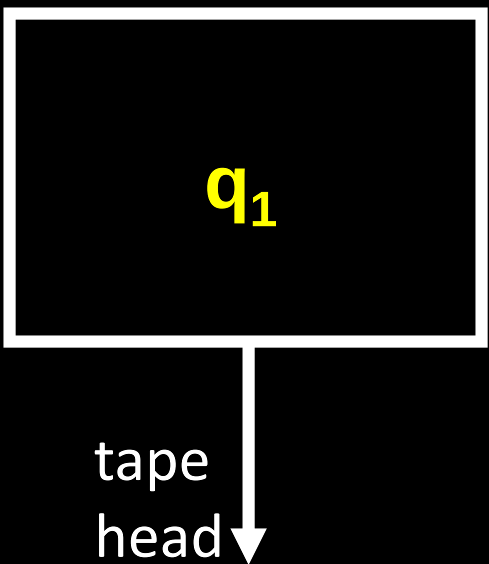
6.045

Lecture 9
Turing Machines: Recognizability, Decidability, The Church-Turing Thesis

Turing Machine (1936)



In each step:
- Reads a symbol
- Writes a symbol
- Changes state
- Moves Left or Right

INFINITE REWRITABLE TAPE

Turing Machine (1936)

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

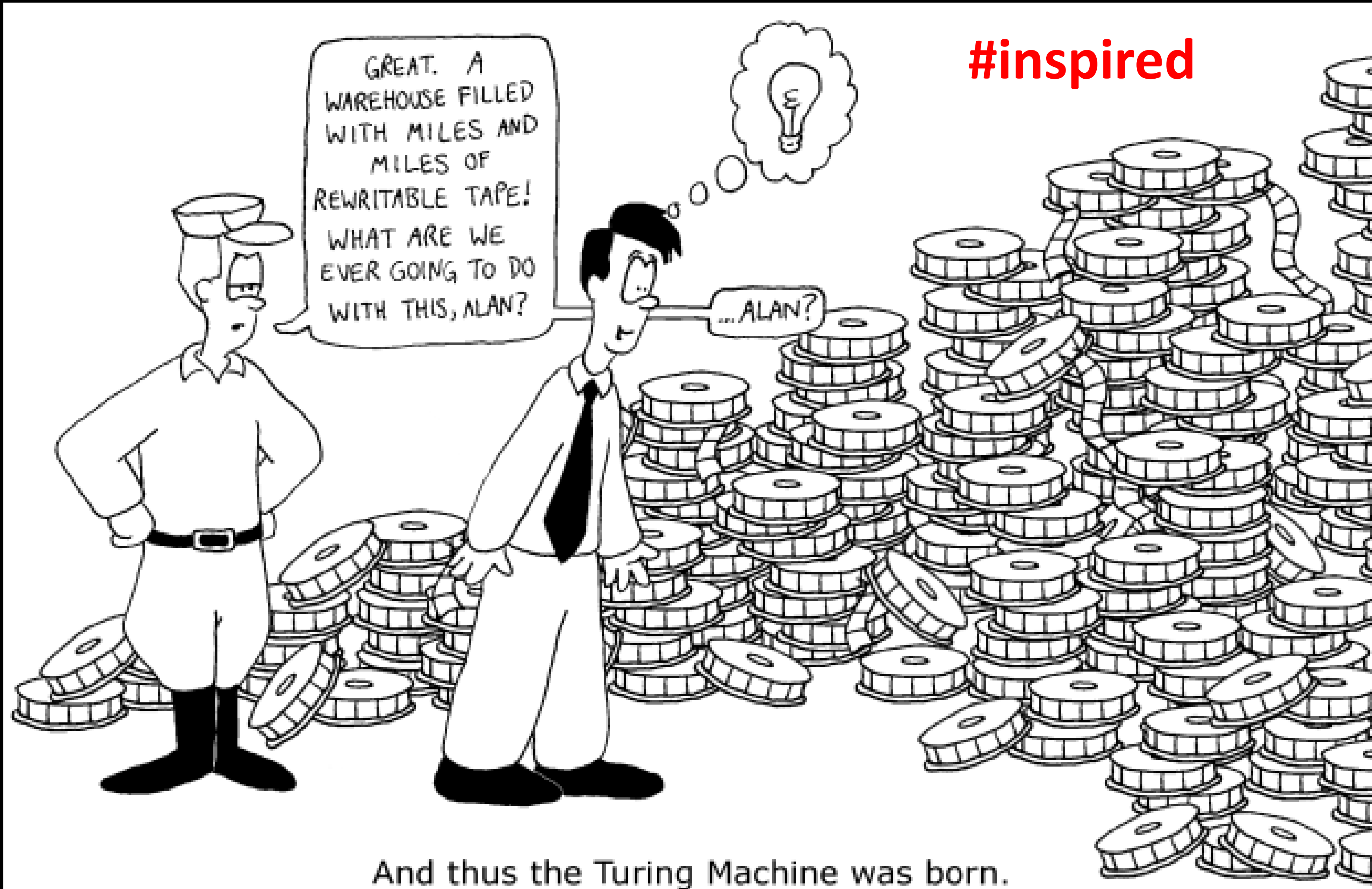
By A. M. TURING.

[Received 28 May, 1936.—Read 12 November, 1936.]

The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers,

…

#inspired



And thus the Turing Machine was born.

https://www.cs.utah.edu/~draperg/cartoons/2005/turing.html

Turing Machines versus DFAs

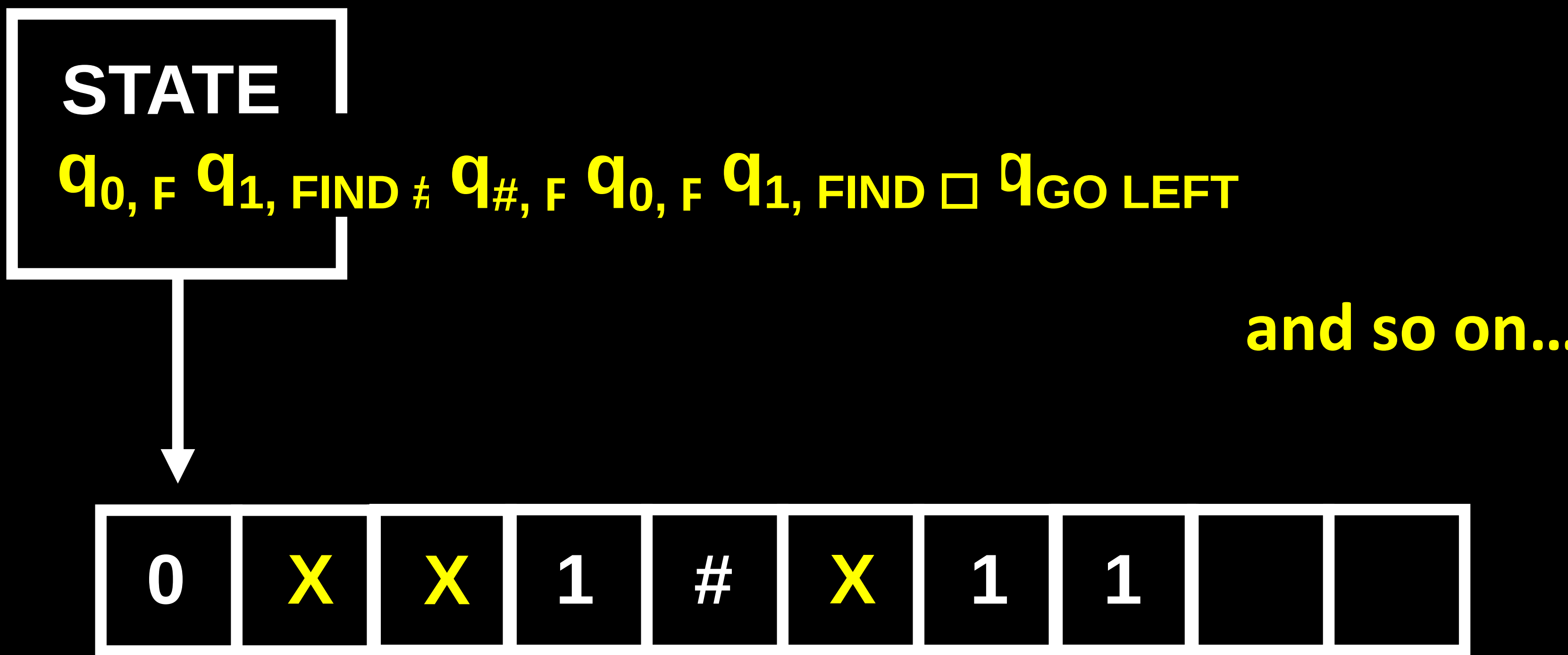
The input is written on an infinite tape
with "blank" symbols after the input

The "tape head" can move *right and left*

The TM can both *write to* and *read from* the tape,
and can write symbols that aren't part of input

Accept and Reject take immediate effect

A TM for $L = \{ w\#w \mid w \in \{0,1\}^* \}$ over $\Sigma=\{0,1,\#\}$



1. If there's no # on the tape (or more than one #), ***reject***.
2. While there is a bit to the left of #,

   Replace the first bit **b** with **X**, and check if the first bit b' to the right of the # is identical to **b**. (If not, ***reject***.)

