dynamical zeta functions: what, why and what are the good for?

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I accept chaos
I am not sure that it accepts me
—Bob Dylan, *Bringing It All Back Home*

in physics, no problem is tractable

requires summing up exponentially increasing # of exponentially decreasing terms

yet

in practice

every physical problem must be tractable

"can't do" doesn't cut it

dynamical systems

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

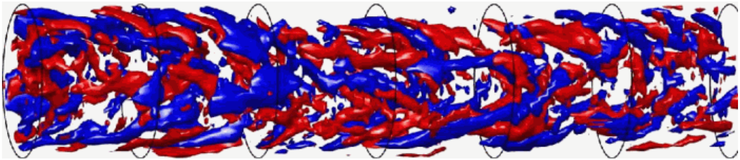
$x(t) \in \mathcal{M}$
a state of physical system at instant in time

today's experiments

example of a representative point

$x(t) \in \mathcal{M}$, $d = \infty$
a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ 3-d velocity field over the entire pipe[1]

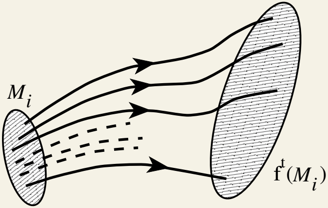


[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

dynamics

map $f^t(x_0)$ = representative point time t later

evolution



f^t maps a region $\mathcal{M}_i$ of the state space into the region $f^t(\mathcal{M}_i)$.

dynamics defined

dynamical system

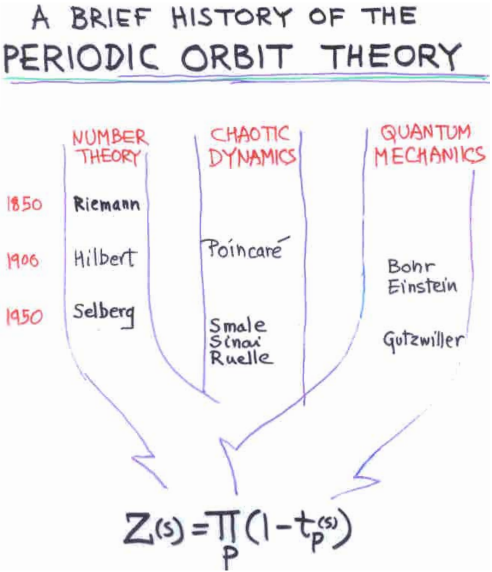
the pair $(\mathcal{M}, f)$

the problem

enumerate, classify all solutions of $(\mathcal{M}, f)$

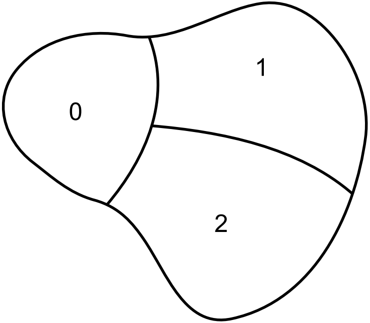
one needs to enumerate $\rightarrow$ hence zeta functions !

dynamical systems



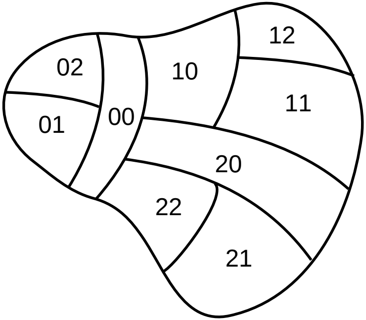
partition into regions of similar states

state space coarse partition



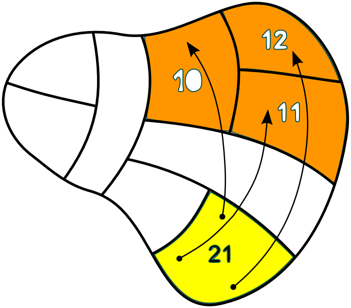
$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$
ternary alphabet
$\mathcal{A} = \{1, 2, 3\}$.

