6.045

Lecture 16:
NP-Complete Problems:
THEY'RE EVERYWHERE!

Polynomial Time Reducibility

$f : \Sigma^* \to \Sigma^*$ is a **polynomial time computable function** if there is a poly-time Turing machine M that on every input w, halts with just f(w) on its tape

Language A is **poly-time reducible** to language B, written as $A \leq_P B$, if there is a poly-time computable $f : \Sigma^* \to \Sigma^*$ so that:

$$w \in A \Leftrightarrow f(w) \in B$$

f is a polynomial time reduction from A to B

Note there is a k such that for all w, $|f(w)| \leq k|w|^k$

Definition: A language B is **NP-complete** if:

1. $B \in NP$
2. Every A in NP is poly-time reducible to B
   That is, $A \leq_P B$
   When this is true, we say "B is NP-hard"

The Cook-Levin Theorem:
3SAT is NP-complete
"Simple Logic can encode any NP problem!"

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“Simple Logic can encode any NP problem!”

Today we’ll see many more NP-complete problems:

NHALT, 3SAT, CLIQUE, IS, VC, SUBSET-SUM, KNAPSACK, PARTITION, BIN-PACKING, …

(There are entire classes at MIT on this kind of stuff)

And even more on pset/pests…

For all of these problems, assuming P $\neq$ NP, they are not in P

There are thousands of *natural* NP-complete problems!

Your favorite topic certainly has an NP-complete problem somewhere in it

Even the other sciences are not safe: biology, chemistry, physics have NP-complete problems too!

Reading Assignment

Read Luca Trevisan's notes for an alternative proof of the Cook-Levin Theorem!

Sketch:

1. **Define CIRCUIT-SAT:** ***Given a logical circuit C, is there an input a such that C(a)=1?***
2. **Show that CIRCUIT-SAT is NP-hard:**
   The n^k x n^k tableau for N on w can be simulated using a logical circuit of $O(n^{2k})$ gates
3. **Reduce CIRCUIT-SAT to 3SAT in polytime**
4. **Conclude 3SAT is also NP-hard**

Theorem (Cook-Levin): 3SAT is NP-complete

Corollary: 3SAT $\notin$ P if and only if P $\neq$ NP

Given a new problem L $\in$ NP, how can we prove it is NP-hard?

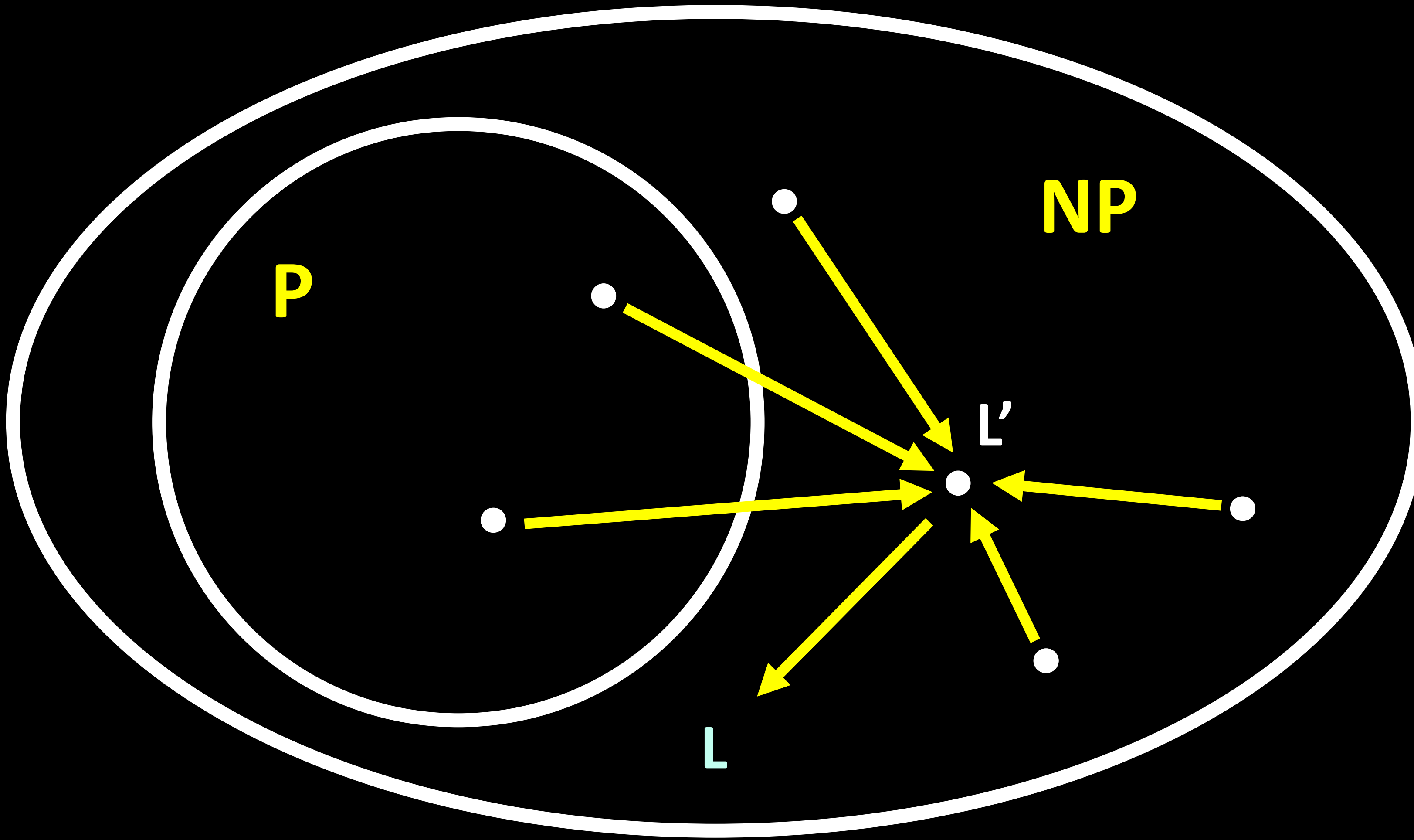
Generic Recipe:

