Topics in TCS: Problem Set # 1

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Due: April 2nd

If you work with other students, you must write-up your solutions by yourself and indicate at the top who you worked with!

Problem 1  [Barvinok, page 19]

Give an example of an infinite family \( \{A_i, i = 1, 2, \ldots\} \) of convex sets in \( \mathbb{R}^d \) such that every \( d + 1 \) sets have a common point but there are no points in common to all of the sets \( A_i \). (Hint: Helly’s theorem holds for infinite families of compact sets, so you will have to look for non-compact sets)

Problem 2  [Matousek, page 12]

In the situation of Radon’s lemma (\( A \) is a \( (d + 2) \)-point set in \( \mathbb{R}^d \)), call a point \( x \in \mathbb{R}^d \) a Radon point of \( A \) if it is contained in convex hulls of two disjoint subsets of \( A \). Prove that if \( A \) is in general position (no \( d + 1 \) points affinely dependent), then its Radon point is unique.

Problem 3  [Matousek, page 12] Kirchberger’s Theorem

(a) Let \( X, Y \subset \mathbb{R}^2 \) be finite point sets, and suppose that for every subset \( S \subseteq X \cup Y \) of at most 4 points, \( S \cap X \) can be linearly-separated from \( S \cap Y \). Prove that \( X \) and \( Y \) are linearly-separable.

(b) Extend (a) to sets \( X, Y \subset \mathbb{R}^d \), with \( |S| \leq d + 2 \).

Problem 4  [Barvinok, page 144]

Let \( A \subset \mathbb{R}^d \) be a non-empty set such that \( A^o = A \). Prove that \( A \) is the unit ball, i.e.

\[
A = \{ x \in \mathbb{R}^d : \|x\| \leq 1 \}
\]
Problem 5  **Seidel’s Algorithm**

Let $A = \{ x \in \mathbb{R}^d : \langle c_i, x \rangle \leq 1 \text{ for } i = 1, \ldots, m \}$ be a polytope (namely it is bounded). Let $A'$ be obtained from $A$ by removing one of its constraints at random. Then prove:

$$\Pr[\max_{x \in A} u^T x < \max_{x \in A'} u^T x] \leq \frac{d}{m}$$

Problem 6  *[Barvinok, page 8] **Guass-Lucas Theorem**

Let $f(z)$ be a non-constant polynomial in one complex variable $z$ and let $z_1, \ldots, z_m$ be the roots of $f$ (that is, the set of all solutions to the equation $f(z) = 0$). Let us interpret a complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$. Prove that each root of the derivative $f'(z)$ lies in the convex hull $\text{conv}(z_1, \ldots, z_m)$.

*Hint:* Without loss of generality we may suppose $f(z) = (z - z_1) \ldots (z - z_m)$. If $w$ is a root of $f'(z)$, then $\sum_{i=1}^m \prod_{j \neq i} (w - z_j) = 0$, and, therefore, $\sum_{i=1}^m \prod_{j \neq i} (\overline{w} - \overline{z}_j) = 0$ (where $\overline{z}$ is complex conjugate of $z$). Multiply both sides of the last identity by $(w - z_1) \ldots (w - z_m)$ and express $w$ as a convex combination of $z_1, \ldots, z_m$. 