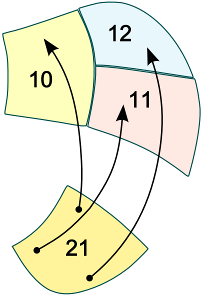
1-step memory refinement

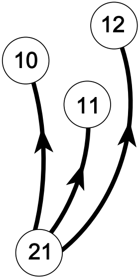


$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$
labeled by nine 'words'
$\{00, 01, 02, \cdots, 21, 22\}$.

topological dynamics







one time step

points from $\mathcal{M}_{21}$
reach $\{\mathcal{M}_{10}, \mathcal{M}_{11}, \mathcal{M}_{12}\}$
and no other regions

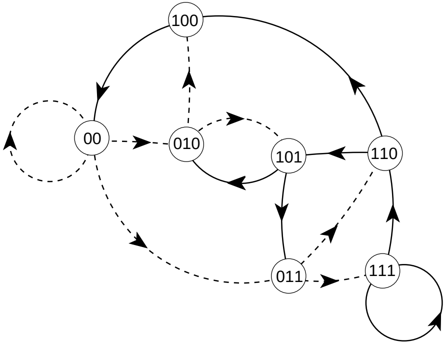
each region = node

allowed transitions
$T_{10,21} = T_{11,21} = T_{12,21} \neq 0$
directed links

topological dynamics

Transition graph T_{ba}

regions reached in one time step



example: state space resolved into 7 neighborhoods

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\}$$

how many ways to get there from here?

$$
\begin{aligned}
(T^n)_{ij} &= \sum_{k_1,k_2,\ldots,k_{n-1}} T_{ik_1} T_{k_1 k_2} \ldots T_{k_{n-1} j} \\
&\propto \lambda_0^n , \qquad \lambda_0 = \text{leading eigenvalue}
\end{aligned}
$$

counts topologically distinct n-step paths
starting in $\mathcal{M}_j$ and ending in partition $\mathcal{M}_i$.

compute eigenvalues by evaluating the determinant

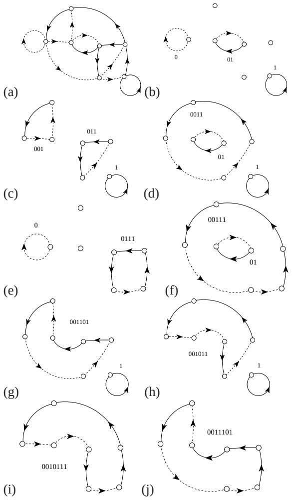
topological (or Artin-Mazur) zeta function

$$1/\zeta_{\mathrm{top}} = \det(1 - zT)$$

topological dynamics

$\det(1 - zT)$

$$= \sum \text{[non-(self)-intersecting loops]}$$

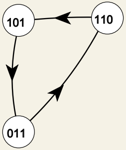


periodic orbit

loop, periodic orbit, cycle

walk that ends at the starting node, for example

$$t_{011} = L_{110,011}L_{011,101}L_{101,110} =$$



.

zeta function ("partition function")

$$\det(1 - zT)$$

can be read off the graph, expanded as a polynomial in z, with coefficients given by products of non-intersecting loops (traces of powers of T)

cycle expansion of a zeta function

$$
\begin{aligned}
\det(1 - zT) = 1 &- (t_0 + t_1)z - (t_{01} - t_0 t_1)\, z^2 \\
&- (t_{001} + t_{011} - t_{01} t_0 - t_{01} t_1)\, z^3 \\
&- (t_{0011} + t_{0111} - t_{001} t_1 - t_{011} t_0 - t_{011} t_1 + t_{01} t_0 t_1)\, z^4 \\
&- (t_{00111} - t_{0111} t_0 - t_{0011} t_1 + t_{011} t_0 t_1)\, z^5 \\
&- (t_{001011} + t_{001101} - t_{0011} t_{01} - t_{001} t_{011})\, z^6 \\
&- (t_{0010111} + t_{0011101} - t_{001011} t_1 - t_{001101} t_1 \\
&\qquad - t_{00111} t_{01} + t_{0011} t_{01} t_1 + t_{001} t_{011} t_1)\, z^7
\end{aligned}
$$

if there is one idea that one should learn about chaotic dynamics

it is this

there is a fundamental local $\leftrightarrow$ global duality which says that

eigenvalue spectrum is dual to periodic orbits spectrum

for dynamics on the circle, this is called Fourier analysis

for dynamics on well-tiled manifolds, Selberg traces and zetas

for generic nonlinear dynamical systems the duality is embodied in the trace formulas and zeta functions

global eigenspectrum ⇔ local periodic orbits

Twenty years of schooling
and they put you on the day shift
Look out kid, they keep it all hid