1. Take a problem L' that you know to be NP-hard (e.g., 3SAT)
2. Prove that L' $\leq_P$ L

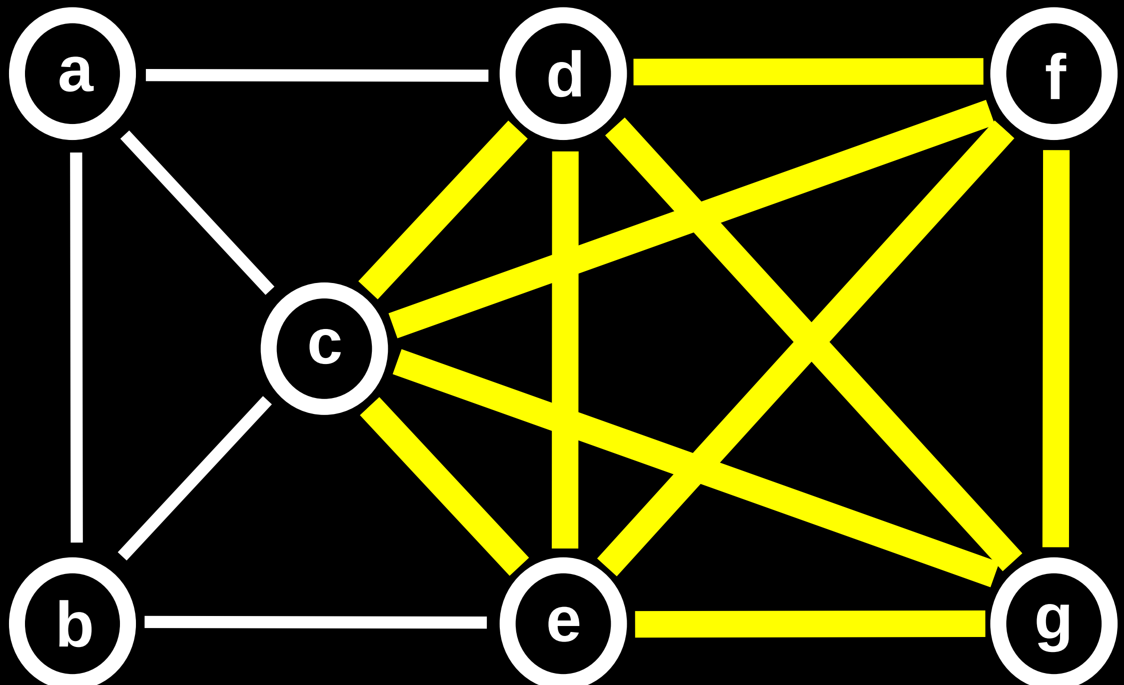
Then for all A $\in$ NP, A $\leq_P$ L' by (1), and L' $\leq_P$ L by (2)

This implies A $\leq_P$ L. Therefore L is NP-hard!