   Replace that bit b' with an **X** too.
3. If there's a bit to the right of #, then ***reject*** else ***accept***

Definition: A Turing Machine is a 7-tuple $T = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where:

$\square$ = "blank"

Q is a finite set of states

Σ is the input alphabet, where $\square \notin \Sigma$

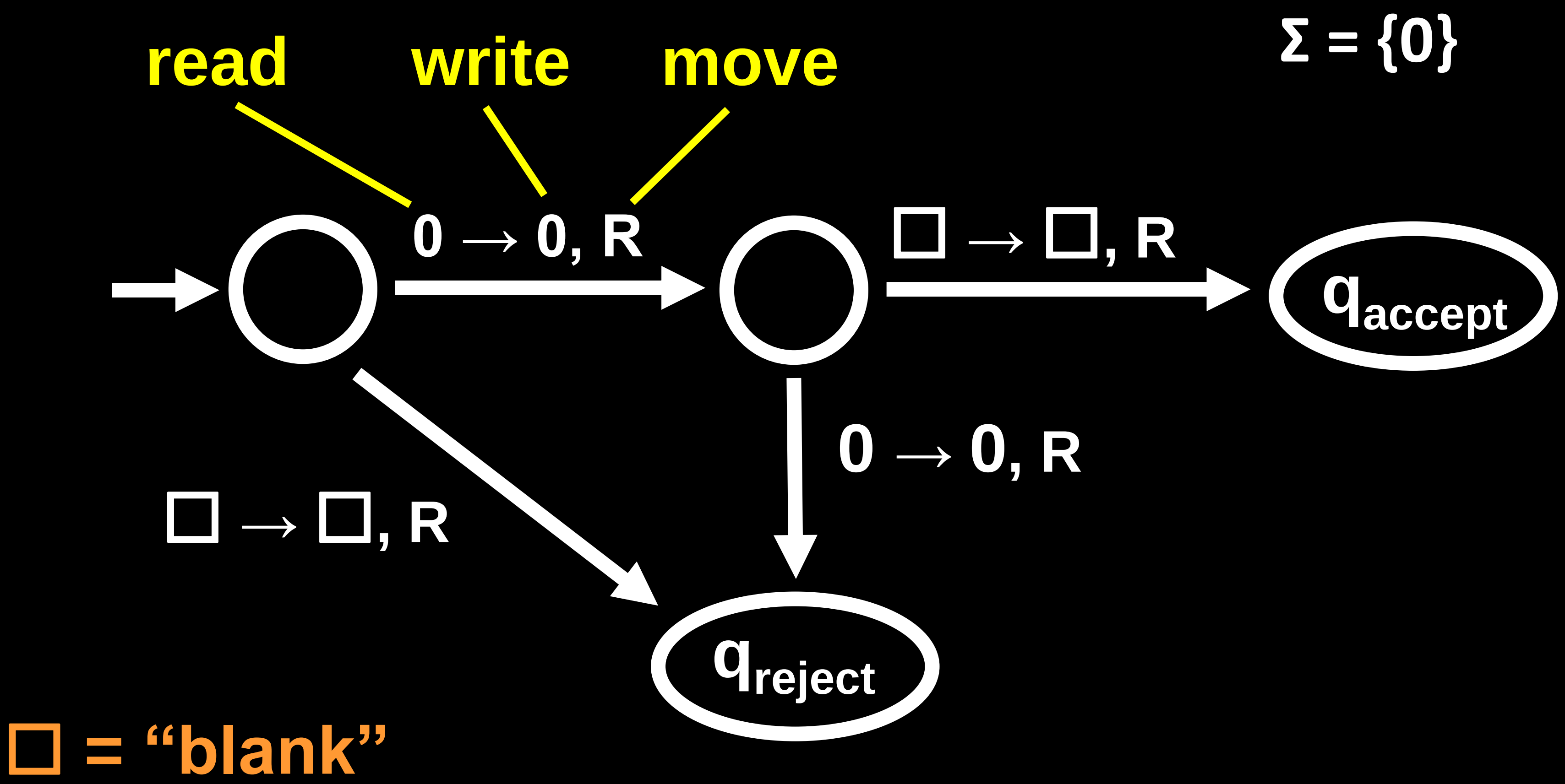
Γ is the tape alphabet, where $\square \in \Gamma$ and $\Sigma \subseteq \Gamma$

