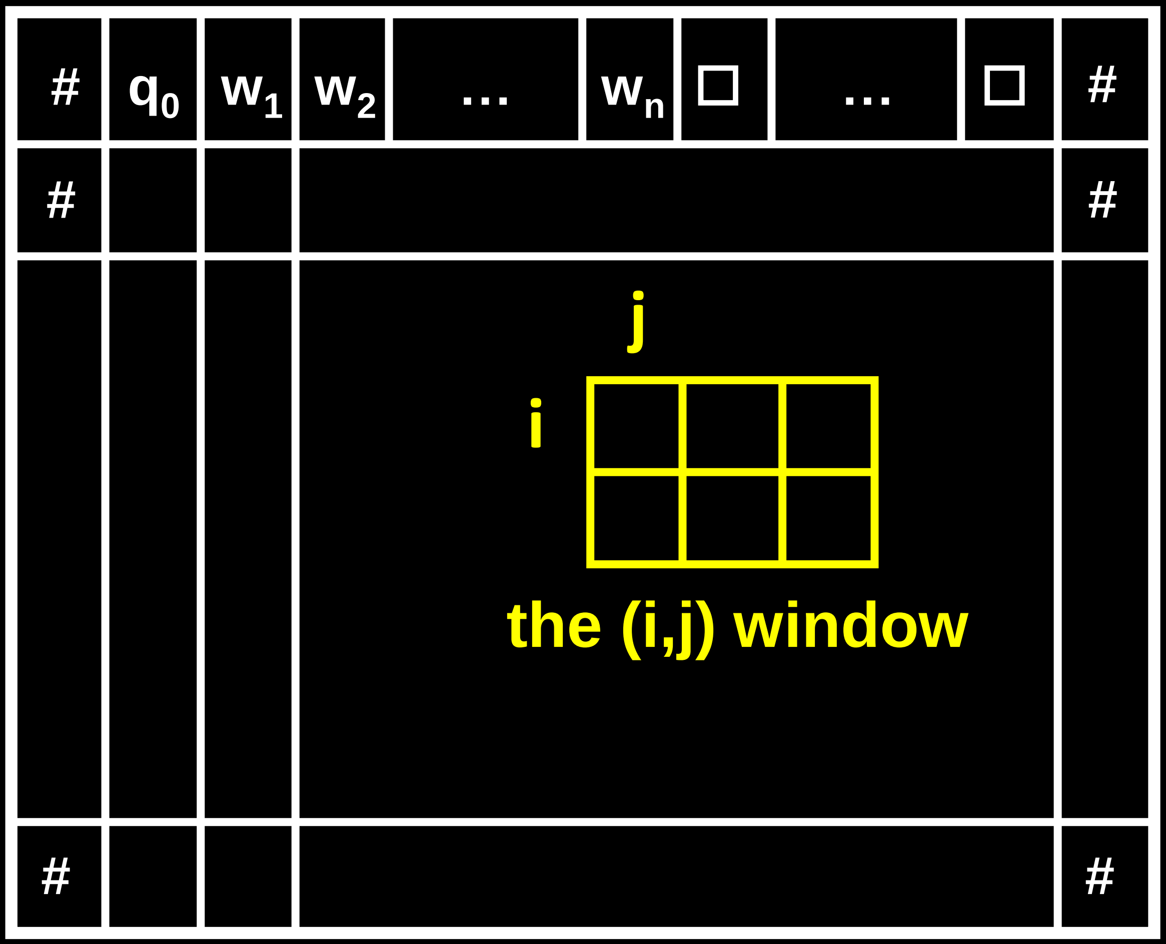
Key Question: If one row yields the next row, how many cells can be different between the two rows?

Answer: AT MOST THREE CELLS!