L is NP-Complete



The Clique Problem



Given a graph G and positive k, does G contain a complete subgraph on k nodes?

CLIQUE = { (G,k) | G is an undirected graph with a k-clique }

The Clique Problem

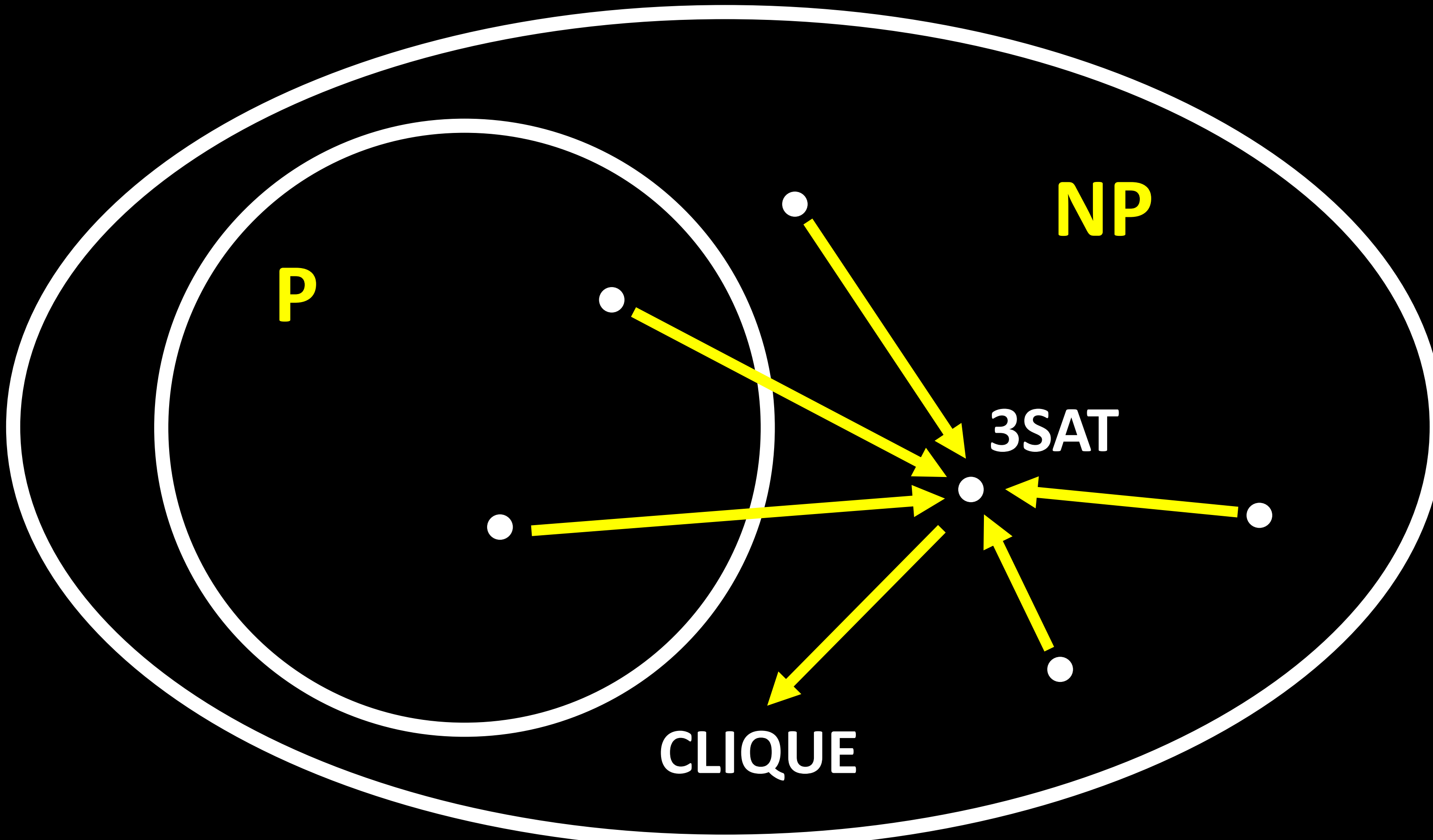
Given a graph G and positive k, does G contain a complete subgraph on k nodes?