$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$

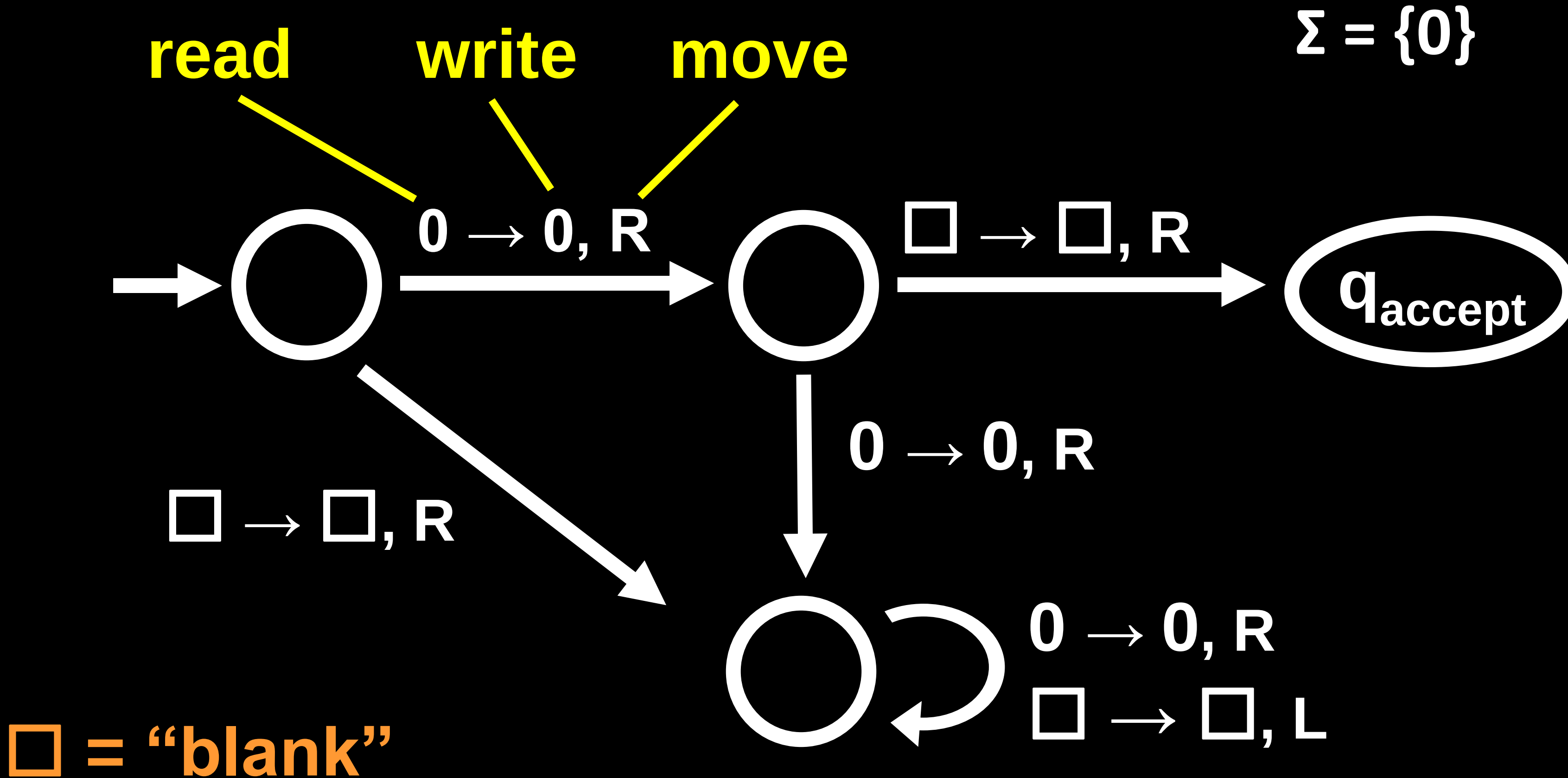
$q_0 \in Q$ is the start state

$q_{accept} \in Q$ is the accept state

$q_{reject} \in Q$ is the reject state, and $q_{reject} \neq q_{accept}$



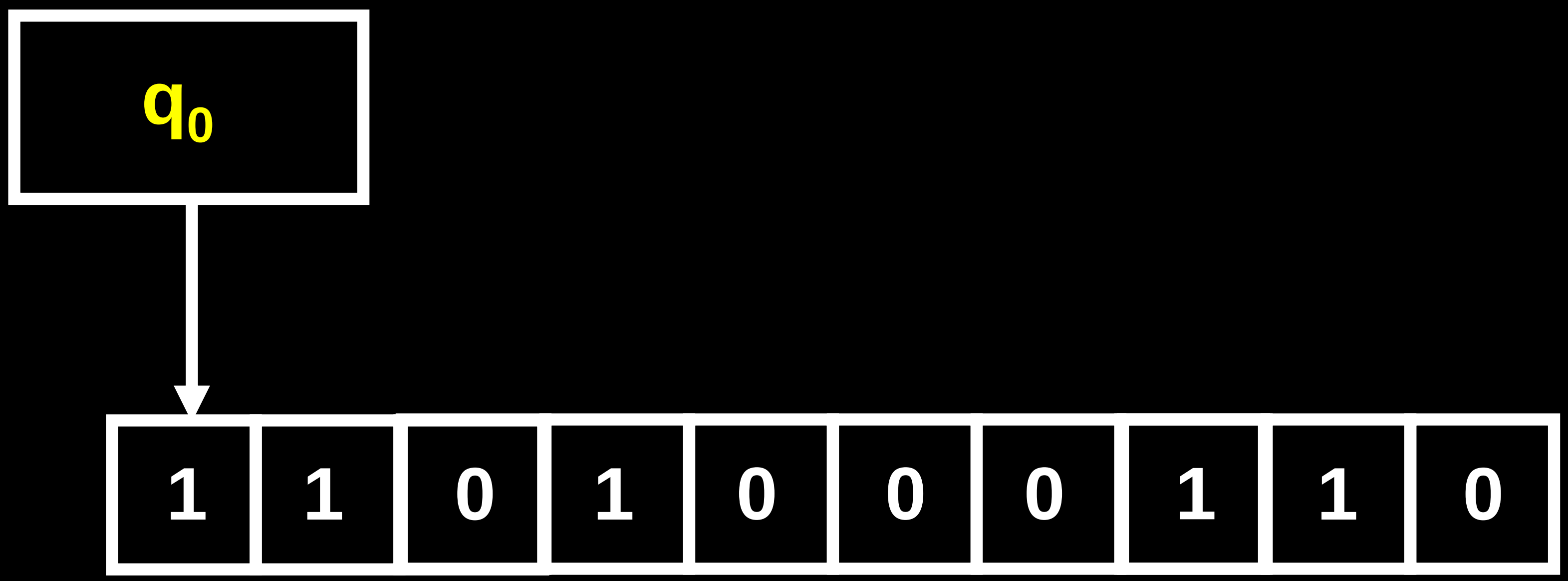
This Turing machine *decides* the language {0}



This Turing machine *recognizes* the language {0}

Three kinds of behaviors:
accepting, rejecting, and running forever!

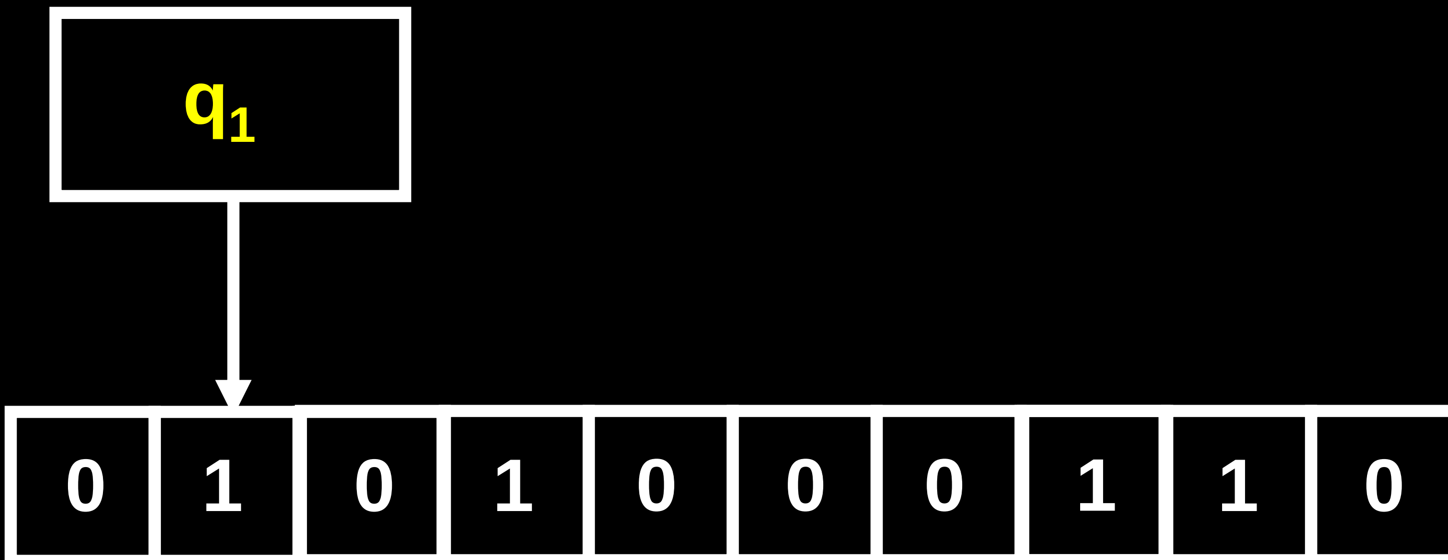
Turing Machine Configurations



corresponds to the *configuration*:

$$q_0 1101000110 \in (Q \cup \Gamma)^*$$

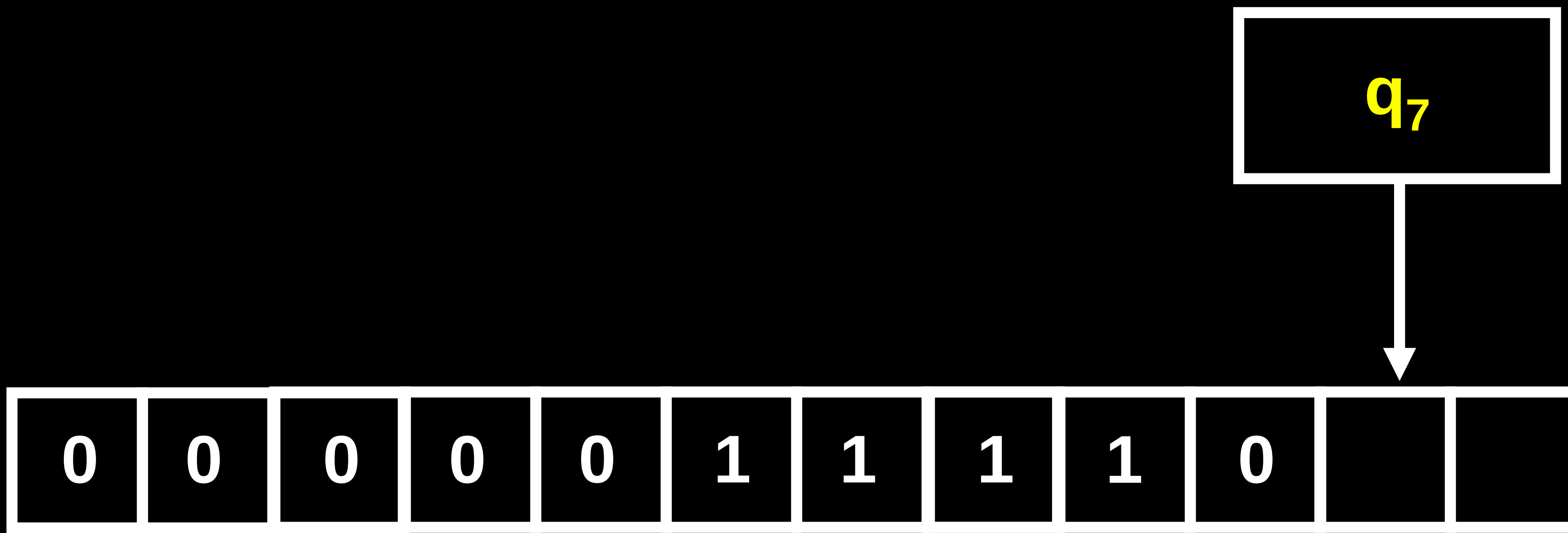
Turing Machine Configurations



corresponds to the *configuration*:

$$0q_1101000110 \in (Q \cup \Gamma)^*$$

Turing Machine Configurations



corresponds to the *configuration*:

$$0000011110q_7\square \in (Q \cup \Gamma)^*$$

Defining Acceptance and Rejection for TMs

Let C_1 and C_2 be configurations of a TM M

Definition. C_1 *yields* C_2 if M is in configuration C_2 after running M in configuration C_1 for one step

Example. Suppose $\delta(q_1, b) = (q_2, c, L)$

Then aq_1bb yields q_2acb

Suppose $\delta(q_1, a) = (q_2, c, R)$

Then abq_1a yields $abcq_2\square$

accepting computation history of M on x

Let $w \in \Sigma^*$ and M be a Turing machine.