—Bob Dylan, *Subterranean Homesick Blues*

the eigenspectrum $s_0, s_1, \cdots$ of the classical evolution operator

trace formula, infinitely fine partition

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} \quad = \quad \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - s T_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|} \, .$$

the beauty of trace formulas lies in the fact that everything on the right-hand-side

–prime cycles p, their periods T_p and the eigenvalues of M_p–

is a coordinate independent, invariant property of the flow

deterministic chaos vs. noise

any physical system:

noise limits the resolution that can be attained in partitioning the state space

noisy orbits

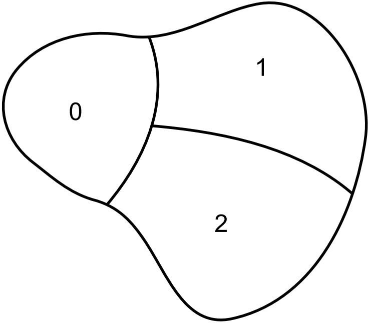
probabilistic densities smeared out by the noise:
a finite # fits into the attractor

goal: determine

the **finest attainable** partition

deterministic partition

state space coarse partition



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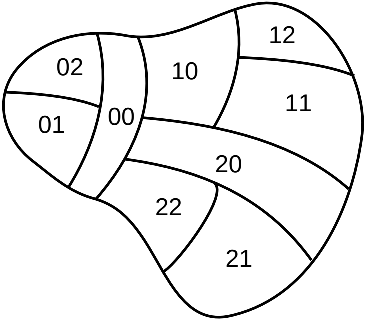
1-step memory refinement



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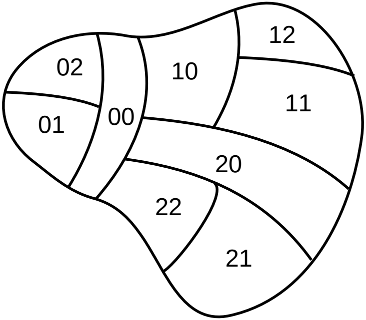
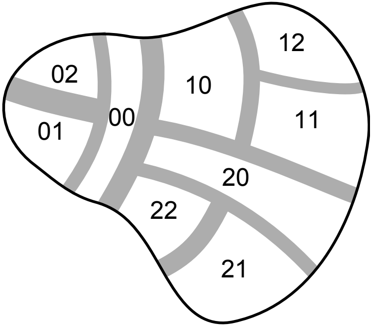
deterministic vs. noisy partitions



deterministic partition

can be refined
ad infinitum



noise blurs the boundaries

when overlapping, no further refinement of partition

periodic points instead of boundaries

- each partition contains a short periodic point smeared into a 'cigar' by noise

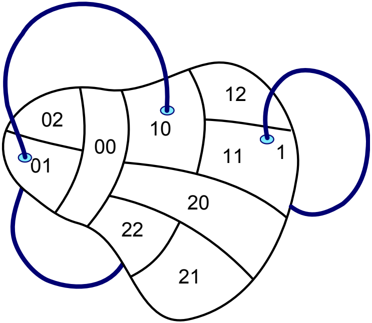
periodic points instead of boundaries

- each partition contains a short periodic point smeared into a 'cigar' by noise

compute the size of a noisy periodic point neighborhood

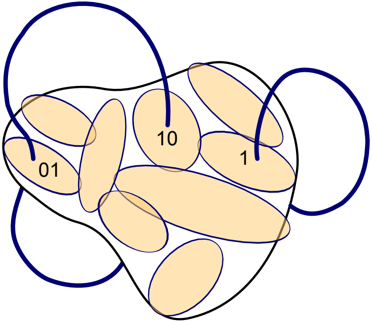
periodic orbit partition

deterministic partition



some short periodic points:
fixed point $\overline{1} = \{x_1\}$
two-cycle $\overline{01} = \{x_{01}, x_{10}\}$

noisy partition

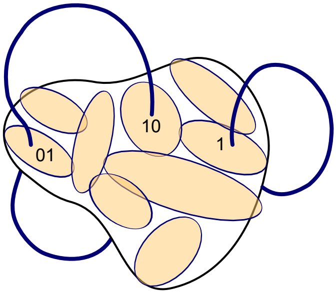


periodic points blurred by the
Langevin noise into
cigar-shaped densities

- successive refinements of a deterministic partition: exponentially shrinking neighborhoods
- as the periods of periodic orbits increase, the diffusion always wins:

partition stops at the finest attainable partition, beyond which the diffusive smearing exceeds the size of any deterministic subpartition.