CLIQUE = { (G,k) | G is an undirected graph with a k-clique }

Theorem (Karp): CLIQUE is NP-complete

Why is it in NP?

Theorem: CLIQUE is NP-Complete



$3SAT \leq_P CLIQUE$

Transform every 3-cnf formula ϕ into (G,k) such that

$$\phi \in 3SAT \Leftrightarrow (G,k) \in CLIQUE$$

Want transformation that can be done in time that is polynomial in the length of ϕ

How can we encode a *logic* problem as a *graph* problem?

$3SAT \leq_P$ CLIQUE

We transform any 3-cnf formula ϕ into (G,k) such that

$$\phi \in 3SAT \Leftrightarrow (G,k) \in CLIQUE$$

Let $C_1, C_2, \ldots, C_m$ be clauses of ϕ, let $x_1, \ldots, x_n$ be vars.

Set $k := m$

Make a graph G with m *groups* of 3 nodes each.

Idea: Group *i* corresponds to clause C_i of ϕ

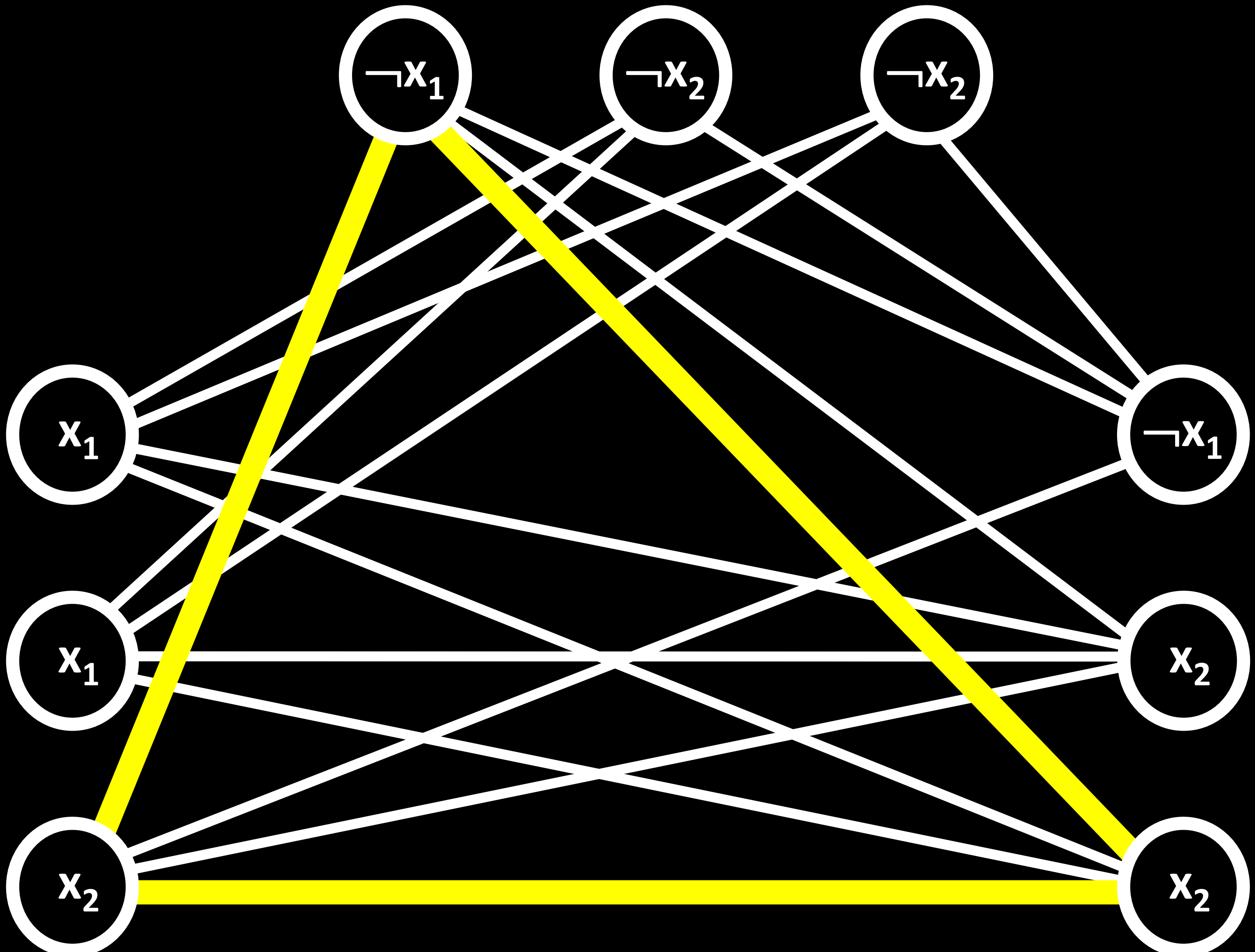
Each node in group *i* is "labeled" by a literal of C_i