#	b	a	a	q_1	b	c	b	#
#	b	a	q_2	a	c	c	b	#

ϕ_{move} : every row is a configuration that yields the configuration on the next row

Idea: check that every 2×3 "window" of cells is **legal:** consistent with the transition function of N



If $\delta(q_1,a) = \{(q_1,b,R)\}$ and $\delta(q_1,b) = \{(q_2,c,L), (q_2,a,R)\}$ which of the following windows are legal?

a	q_1	b
q_2	a	c

a	q_1	b
q_1	a	a

a	a	q_1
a	a	b

#	b	a
#	b	a

a	b	a
a	b	q_2

b	q_1	b
q_2	b	q_2

a	b	a
a	a	a

a	q_1	b
a	a	q_2

b	b	b
c	b	b

Key Lemma:
IF Every window of the tableau is legal, and
The 1st row is the start configuration of N on w
THEN for all i = 1,…,n^k – 1, the ith row of the tableau is a configuration which yields the (i+1)th row.

Proof Sketch: (Strong) induction on i.
The 1st row is a configuration. If it *didn't* yield the 2nd row, there's a 2 x 3 "illegal" window on 1st and 2nd rows
Assume rows 1,…,L are all configurations which yield the next row, and assume every window is legal.
If row L+1 did *not* yield row L+2, then there's a 2 x 3 window along those two rows which is "illegal"

The (i, j) window of a tableau is the tuple $(a_1,\dots,a_6) \in C^6$ such that

	col. j	col. j+1	col. j+2
row i	a_1	a_2	a_3
row i+1	a_4	a_5	a_6

ϕ_{move} : every row is a configuration that legally follows from the previous configuration

$$\phi_{move} = \bigwedge_{\substack{1 \le i \le n^k - 1 \\ 1 \le j \le n^k - 2}} (\text{ the (i, j) window is legal })$$

(the (i, j) window is legal) =

$$\bigvee_{\substack{(a_1, \ldots, a_6) \\ \text{is a legal window}}} (x_{i,j,a_1} \wedge x_{i,j+1,a_2} \wedge x_{i,j+2,a_3} \wedge x_{i+1,j,a_4} \wedge x_{i+1,j+1,a_5} \wedge x_{i+1,j+2,a_6})$$

$$\equiv \bigwedge_{\substack{(a_1, \ldots, a_6) \\ \text{is NOT a legal window}}} (\overline{x_{i,j,a_1}} \vee \overline{x_{i,j+1,a_2}} \vee \overline{x_{i,j+2,a_3}} \vee \overline{x_{i+1,j,a_4}} \vee \overline{x_{i+1,j+1,a_5}} \vee \overline{x_{i+1,j+2,a_6}})$$

$$\phi_{move} = \bigwedge_{1 \leq i, j \leq n^k} (\text{ the (i, j) window is "legal" })$$

the (i, j) window is "legal" =

$$\equiv \bigwedge_{\substack{(a_1, \ldots, a_6) \\ \text{ISN'T "legal"}}} (\overline{x}_{i,j,a_1} \vee \overline{x}_{i,j+1,a_2} \vee \overline{x}_{i,j+2,a_3} \vee \overline{x}_{i+1,j,a_4} \vee \overline{x}_{i+1,j+1,a_5} \vee \overline{x}_{i+1,j+2,a_6})$$

$O(n^{2k})$ clauses

Summary. Our goal was to prove:

Every A in NP has a polynomial time reduction to 3SAT

For every A ∈ NP, we know A is decided by some nondeterministic n^k time Turing machine N

We gave a generic method to reduce **N** and a string **w** to a **3cnf formula ϕ** of **$O(|w|^{2k})$ clauses** such that