noisy periodic orbit partition



optimal partition hypothesis

optimal partition:
the maximal set of resolvable periodic point neighborhoods

why care?

if the high-dimensional flow has only a few unstable directions, the overlapping stochastic 'cigars' provide a *compact cover* of the noisy chaotic attractor, embedded in a state space of arbitrarily high dimension

strategy

- use periodic orbits to partition state space
- compute local eigenfunctions of the Fokker-Planck operator to determine their neighborhoods
- done once neighborhoods overlap

how big is the neighborhood blurred by the Langevin noise?

the (well known) **key formula**

composition law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

density covariance matrix at time a: Q_a
Langevin noise covariance matrix: Δ_a
Jacobian matrix of linearized flow: M_a

roll your own cigar

evolution law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

in one time step a Gaussian density distribution with covariance matrix Q_a is smeared into a Gaussian 'cigar' whose widths and orientation are given by eigenvalues and eigenvectors of Q_{a+1}

(1) deterministically advected and deformed local density covariance matrix $Q \to MQM^T$

(2) add noise covariance matrix Δ

add up as sums of squares

noise along a trajectory

iterate $Q_{a+1} = M_a Q_a M_a^T + \Delta_a$ along the trajectory

if M is contracting, over time the memory of the covariance Q_{a-n} of the starting density is lost, with iteration leading to the limit distribution

$$Q_a = \Delta_a + M_{a-1}\Delta_{a-1}M_{a-1}^T + M_{a-2}^2\Delta_{a-2}(M_{a-2}^2)^T + \cdots .$$

diffusive dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow induces a history dependent effective noise: Always!

noise and a single attractive fixed point

if all eigenvalues of M are strictly contracting, any initial compact measure converges to the unique invariant Gaussian measure $\rho_0(z)$ whose covariance matrix satisfies

time-invariant measure condition (Lyapunov equation)

$$Q = MQM^T + \Delta$$

[A. M. Lyapunov 1892, doctoral dissertation]

example : Ornstein-Uhlenbeck process

width of the natural measure concentrated at the deterministic fixed point $z = 0$

$$Q = \frac{2D}{1 - |\Lambda|^2}\,, \qquad \rho_0(z) = \frac{1}{\sqrt{2\pi\, Q}} \exp\left(-\frac{z^2}{2\, Q}\right)\,,$$

- is balance between contraction by Λ and diffusive smearing by $2D$ at each time step
- for strongly contracting Λ, the width is due to the noise only
- As $|\Lambda| \to 1$ the width diverges: the trajectories are no longer confined, but diffuse by Brownian motion

things fall apart, centre cannot hold

but what if M has *expanding* Floquet multipliers?

both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future

things fall apart, centre cannot hold

but what if M has *expanding* Floquet multipliers?

Fokker-Planck operator is non-selfadjoint

If right eigenvector is peaked (attracting fixed point)
the left eigenvector is flat (probability conservation)

case of *repelling* fixed point

if M has only *expanding* Floquet multipliers, both deterministic dynamics and noise tend to smear densities away from the fixed point

balance between the two is described by the *adjoint Fokker-Planck operator*, and the evolution of the covariance matrix Q is now given by

$$Q_a + \Delta = M_a Q_{a+1} M_a^T \,,$$

[aside to control freaks: reachability and observability Gramians]

optimal partition challenge

finally in position to address our challenge:

determine the finest possible partition for a given noise

resolution of a one-dimensional chaotic repeller

As an illustration of the method, consider the chaotic repeller on the unit interval

$$x_{n+1} = \Lambda_0 \, x_n(1 - x_n)(1 - b x_n) + \xi_n \,, \quad \Lambda_0 = 8,\ b = 0.6 \,,$$

with noise strength $2D = 0.002$

'the best possible of all partitions' hypothesis formulated as an algorithm

- calculate the local adjoint Fokker-Planck operator eigenfunction width Q_a for every unstable periodic point x_a
- assign one-standard deviation neighborhood $[x_a - Q_a, x_a + Q_a]$ to every unstable periodic point x_a
- cover the state space with neighborhoods of orbit points of higher and higher period n_p
- stop refining the local resolution whenever the adjacent neighborhoods of x_a and x_b overlap:

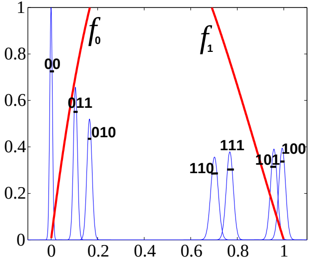
$$|x_a - x_b| < Q_a + Q_b$$

optimal partition, 1 dimensional map

f_0, f_1: branches of deterministic map

local eigenfunctions $\tilde{\rho}_a$ partition state space by neighborhoods of periodic points of period 3

neighborhoods $\mathcal{M}_{000}$ and $\mathcal{M}_{001}$ overlap, so $\mathcal{M}_{00}$ cannot be resolved further



all neighborhoods $\{\mathcal{M}_{0101}, \mathcal{M}_{0100}, \cdots\}$ of period $n_p = 4$ cycle points overlap, so

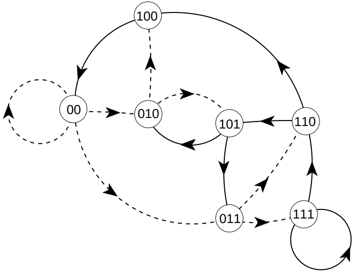
state space can be resolved into 7 neighborhoods

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\}$$

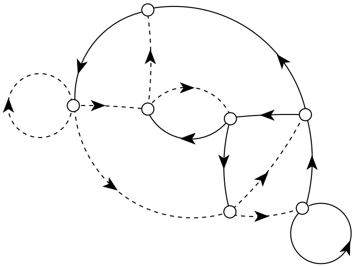
Markov partition

evolution in time maps intervals
$\mathcal{M}_{011} \to \{\mathcal{M}_{110}, \mathcal{M}_{111}\}$
$\mathcal{M}_{00} \to \{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}\}$, etc..

summarized by the transition graph (links correspond to elements of transition matrix T_{ba}): the regions b that can be reached from the region a in one time step



transition graph



7 nodes = 7 regions of the optimal partition

dotted links = symbol 0 (next region reached by f_0)

full links = symbol 1 (next region reached by f_1)

region labels in the nodes can be omitted, with links keeping track of the symbolic dynamics

(1) deterministic dynamics is full binary shift, but

(2) noise dynamics nontrivial and *finite*

predictions

escape rate and the Lyapunov exponent of the repeller

are given by the leading eigenvalue of this $[7 \times 7]$ graph / transition matrix

tests : numerical results are consistent with the full Fokker-Planck PDE simulations

what is novel?

- we have shown how to compute the **locally optimal partition**, for a given dynamical system and given noise, in terms of local eigenfunctions of the forward-backward actions of the Fokker-Planck operator and its adjoint

what is novel?

- **A handsome reward:** as the optimal partition is always finite, the dynamics on this 'best possible of all partitions' is encoded by a finite transition graph of finite memory, and the Fokker-Planck operator can be represented by a finite matrix

the payback

claim:

optimal partition hypothesis

- the best of all possible state space partitions
- optimal for the given noise

the payback

claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices

the payback

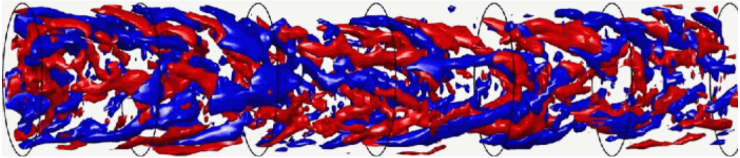
claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
- finite matrix calculations, finite cycle expansions $\Rightarrow$ optimal estimates of long-time observables (escape rates, Lyapunov exponents, etc.)

summary

- Computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes,



is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?

summary

- Intuitively, as we look at longer and longer periodic orbits, their neighborhoods shrink exponentially with time, while the variance of the noise-induced orbit smearing remains bounded; there has to be a *turnover time*, a time at which the noise-induced width overwhelms the exponentially shrinking deterministic dynamics, so that no better resolution is possible.

summary

- We have described here the *optimal partition hypothesis*, a new method for partitioning the state space of a chaotic repeller in presence of weak Gaussian noise, and tested the method in a 1-dimensional setting against direct numerical Fokker-Planck operator calculation.

references

- D. Lippolis and P. Cvitanović, *How well can one resolve the state space of a chaotic map?*, Phys. Rev. Lett. 104, 014101 (2010); arXiv.org:0902.4269
- D. Lippolis and P. Cvitanović, *Optimal resolution of the state space of a chaotic flow in presence of noise* (in preparation)