(Note these labels do not actually appear in the graph!)

Put edges between all pairs of nodes in different groups, *except for pairs of nodes with labels x_i and $\neg x_i$*

Put no edges between nodes in the same group

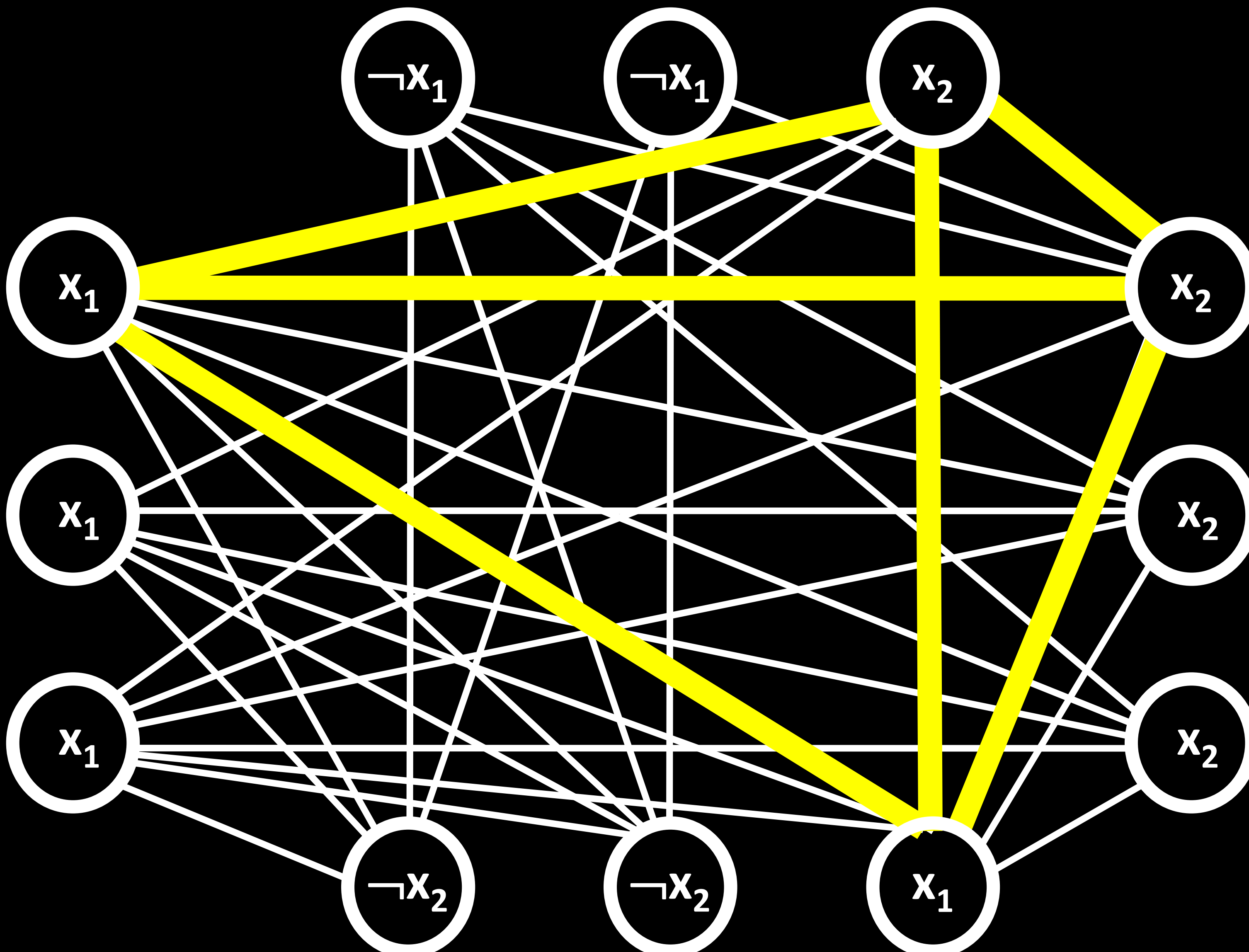
$(x_1 \vee x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_2) \wedge (\neg x_1 \vee x_2 \vee x_2)$



