Lecture #10

Last Time: Min Cost Flow and Goldberg-Tarjan

Augment along cycle of min mean cost

Today: Introduction to Linear Programs

Many examples so far of pattern

Optimization problem \( \rightarrow \) certificate of optimality

\[
\begin{align*}
\text{max flow} & \quad \rightarrow \quad \text{min cut} \\
\text{min cost flow} & \quad \rightarrow \quad \text{potentials with nonnegative reduced costs}
\end{align*}
\]

Many more examples in combinatorial optimization, most derived from linear programming:

Canonical form

\[
\begin{align*}
(P) & \quad \text{max } c^T x \\
(D) & \quad \text{min } y^T b \\
\text{s.t. } A x & \leq b \\
x & \geq 0 \quad \text{s.t. } y^T A \geq c^T \\
y & \geq 0
\end{align*}
\]

Here "\( \leq \)" is a componentwise constraint

We say that \( x \) is feasible if it meets constraints in \( P \), similarly for \( y \).
Lemma: If $x$ and $y$ are feasible then $c^T x \leq y^T b$

Proof: Using that $x \geq 0$ and $y^T A \geq c^T$ we have

$$y^T A x \geq c^T x$$

Now using that $Ax \leq b$ and $y \geq 0$ we have

$$y^T A x \leq y^T b$$

Combining inequalities completes proof $\square$

Let's apply this to max flow. Let $P_{s,t}$ be all $s-t$ paths in $G$.

$$\max \sum_{e \in P_{s,t}} x_e \quad \min \sum_{e \in P_{s,t}} u(e) y(e)$$

$s.t.$ $\sum_{e \in P_{s,t}} x_e \leq u(e)$ $s.t.$ $\sum_{e \in P_{s,t}} y(e) \geq 1$ $\forall P_{s,t}$

$$x_e \geq 0$$

Here $A$ is the edge-by-path matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

How is on $s-t$ cut a feasible solution to the dual? Let $e = (uv)$

Let $y(e) = \begin{cases} 1 & \text{if } u \in S, v \in V \setminus S \\ 0 & \text{else} \end{cases}$
Then $\Sigma_{e \in P} y(e) = 1$ if $P = P_{st}$, because the $s-t$ path must cross at least one edge, $y(e) = 1$.

Moreover $\Sigma_{e \in P} w(e) = \text{cap}(S, V \setminus S)$.

In fact, duality gives us a more general way to prove c.s.s. via fractional cuts, but we do not need them here.

Let's draw a picture:

![Diagram of a polyhedron in 2D space. The feasible solutions are shown within the constraints $P = \{ x \mid Ax \leq b, x \geq 0 \}$, where $A$ is a matrix and $b$ is a vector.]

We call a finite intersection of halfspaces a polyhedron.

It is convex:

for any $x, y \in P$ and $\lambda \in [0, 1]$ we have

$$\lambda x + (1-\lambda) y \in P$$

Basic feasible solution = vertex = extreme point
projection thm

Thm. If P is a nonempty, closed convex set then

1. For any \( b \), there is a unique minimizer to

\[
(\mathbf{z} - b)^2
\]

over \( P \). (all the optimal point \( p \)

2. \( P \) is the projection of \( b \) if

\[
(\mathbf{z} - p)^\top (b - p) \leq 0 \quad \forall \mathbf{z} \in P
\]

Idea: Follows from fact that \( f(z) \) is strictly convex:

\[
f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y)
\]

for all \( \lambda \in (0,1) \), and optimality condition \( \nabla f(p)^\top (z - p) = 0 \)

We are now ready to prove Farkas' Lemma:

Lemma: Exactly one of the following holds

1. \( \exists \mathbf{x} \) s.t. \( A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 \)
2. \( \exists \mathbf{y} \) s.t. \( \mathbf{y}^\top A \geq 0 \) and \( \mathbf{y}^\top \mathbf{b} < 0 \)

Proof: If both (1) and (2) hold set \( c = 0 \).

Then \( \mathbf{y}^\top \mathbf{b} < \mathbf{c}^\top \mathbf{x} \), but since \( \mathbf{y} \) and \( \mathbf{y} \) are feasible, this violates weak duality for "standard form"
Now assume $\exists x$ that satisfies (1). We will construct $y$ that satisfies (2).

Let $P = \{Ax \; s.t. \; x \geq 0 \}$, by assumption by $P$.

Let $p = \text{Proj}_P(b)$, $p = Ax$ for $x \geq 0$ and $y = p - b$.

Claim #1 $y^TA \succeq 0$

Proof: Projection thm $\Rightarrow (z-p)^T(b-p) \leq 0 \; \forall z \in P$

Let $z = A(x + e_i)$, then

$$(z-p)^T(b-p) = (A(x + e_i) - Ax)^T(b-p)$$

$$= e_i^TA^T(b-p) \leq 0$$

This is equivalent to $e_i^TA^Ty \geq 0$, and this holds for all $i$. $\Box$

Claim #2 $b^Ty < 0$

Proof: $b^Ty = (p-y)^Ty = p^Ty - y^Ty$

Now $0 \notin P$, then again by projection thm

$$(z-p)^T(b-p) = p^Ty \leq 0$$

And since $y^Ty > 0$, we're done $\Box$

This establishes both constraints in (2). $\Box$
There are many equivalent formulations.

\[(1') \begin{bmatrix} A, -A, I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b, \quad \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} \geq 0\]

This is a system of inequalities as in (1), equivalent to

\[\text{unconstrained} \]
\[Ax \leq b\]

Thus Farkas' lemma yields that exactly one of (1') and

\[(2') \quad y^+ \begin{bmatrix} A - A^T \end{bmatrix} \geq 0, \quad y^T b < 0\]

holds. Moreover (2') is equivalent to

\[y^T A = 0, \quad y \geq 0 \quad \text{and} \quad y^T b < 0\]

Alternate Farkas' lemma: Exactly one of the following conditions holds:

\[(1') \exists x \text{ s.t. } Ax \leq b\]

\[(2') \exists y \text{ s.t. } y^T A = 0, \quad y \geq 0 \quad \text{and} \quad y^T b < 0\]