Tensor networks and Probability Amplification for $\mathbb{C}^{IP(3)}$

by

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Tensor Network:

A tensor network is one way of visualizing a tensor. Consider a tensor $T \in \mathcal{H}^{\otimes k}$, where it has a fixed basis $|0\rangle, |1\rangle, \ldots, |n-1\rangle$. Then $T$ can be written as:

$$
T = \sum_{\bar{e}} \alpha_{\bar{e}} |l_1\rangle|l_2\rangle\cdots|l_k\rangle
$$

where $\bar{e}$ varies over $\{0, 1, 2, \ldots, n-1\}^k$.

In the Tensor network notation, we denote this by a vertex $T$, attached to $k$ edges, labelled from $1$ to $k$:

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The Tensor "outputs" the number $\alpha_{\bar{e}}$ when the $\bar{e}$th edge gets "input" $l_\bar{e}$, for every tuple $\bar{e}$.

The utility of this notation is its versatility. For example, we can easily visualize how the tensor represents a multi-linear map. Consider a tensor $T \in \mathcal{H}^{\otimes 4}$. We see how we can interpret its diagram as a map from $\mathcal{H}^{\otimes 2}$ to $\mathcal{H}^{\otimes 2}$.
We define \( \langle m_T | (i > \otimes j >) @ k > \otimes l > \rangle \) as \( T(i_{ij}, k, l) \), which is obtained by "inputting" \( i, j, k, l \) on "inputs" 1, 2, 3, 4 above. Now, the map is clearly defined:

\[
m_T(i > \otimes j >) = \sum_{k,l} T(i_{ij}, k, l) \; k > \otimes l >
\]

**Operations with Tensor diagrams:**

The real beauty of the Tensor network approach lies in how these diagrams can be composed in natural ways to visualize tensor operations.

1. **Tensor product:** The given tensor diagrams for \( T \) and \( S \), the tensor diagram for \( T \otimes S \) is obtained by:
   - Writing the two diagrams together and
   - Taking the "output" as the product of the "outputs" of \( T \) and \( S \) when all inputs of both \( T \) and \( S \) have been set. Example:

   \[
   T_i \; \otimes \; S_j = \sum_k (T \otimes S)_{ijk} \; T_i \; \otimes \; S_j
   \]
This clearly preserves the semantics of the tensor product.

2. **Contraction:** This operation is a generalized trace, and can be used to represent various operations such as inner products and traces. It is best illustrated through an example: Consider tensors $V$ and $W$ as follows:

\[ V \otimes W \]

Then a possible contraction is:

\[ S(l_1, r_1, r_2) = \sum_{l_3, l_4} V(l_1, l_2, l_3, l_4) S(l_2, r_2, l_3) \]

For our convenience, we also define conjugation, so that $V^*$ in the natural way, so that if $V$ is a tensor whose output is the complex conjugate of the output of $V$
Some examples of the use of these operations:

1. Inner product:
   Let \( \mathbf{v}, \mathbf{w} \in \mathbb{R} \). So, their diagrams are:
   \[
   \mathbf{v} \xrightarrow{1} \quad \mathbf{w} \xrightarrow{1}
   \]
   The inner product \( \langle \mathbf{v}, \mathbf{w} \rangle \) is thus \( \mathbf{v} \times \mathbf{w} \).

2. Matrix trace:
   Let \( \mathbf{V} \in \mathbb{R}^{n \times n} \) be a matrix tensor denoting the corresponding matrix:
   \[
   \mathbf{V} \llcorner_j \llcorner_i
   \]
   Then \( \text{Tr}(\mathbf{V}) \) is clearly denoted by \( \sum_j V_{ji} \), the contraction:
   \[
   \text{Tr}(\mathbf{V}) = \sum_j V_{ji}
   \]