|V| = 3(number of clauses)

k = number of clauses

$(x_1 \vee x_1 \vee x_1) \wedge (\neg x_1 \vee \neg x_1 \vee x_2) \wedge$
$(x_2 \vee x_2 \vee x_2) \wedge (\neg x_2 \vee \neg x_2 \vee x_1)$



Claim: $\phi \in$ 3SAT $\Leftrightarrow$ $(G,m) \in$ CLIQUE

Claim: If $\phi \in$ 3SAT then $(G,m) \in$ CLIQUE

Proof: Let A be a SAT assignment of ϕ.

For each clause C of ϕ, there is a literal in C set true by A

Let v_C be that literal's corresponding vertex in G.

Claim: $S = \{v_C \mid C \text{ is a clause in } \phi\}$ is an m-clique in G.

Proof: Let $v_C \neq v_{C'}$ be in S. Suppose $(v_C, v_{C'}) \notin E$.

Note v_C and $v_{C'}$ are from different groups. So they must label *inconsistent* literals, call these literals x and $\neg x$

But assignment A cannot set true both x and $\neg x$!

Contradiction. So $(v_C, v_{C'}) \in E$, for all $v_C, v_{C'} \in S$.

Hence S is an m-clique, and $(G,m) \in$ CLIQUE

Claim: $\phi \in$ 3SAT $\Leftrightarrow$ (G,m) $\in$ CLIQUE

Claim: If (G,m) $\in$ CLIQUE then $\phi \in$ 3SAT

Proof: Let S be an m-clique of G.

We'll construct a satisfying assignment A of ϕ.

Claim: S contains *exactly one node* from each group of G.

For each variable x of ϕ, define variable assignment A:

A(x) := 1, if there is a vertex in S with label x,

A(x) := 0, if there is a vertex in S with label $\neg$x,

or no vertices in S are labeled x or $\neg$x

For all i = 1,…,m, one vertex from the i-th group is in S.

$\Rightarrow$ one literal from the i-th clause of ϕ is a vertex in S

So for all i = 1,…,m, A sets at least one literal true in i-th clause of ϕ. Therefore A is a satisfying assignment to ϕ.

Independent Set is NP-hard

IS: Given a graph $G = (V, E)$ and integer k,
is there $S \subseteq V$ such that $|S| \geq k$ and
no pair of vertices in S have an edge?

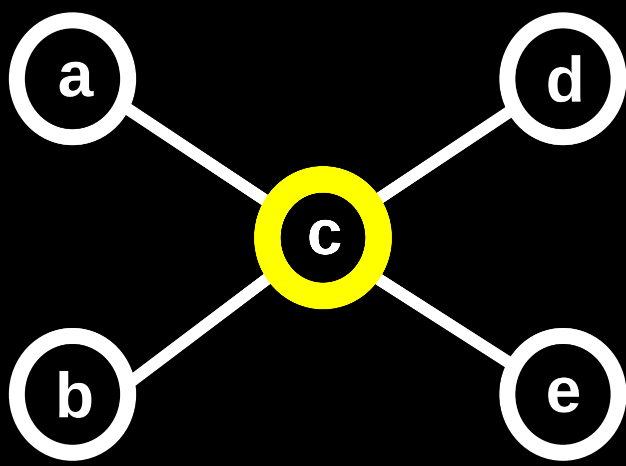
CLIQUE: Given $G = (V, E)$ and integer k,
is there $S \subseteq V$ such that $|S| \geq k$
and *every pair* of vertices in S have an edge?