M *accepts* w if there are configs $C_0, C_1, \ldots, C_k$, s.t.

- $C_0 = q_0w$ [the initial configuration]
- C_i yields C_{i+1} for i = 0, ..., k-1, and
- C_k contains the accept state q_{accept}

A TM *M* *recognizes* a language L
if *M* accepts exactly those strings in L

A language L is *recognizable*
(a.k.a. recursively enumerable)
if some TM recognizes L

A TM *M* *decides* a language L if *M* accepts all strings in L and rejects all strings not in L

A language L is *decidable (a.k.a. recursive)*
if some TM decides L

L(M) := set of strings M accepts

A Turing machine for deciding $\{ 0^{2^n} \mid n \geq 0 \}$

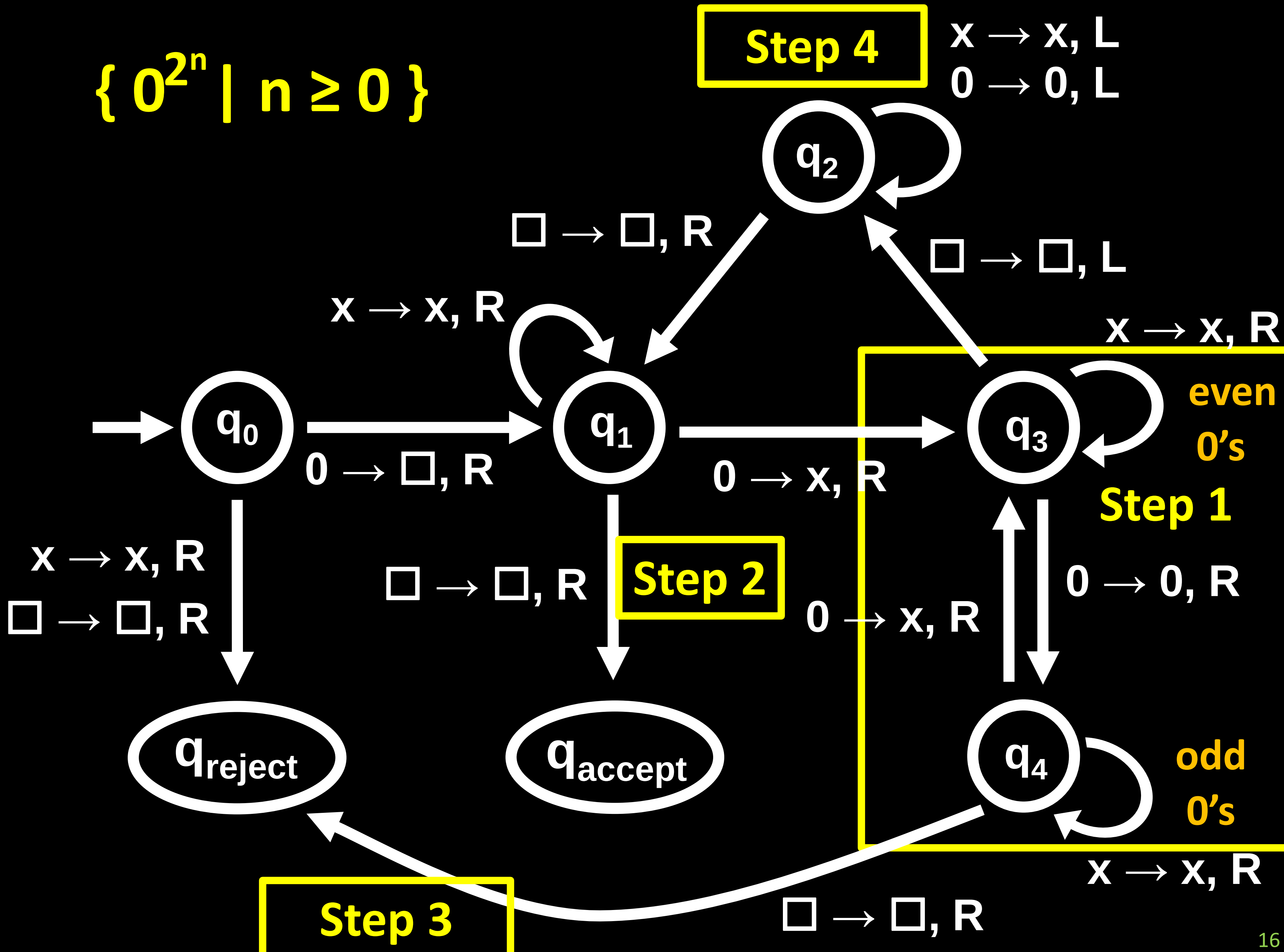
Turing Machine PSEUDOCODE:

1. Sweep from left to right, **x**-out every other **0**
2. If in step 1, the tape had only one **0**, *accept*
3. If in step 1, the tape had an **odd number** of **0**'s (at least 3), *reject*
4. Move the head left to the first input symbol.
5. Go to step 1.

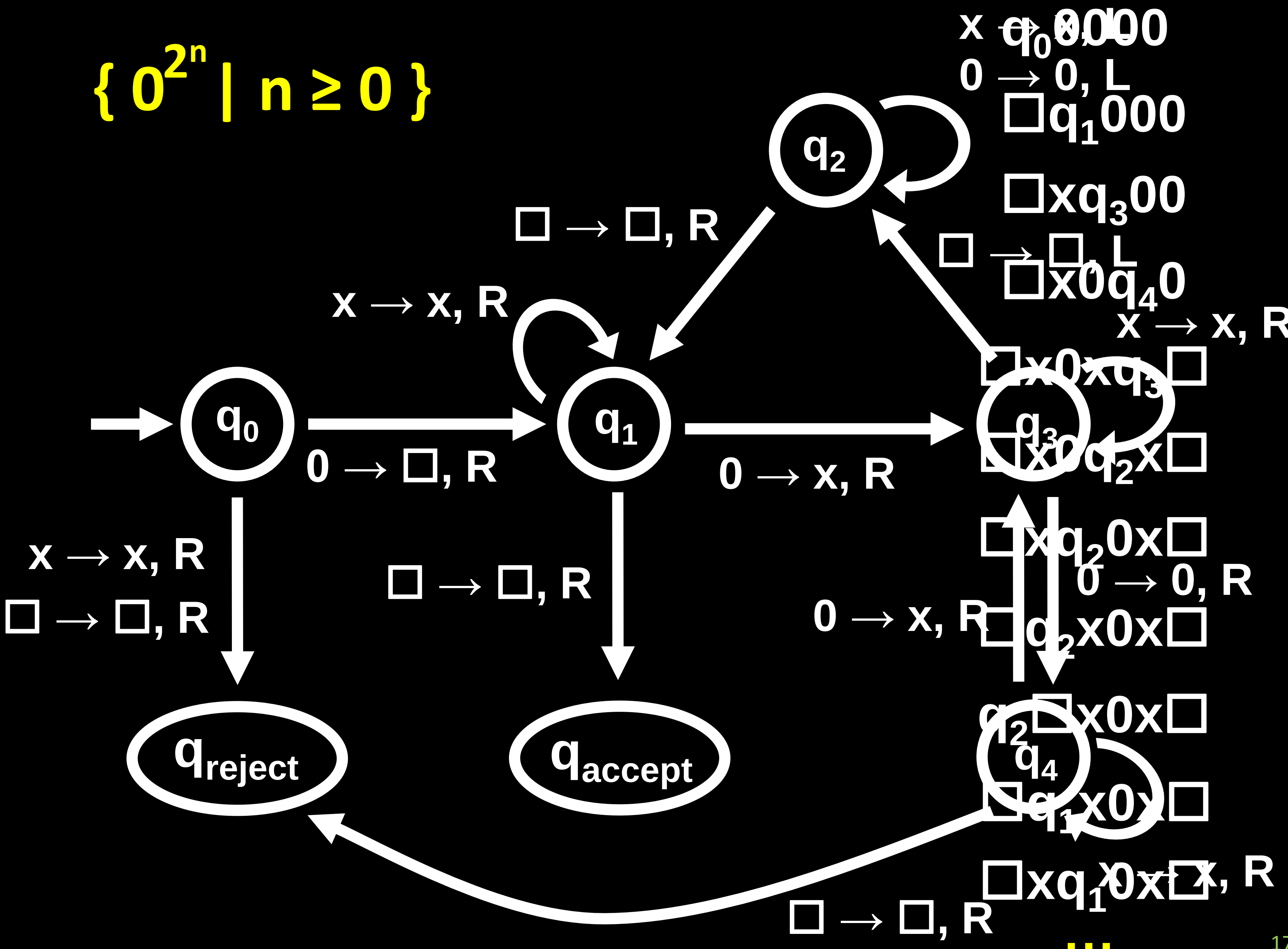
Why does this work?