3. Suppose a tensor \( \mathbf{X} \) has the Schmidt decomposition:
   \[
   \mathbf{X} = \sum_i \alpha_i |\mathbf{v}_i \rangle \langle \mathbf{w}_i|
   \]
   Then:
   \[
   |\mathbf{X} \rangle = \sum_k |\mathbf{L}_k \rangle
   \]
   This is the Schmidt decomposition.
   Where \( \alpha_{ii} = \alpha_i \) and \( \alpha_{ij} = 0 \) if \( i \neq j \).
4) Matrix multiplication: let $u \in H \otimes H$, $v \in H \otimes H$ be tensors representing matrices:

![Diagram](image)

Then the $m$-tensor $S = uv$ is

![Diagram](image)

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**Parallel Repetition**

We will now see how to use the Tensor Network framework to do probability amplification for QIP(3). Specifically, we want to show that if a given QIP(3) protocol has soundness $s$, then repeating it twice gives a protocol in parallel gives soundness of $s^2$.

Consider a QIP(3) protocol: (Assume the input $x$, which is used to define $V_1, V_2$, is not to be accepted).

![Diagram](image)

Then the soundness of the protocol is $s$ iff for all unitaries $U$, and all vectors $v$ and pure states $w, w'$,

$$
|\langle w | V_2 U V_1 | v \rangle|^2 < s^2.
$$

\( \Box \)
The parallelized protocol looks like:

Thus, the prover can try to cheat by entangling its responses to the two copt parallel runs. This has soundness \( s' \) iff for all states \( \langle V \rangle, \langle W \rangle \) and witness \( U \):

\[
|\langle W | (\mathcal{V}_2 \mathcal{O} \mathcal{V}_1) U (\mathcal{V}_1 \mathcal{O} \mathcal{V}_1) | V \rangle| < \sqrt{s'} \quad - 2
\]

Now, we notice that inner product corresponds to contraction, so (2) above corresponds to the fact that for any two vectors \( V \) and \( W \), if we contract their tensors with the tensor of the protocol as shown in the figure (2), then the value of the tensor should be less than \( \sqrt{s'} \), for all choices of \( U \). We need to use the fact that for the following tensor network as (Fig 3)
The fact that the sig single round protocol in Fig 1 has soundness is equivalent to the fact that the value of the above tensor is \( k' \) for any choice of \( U \) unitary, and \( v, w \) unit norm vectors.

Now we want to `break' the tensor of the parallelized protocol in Fig 2 to get be able to use Fig 3. Consider the following cut.
Let us look at the tensor $X$ inside the curve drawn above, and suppose that its Schmidt decomposition is

$$X = \sum x_i |v_i\rangle \langle w_i|$$

where $x_i$'s are non-negative reals.

Now, we can write the protocol as

Now we can have exactly the situation in figure 0, and so we can see that the value of this tensor is at most:

$$\sum x_i \sqrt{s} = \sqrt{s} \left( \sum x_i \right)$$

for any choice of $U$ unitary.

Now, we need to look at the same tensor in another way to get an estimate on $\sum x_i$.

We do this in two steps:

1. Notice that the value of the Fig 5 is maximized when $V_1 U V_2$ actually takes each $|v_i\rangle$ to $|w_i\rangle$. We
can thus say that the value of the fig. 5 is at most the value of below is $\sum a_i$

(Figure 5)

where $Y$ is a unitary which in particular takes $|v_i\rangle$ to $|w_i\rangle$

Thus, the value is $\sum_k |v_i\rangle\langle v_i| |w_i\rangle\langle w_i| = \sum a_i$

However, Fig. (S) is by definition equivalent to.

And this matches figure 3, so its value is at most $\sqrt{S}$. So, we get that $\sqrt{S} \geq \sum |v_i\rangle\langle v_i|$.

Thus, we get that the value of fig. 4 is at most $\sqrt{S}$. $\sum S = S$, and so the value of fig. 2 is $\sqrt{S'} = S \Rightarrow S' = S^2$. But $S'$
was defined to be the soundness of the parallelized protocol, and thus we are done.

We can repeat the same argument by induction, and finally, we get that if we do parallel repetition \( n \) times, the soundness drops to \( \frac{s}{2^n} \).