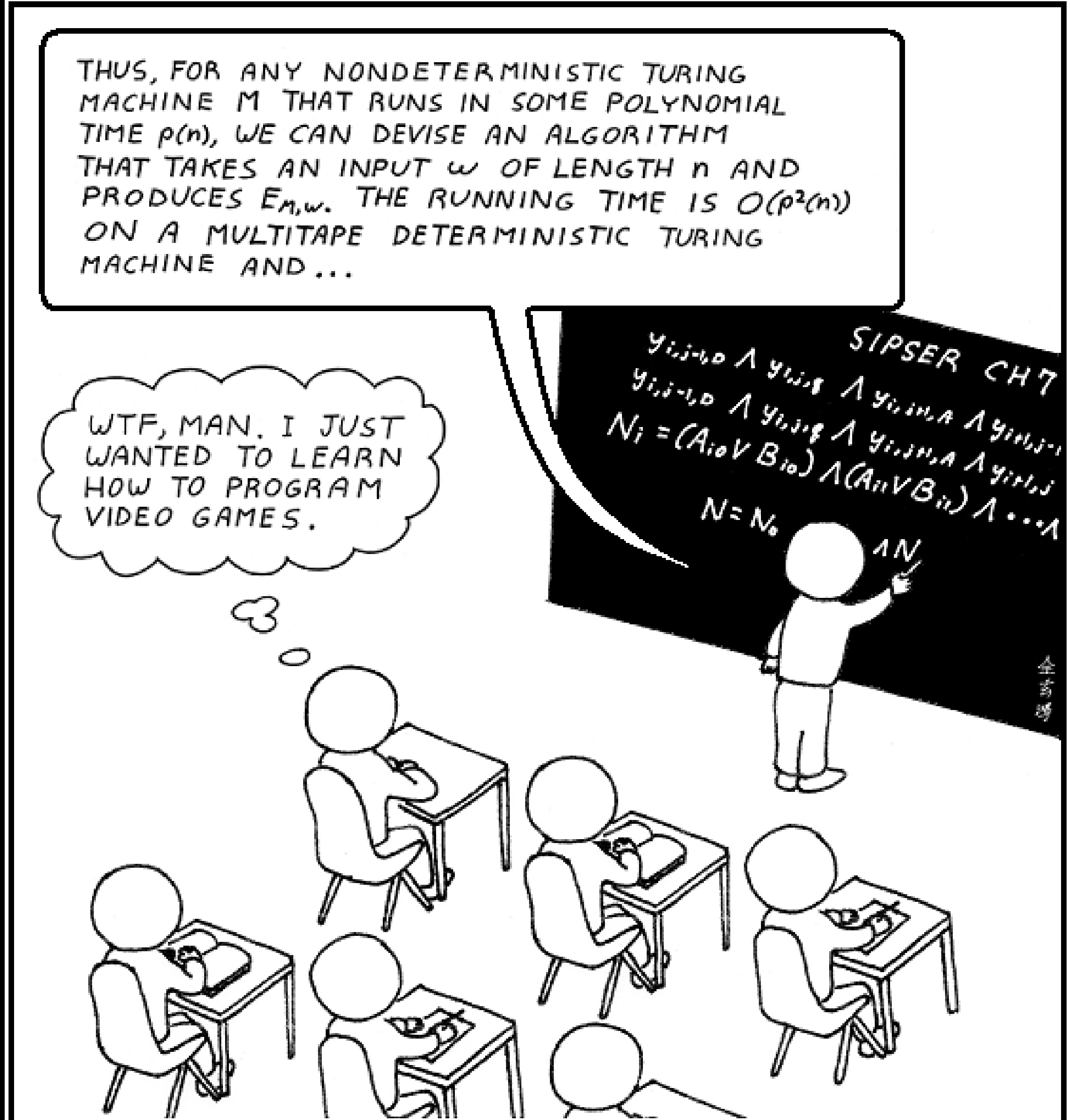
satisfying assignments to the variables of ϕ

directly correspond to

accepting computation histories of N on w

The formula ϕ is the **AND** of four 3cnf formulas:

$$\phi = \phi_{cell} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$$



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Reading Assignment

Read Luca Trevisan's notes for an alternative proof of the Cook-Levin Theorem!

Sketch:

1. **Define CIRCUIT-SAT:** ***Given a logical circuit C, is there an input a such that C(a)=1?***
2. **Show that CIRCUIT-SAT is NP-hard:**
   The n^k x n^k tableau for N on w can be simulated using a logical circuit of $O(n^{2k})$ gates
3. **Reduce CIRCUIT-SAT to 3SAT in polytime**
4. **Conclude 3SAT is also NP-hard**