Observation: Every time we return to step 1, the number of 0's on the tape has been halved.

$\{ 0^{2^n} \mid n \geq 0 \}$



$\{ 0^{2^n} \mid n \geq 0 \}$



MULT = $\{a^i b^j c^k \mid k = i*j, \text{ and } i, j, k \geq 1\}$

TURING MACHINE PSEUDOCODE:

1. If the input doesn't match **a*b*c***, *reject.*
2. Move the head back to the leftmost symbol.
3. Cross off one **a**, scan to the right until see **b**.
   Sweep between **b**'s and **c**'s, crossing off one of each until all **b**'s are crossed off.
   If all **c**'s get crossed off while doing this, *reject.*
4. *Uncross* all the **b**'s.
   If there is some **a** left, then repeat stage 3.
   If all **a**'s are crossed off,
      Check if all **c**'s are crossed off.
      If yes, then *accept*, else *reject.*

MULT = {$a^ib^jc^k$ | k = i*j, and i, j, k ≥ 1}

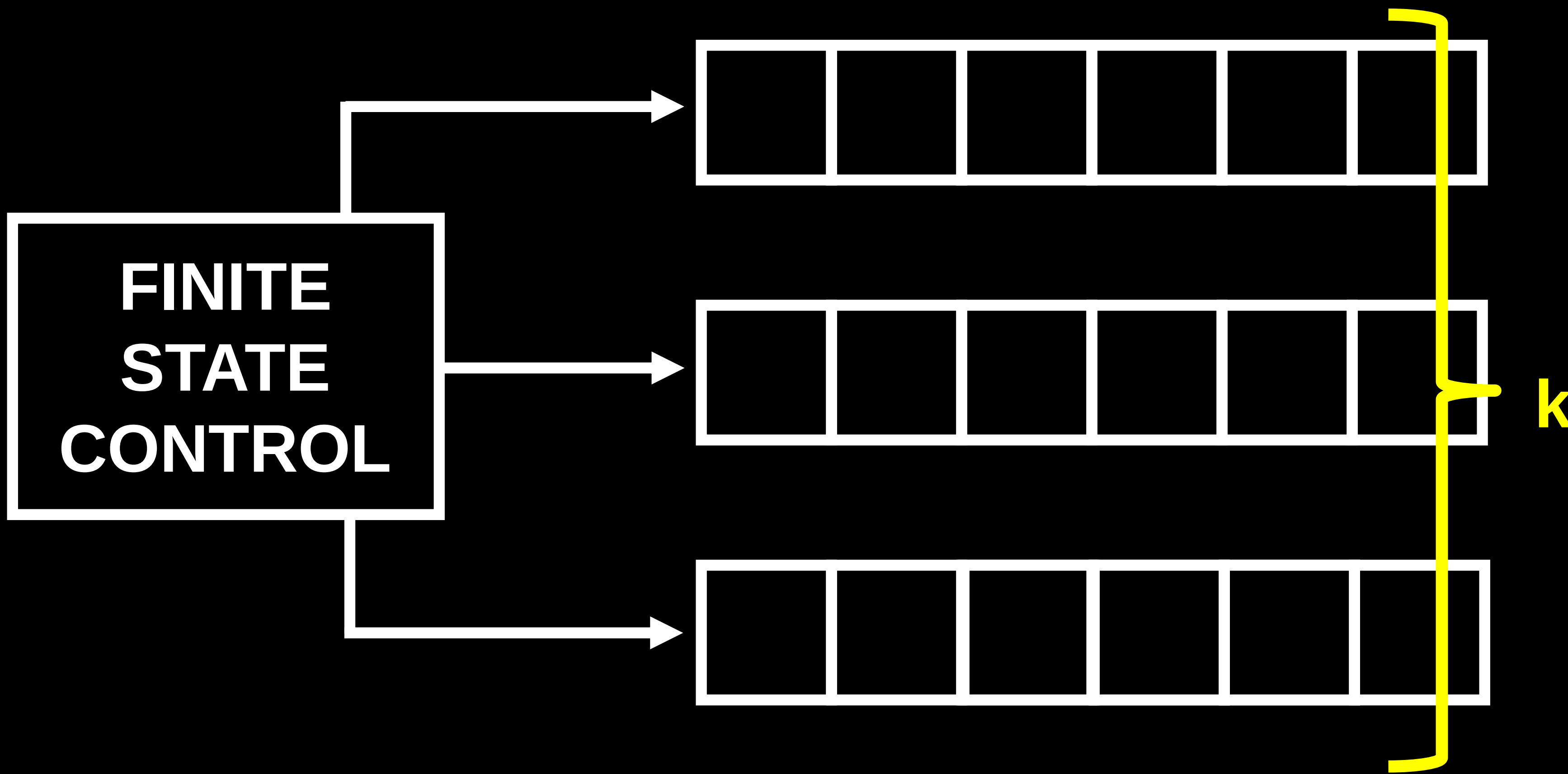
Check matches a*b*c*	aabbbcccccc
Cross off an a	ảabbbcccccc
Cross off one c for each b	ảab̍b̍b̍ḉḉḉccc
"Uncross" the b's	ảabbbḉḉḉccc
Repeat the crossing, until all a's crossed (or reject early)	ảảb̍b̍b̍ḉḉḉḉḉḉ
	Accept

Turing Machines are Robust!

Many variants and models can be defined. As long as your favorite model reads and writes a finite number of symbols in each step, it doesn't matter!

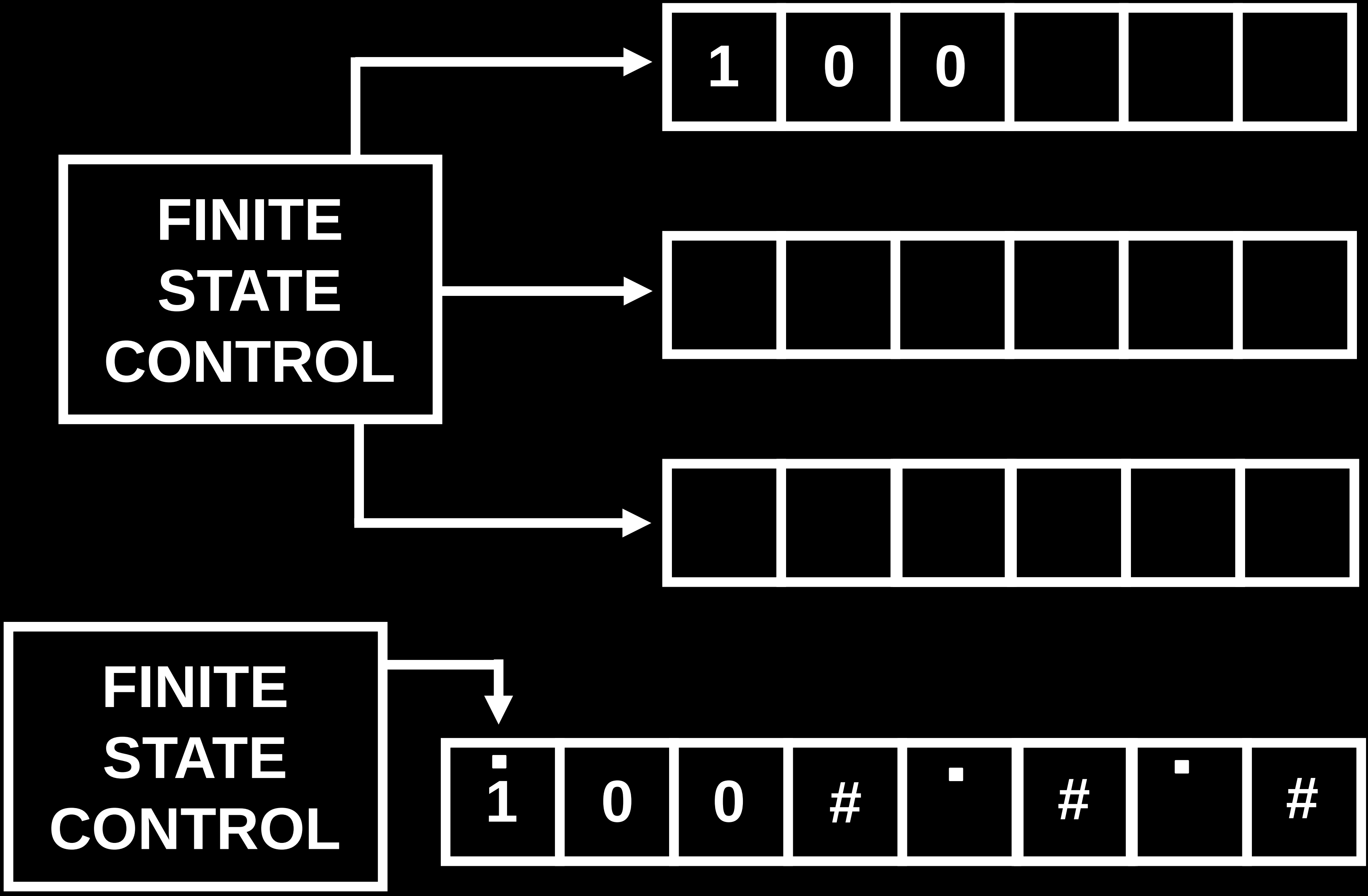
A good ole TM can still simulate it!

