Oracles

**Notation:** \( M^B \) denotes the oracle TM \( M \) with oracle access to problem \( B \) (constant time solver for \( B \))

**Notation:** \( P^B = \{ A \mid A \text{ is poly} \text{- time decidable with an oracle for } B \} \)

**Theorems**

\( \exists A, B \text{ such that } P^A = NP^A \text{ and } P^B \neq NP^B \) (contradictory relativizations)

**Part I:** Prove that \( P^A = NP^A \) for \( A = TQBF \)

\[ NP^{TQBF} \subseteq NPSPACE^{TQBF} \] because \( NP \subseteq NPSPACE \), and \( TQBF \in NPSPACE \)

Also, \( NPSPACE^{TQBF} \subseteq NPSPACE \) because a TQBF oracle is useless in NPSPACE, where you can just solve the problem.

So, \( NP^{TQBF} \subseteq NPSPACE \).

Savitch’s theorem tells us that \( NPSPACE = PSPACE \), and with a \( TQBF \) oracle in \( P \) we can solve any PSPACE problem so \( NPSPACE = PSPACE \subseteq P^{TQBF} \subseteq NP^{TQBF} \)

Thus, we have \( NP^{TQBF} \subseteq NPSPACE \subseteq P^{TQBF} \subseteq NP^{TQBF} \Rightarrow P^{TQBF} = NP^{TQBF} \)

**Part II:** Prove that \( P^B \neq NP^B \) for \( A = \emptyset \), because an “empty” oracle will not give much power to \( P \), assuming \( P \neq NP \) (I think more complicated proof for \( P = NP \) can be found in textbook).

\( NP \subseteq P^{SAT} \), \( coNP \subseteq P^{SAT} \)

Note that now you can just flip the answer to the \( SAT \) oracle so in effect you also have an \( UNSAT \) oracle, thus \( coNP \subseteq P^{SAT} = P^{UNSAT} \)

\( NP \subseteq P^{NP} \)

Not sure of the notation \( P^{NP} \), but if it means “\( P \) with oracle access to all languages in \( NP \)” or “with oracle access to an NP-complete language”, then it’s obvious that \( NP \subseteq P^{NP} \), since \( \forall A \in NP \) we can use the NP-complete oracle to solve it in \( P \).

**Open problems**

\( NP \neq NP^{NP} \) (believed)

\( coNP \neq NP^{NP} \) (believed)

\( P^{SAT} \subseteq NP \cup coNP \) is unknown because it would imply \( NP = coNP \), since \( NP \cup coNP \subseteq P^{SAT} \)

Is \( P^{SAT} \) closed under complement?

Is \( P^{SAT} = NP^{SAT} \)?