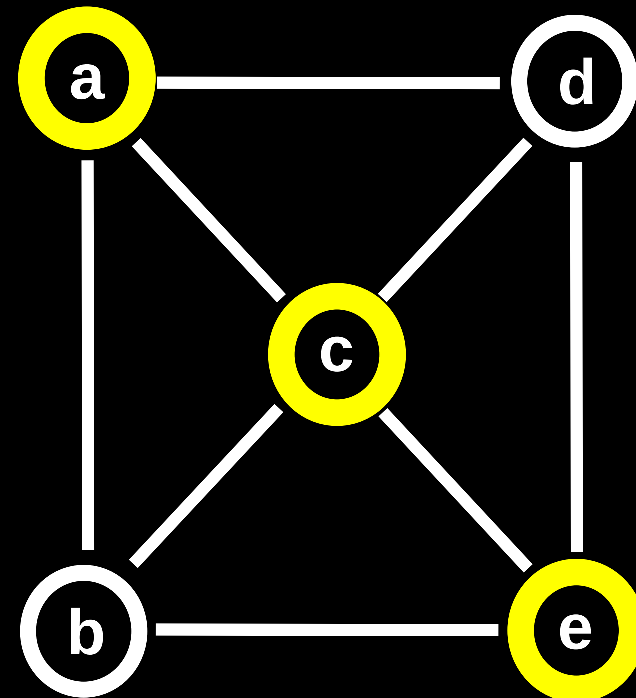
CLIQUE $\leq_P$ IS:
Given $G = (V, E)$, output $G' = (V, E')$ where
$E' = \{(u,v) \mid (u,v) \notin E\}$.

$(G, k) \in$ CLIQUE iff $(G', k) \in$ IS

each k-Clique in G is an k-IS in G'

The Vertex Cover Problem





vertex cover = set of nodes C that cover all edges

For all edges, at least one endpoint is in C

VERTEX-COVER = { (G,k) | G is a graph with a vertex cover of size at most k}

Theorem: VERTEX-COVER is NP-Complete

(1) VERTEX-COVER $\in$ NP

(2) IS $\leq_P$ VERTEX-COVER

Want to transform a graph G and integer k into G’ and k’ such that

$$(G,k) \in \text{IS} \Leftrightarrow (G',k') \in \text{VERTEX-COVER}$$

IS $\leq_P$ VERTEX-COVER

Claim: For every graph $G = (V,E)$, and subset $S \subseteq V$,

S is an independent set
if and only if $(V - S)$ is a vertex cover

Proof: S is an independent set

$\Leftrightarrow (\forall\, u, v \in V)[\, (u \in S \text{ and } v \in S) \Longrightarrow (u,v) \notin E\,]$

$\Leftrightarrow (\forall\, u, v \in V)[\, (u,v) \in E \Longrightarrow (u \notin S \text{ or } v \notin S)\,]$

$\Leftrightarrow (V - S)$ is a vertex cover!

Therefore $(G,k) \in \text{IS} \Leftrightarrow (G,|V| - k) \in \text{VERTEX-COVER}$

Our polynomial time reduction: $f(G,k) := (G, |V| - k)$

The Subset Sum Problem

Given: Set $S = \{a_1, \ldots, a_n\}$ of positive integers and a positive integer t

Is there an $A \subseteq \{1, \ldots, n\}$ such that $t = \sum_{i \in A} a_i$?

SUBSET-SUM $= \{(S, t) \mid \exists\, S' \subseteq S \text{ s.t. } t = \sum_{b \in S'} b\}$

A simple summation problem!

Theorem (in algs): There is a $O(n \cdot t)$ time algorithm for solving SUBSET-SUM.

But t can be specified in $(\log t)$ bits… this isn't an algorithm that runs in poly-time in the input!

The Subset Sum Problem

Given: Set $S = \{a_1,\ldots, a_n\}$ of positive integers and a positive integer t

Is there an $A \subseteq \{1, \ldots ,n\}$ such that $t = \sum_{i \in A} a_i$?

SUBSET-SUM = $\{(S, t) \mid \exists\, S' \subseteq S \text{ s.t. } t = \sum_{b \in S'} b \}$

A simple summation problem!