Multitape Turing Machines



$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L,R\}^k$$

Theorem: Every Multitape Turing Machine can be transformed into a single tape Turing Machine



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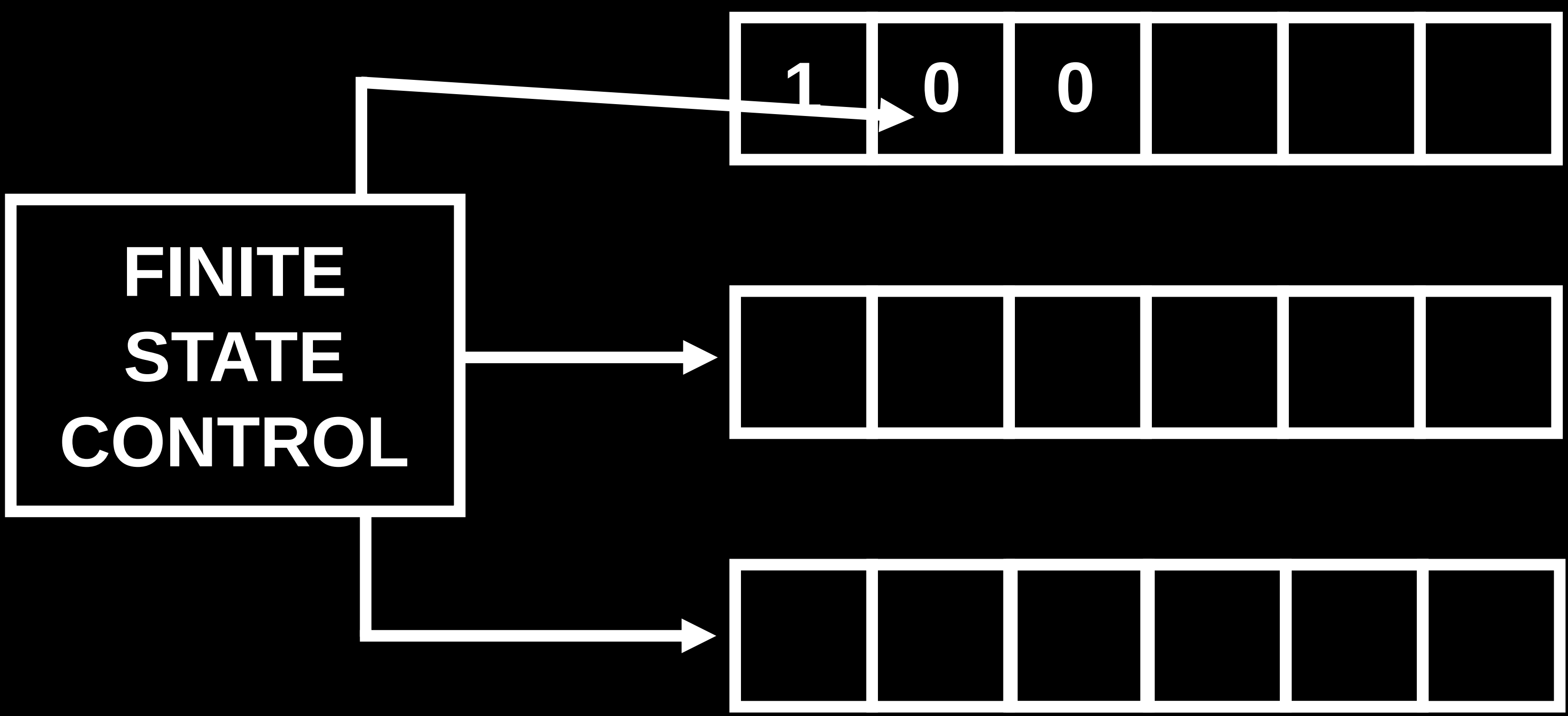

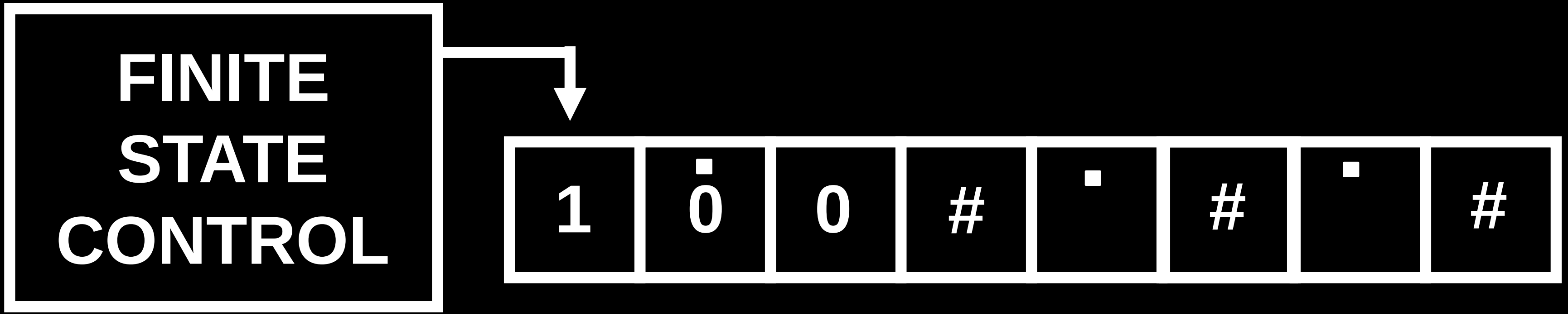

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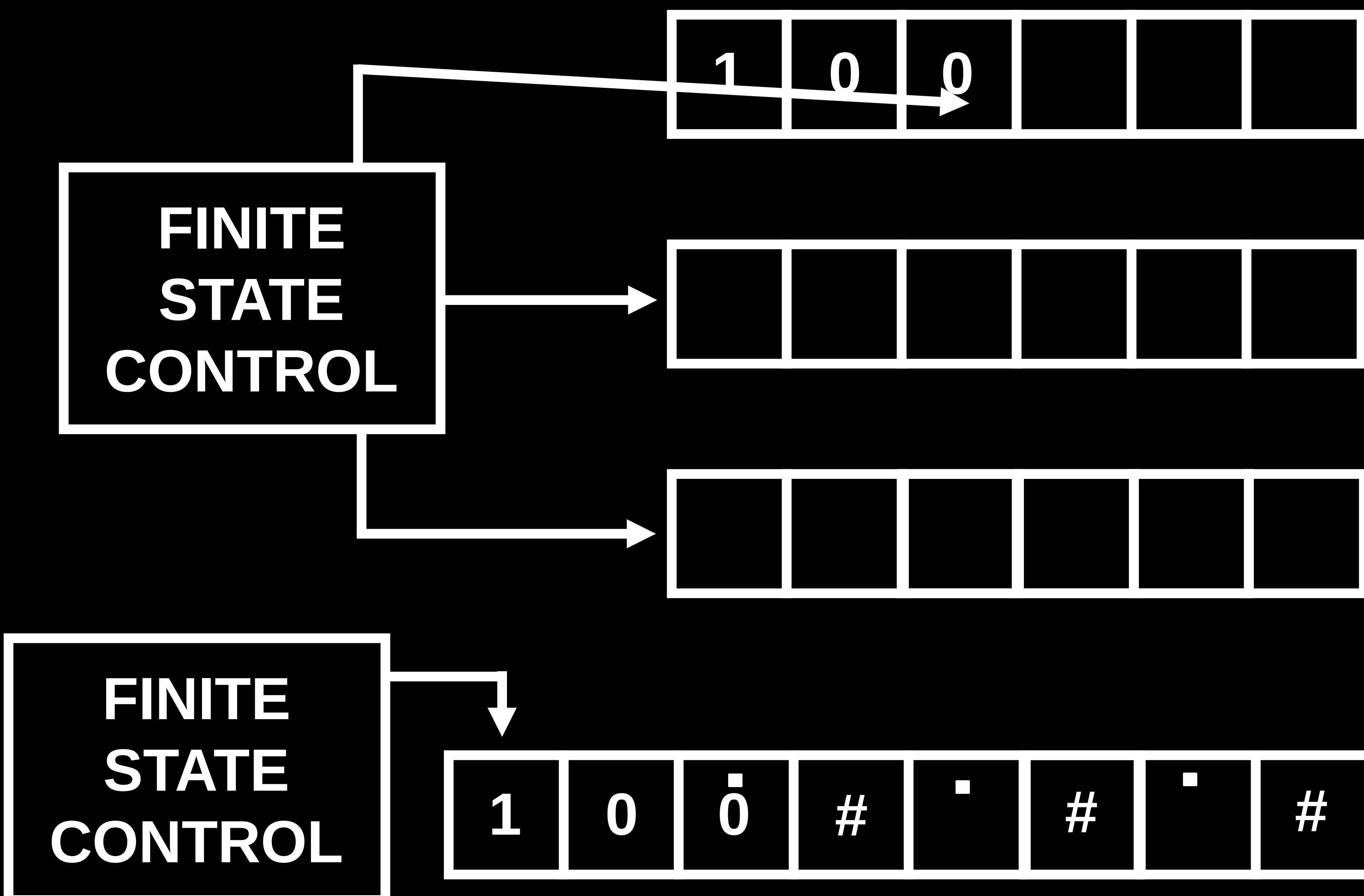

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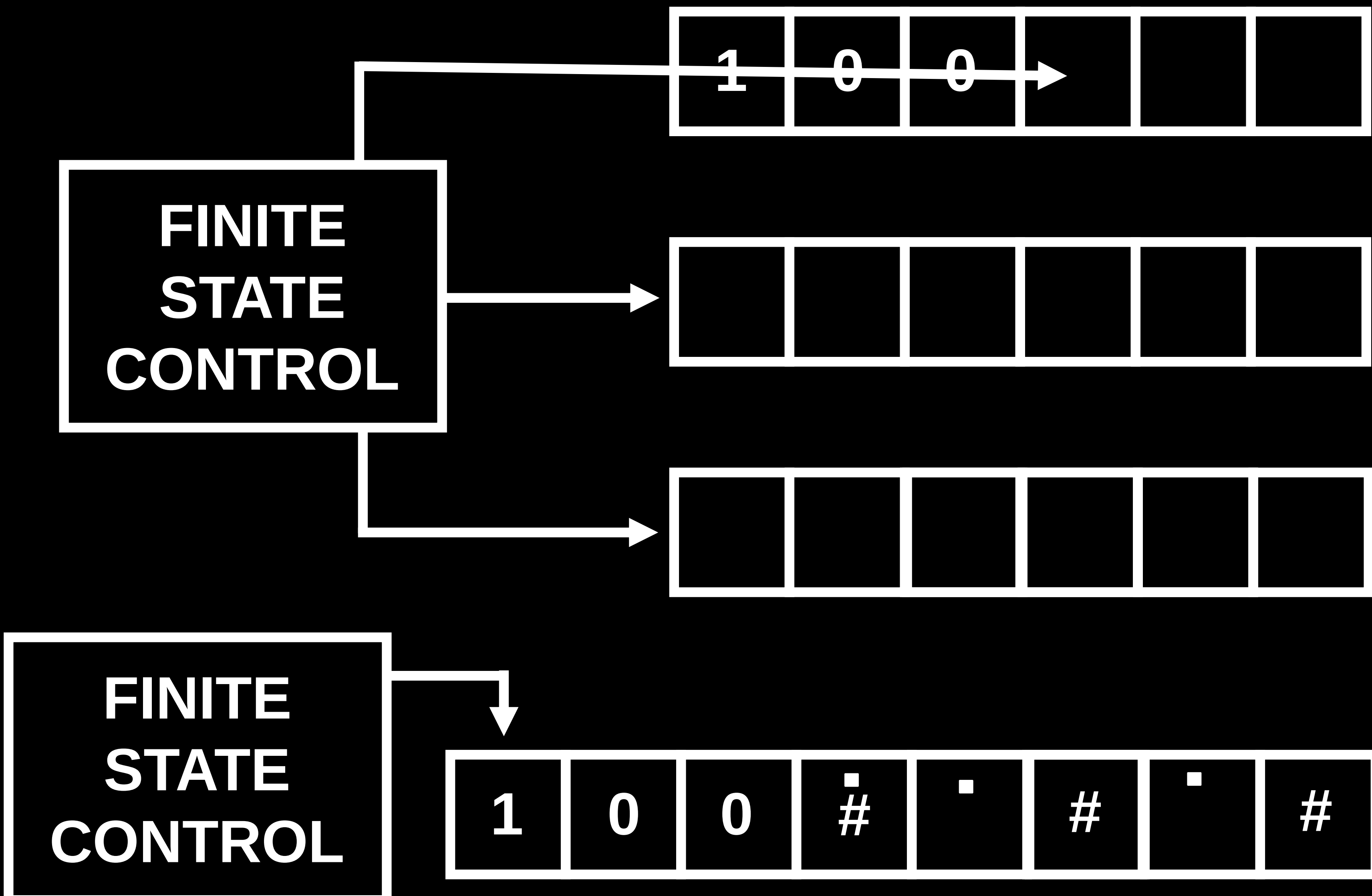
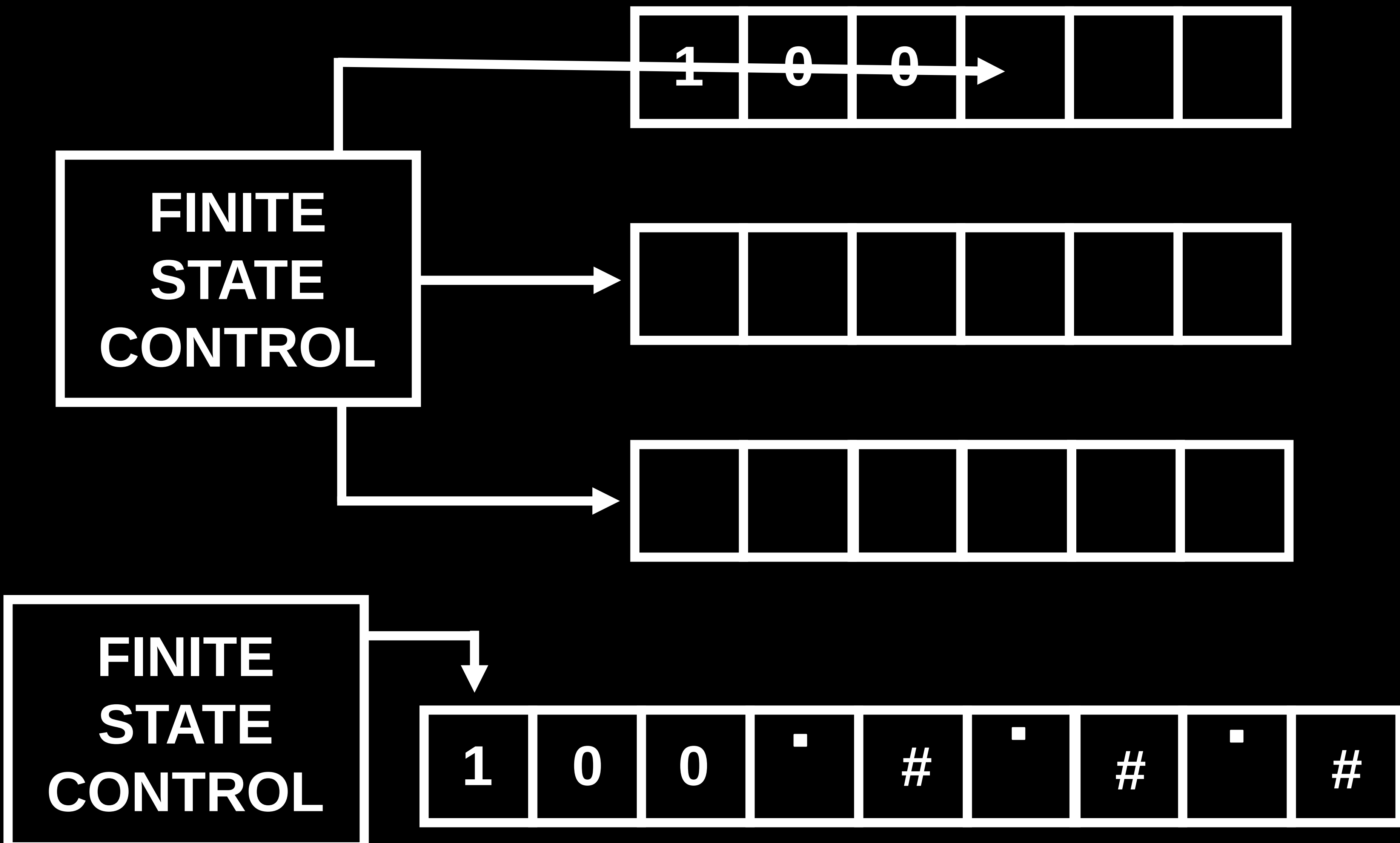