Theorem: SUBSET-SUM is NP-complete

VC $\leq_P$ SUBSET-SUM

Want to reduce a *graph* to a *set of numbers*

Given (G, k), let $E = \{e_0,\ldots,e_{m-1}\}$ and $V = \{1,\ldots,n\}$

Our subset sum instance (S, t) will have $|S| = n + m$

"Edge numbers":

For every $e_j \in E$, put $b_j = 4^j$ in S

"Node numbers":

For every $i \in V$, put $a_i = 4^m + \sum_{j\,:\,i \in e_j} 4^j$ in S

Set the target number: $t = k \cdot 4^m + \sum_{j=0}^{m-1} (2 \cdot 4^j)$

Think of the numbers as being in "base 4"…

as vectors with m+1 components

For every $e_j \in E$ $(j=0,\ldots,m-1)$ put $b_j = 4^j$ in S

For every $i \in V$, put $a_i = 4^m + \sum_{j\,:\,i \in e_j} 4^j$ in S

Set $t = k \cdot 4^m + \sum_{j=0}^{m-1} (2 \cdot 4^j)$

Claim: If $(G,k) \in VC$ then $(S,t) \in$ SUBSET-SUM

Suppose $C \subseteq V$ is a VC with k vertices.
Define $S' = \{a_i : i \in C\} \cup \{b_j : |e_j \cap C| = 1\}$

S' = (*node numbers* corresponding to nodes in C) *plus*
(*edge numbers* corresponding to edges covered *only once* by C)

Claim: The sum of all numbers in S' equals t

$$\sum_{i \in C} a_i = k \cdot 4^m + \sum_{i \in C} \left(\sum_{j\,:\,i \in e_j} 4^j\right)$$
$$= k \cdot 4^m + \sum_{j\,:\,e_j \text{ covered once by } C} 4^j + \sum_{j\,:\,e_j \text{ covered twice by } C} (2 \cdot 4^j)$$

$$\sum_{j\,:\,|e_j \cap C| = 1} b_j = \sum_{j\,:\,e_j \text{ covered once by } C} 4^j$$

Total sum is t

For every $e_j \in E$ ($j=0,\ldots,m-1$) put $b_j = 4^j$ in S

For every $i \in V$, put $a_i = 4^m + \sum_{j\,:\,i \in e_j} 4^j$ in S

Set $t = k \cdot 4^m + \sum_{j=0}^{m-1} (2 \cdot 4^j)$

Claim: If $(S,t) \in$ SUBSET-SUM then $(G,k) \in$ VC

Suppose $C \subseteq V$ and $F \subseteq E$ satisfy

$$\sum_{i \in C} a_i + \sum_{e_j \in F} b_j = t = k \cdot 4^m + \sum_{j=0}^{m-1} (2 \cdot 4^j)$$

Claim: C is a vertex cover of size k.

Proof: Subtract the b_j numbers from the LHS.

Each $b_j = 4^j$. So what remains is a sum of the form:

$$\sum_{i \in C} a_i = k \cdot 4^m + \sum_{j=0}^{m-1} (c_j \cdot 4^j)$$

where each $c_j > 0$. But c_j = number of nodes in C covering e_j

Therefore every e_j is covered by C, so C is a vertex cover!

Moreover, $|C| = k$: each a_i in C adds 4^m to t

The Knapsack Problem

Given: $S = \{(v_1,c_1)\ldots, (v_n,c_n)\}$ of pairs of positive integers (items)

a capacity budget C

a value target V

Is there an $S' \subseteq \{1,\ldots,n\}$ such that

$(\sum_{i \in S'} v_i) \geq V$ and $(\sum_{i \in S'} c_i) \leq C$?

Define: KNAPSACK = {(S, C, V) | the answer is yes}

A classic economics/logistics/OR problem!

Theorem: KNAPSACK is NP-complete

KNAPSACK is NP-complete

KNAPSACK is in NP?

Theorem: SUBSET-SUM $\leq_P$ KNAPSACK

Proof: Given an instance $(S = \{a_1,\ldots,a_n\}, t)$ of SUBSET-SUM, create a KNAPSACK instance:

For all i, set $(v_i, c_i) := (a_i, a_i)$

Define $T = \{(v_1, c_1),\ldots, (v_n, c_n)\}$

Define $C := V := t$

Then, $(S,t) \in$ SUBSET-SUM $\Leftrightarrow$ $(T,C,V) \in$ KNAPSACK

Subset of S that sums to t = Solution to the Knapsack instance!