Theorem: Every Multitape Turing Machine can be transformed into a single tape Turing Machine



Nondeterministic Turing Machines

Have multiple transitions for a state, symbol pair

Theorem: Every nondeterministic Turing machine N can be transformed into a Turing Machine M that accepts precisely the same strings as N. (L(M)=L(N))

Proof Idea (more details in Sipser p.178-179)

Pick a natural ordering on the strings in $(Q \cup \Gamma \cup \#)^*$

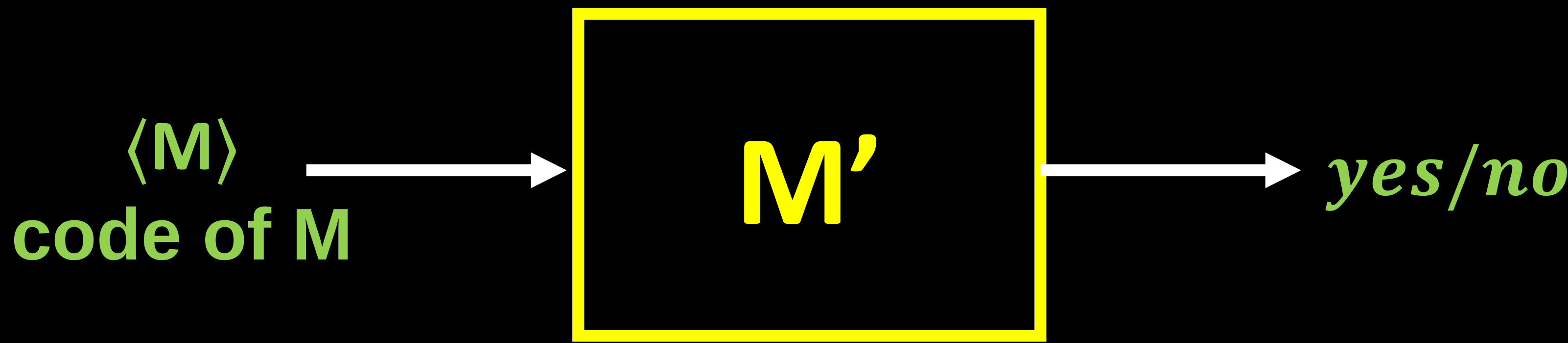
M(w): For all strings $D \in (Q \cup \Gamma \cup \#)^*$ in the ordering,

Check if $D = C_0\# \cdots \#C_k$ where $C_0, \ldots, C_k$ is *an accepting computation history* for N on w.

If so, *accept*.

What else can Turing Machines do?

They can analyze and simulate other TMs



To do that, we need to encode TMs as strings.

Fact: We can encode Turing Machines as bit strings

n states

start state

reject state

$$0^n 1 0^m 1 0^k 1 0^s 1 0^t 1 0^r 1 0^u 1 \ldots$$

m tape symbols (first k are input symbols)

accept state

blank symbol

$$((p, i), (q, j, L)) = 0^p 1 0^i 1 0^q 1 0^j 1 0 1$$

$$((p, i), (q, j, R)) = 0^p 1 0^i 1 0^q 1 0^j 1 0 0 1$$

Can map every TM M to a bit string $\langle M \rangle$

We can also encode DFAs and NFAs as *bit strings*, and $w \in \Sigma^*$ as *bit strings*

For $x \in \Sigma^*$ define $b_\Sigma(x)$ to be its binary encoding

For $x, y \in \Sigma^*$, define the *pair of x and y* as a binary string encoding both x and y

$$\langle x, y \rangle := 0^{|b_\Sigma(x)|} 1\, b_\Sigma(x)\, b_\Sigma(y)$$

Then we define the following languages over $\{0,1\}$:

$A_{DFA} = \{ \langle D, w \rangle \mid$ D encodes a DFA over some Σ, and D accepts $w \in \Sigma^* \}$

$A_{NFA} = \{ \langle N, w \rangle \mid$ N encodes an NFA, N accepts w $\}$

$A_{TM} = \{ \langle M, w \rangle \mid$ M encodes a TM, M accepts w $\}$

Universal Turing Machines

Theorem: There is a Turing machine U
which takes as input:
- the code of an arbitrary TM M
- and an input string w

such that U accepts $\langle M, w\rangle \Leftrightarrow$ M accepts w.

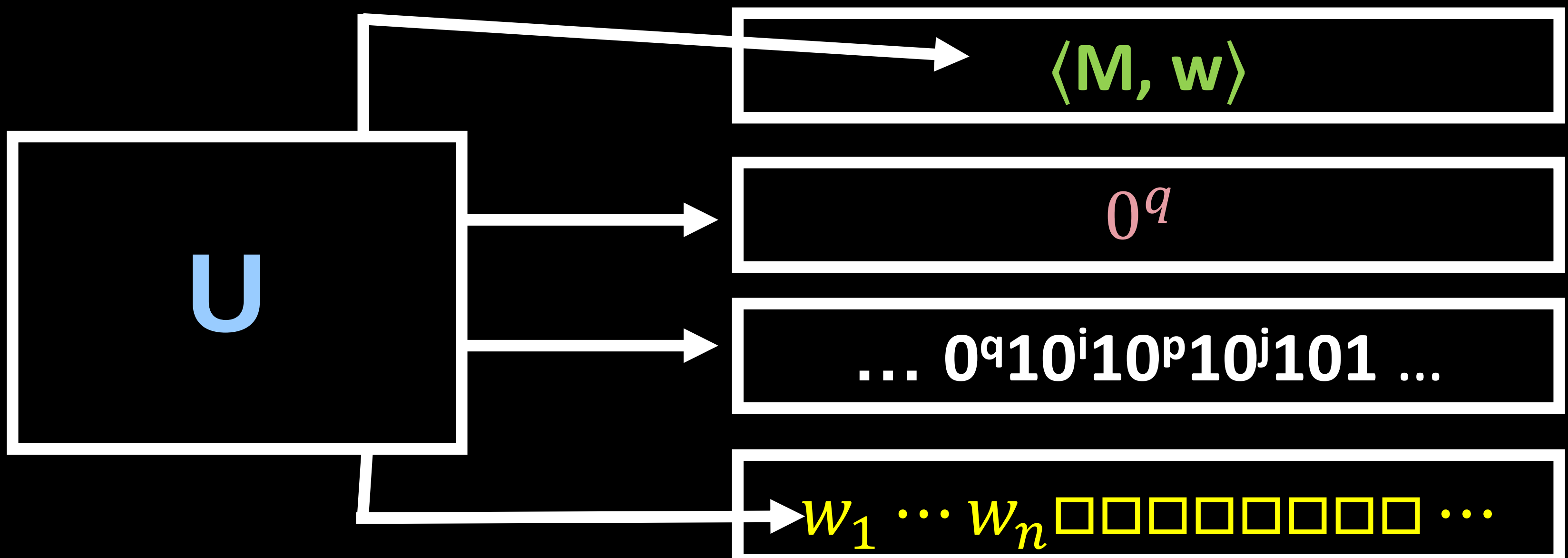
This is a *fundamental* property of TMs:
There is a Turing Machine that
can run arbitrary Turing Machine code!

Note that DFAs/NFAs do *not* have this property.
That is, A_{DFA} and A_{NFA} are not regular.

Want: U accepts ⟨M, w⟩ ⇔ M accepts w.

Can make a multitape TM U with four tapes:

1. Input tape: receives ⟨M, w⟩
2. State tape: holds the current state of M
3. Machine code tape: holds transitions of M
4. Simulation tape: content is identical to M's tape



For each step of M: U looks up the matching transition in machine code tape, updates the state and simulation tape

$A_{DFA} = \{ \langle D, w \rangle \mid D \text{ is a DFA that accepts string } w \}$

Theorem: A_{DFA} is decidable

Proof: A DFA is a special case of a TM.

Run the universal U on $\langle D, w \rangle$ and output its answer!

$A_{NFA} = \{ \langle N, w \rangle \mid N \text{ is an NFA that accepts string } w \}$

Theorem: A_{NFA} is decidable. (Why?)

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM that accepts string } w \}$

Theorem: A_{TM} is recognizable

(Why?)

The Church-Turing Thesis

Everyone's Intuitive Notion of Algorithms **=** Turing Machines

This is not a theorem –
it is a falsifiable scientific hypothesis.

And it has been *thoroughly* tested!

Thm: There are *unrecognizable* languages

Assuming the Church-Turing Thesis, this means there are problems that *NO* computing device will *ever* solve!

We will prove there is no *onto* function from the set of all Turing Machines to the set of all languages over {0,1}.

(But the proof will work for any *finite* Σ)

Therefore, the function mapping every TM M to its language L(M), *fails to cover all possible languages*