Improved Mixing Condition on the Square Lattice for Counting and Sampling Independent Sets

Ricardo Restrepo, **Jinwoo Shin**, Prasad Tetali,
Eric Vigoda, and Linji Yang

Georgia Institute of Technology

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High Level Story

Previous Talk by Pinyan Lu is about

- Counting up to the correlation decay (or spatial mixing) threshold of regular trees
- For general spin models & graphs

This Talk is about

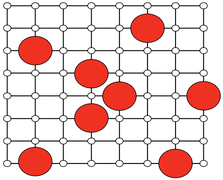
- Counting beyond the correlation decay (or spatial mixing) threshold of regular trees
- For specific spin models & graphs

Hard-core Model

Definition

For given (finite) graph G and activity $\lambda > 0$, define the distribution μ on 2^V as

$$\mu(I) \propto \begin{cases} \lambda^{|I|} & \text{if } I \in \mathcal{I}(G) \\ 0 & \text{otherwise} \end{cases},$$



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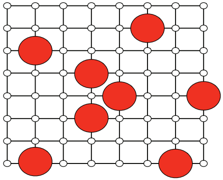

Questions in this Talk

- Computational complexity of computing the partition function (normalizing factor)

$$Z = Z(G;\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$$

  - If $\lambda = 1$, Z is $\#$ of independents sets.

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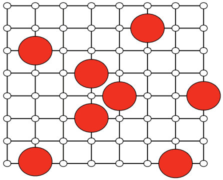

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  - If $\lambda = 1$, Z is $\#$ of independents sets.

- Computational complexity of sampling independent set I from μ?

How much hard to compute Z and sample from μ?

Intuitively

- Not easy since $Z = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$ is the sum over exponentially large $\mathcal{I}(G)$ in $|V| = n$
- Harder if λ is large or the maximum degree Δ is large

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- (Sly 2010) NP-hard to approximate Z for $\Delta \geq 3$ and $\lambda_{reg}(\Delta) < \lambda < \lambda_{reg}(\Delta) + \varepsilon$
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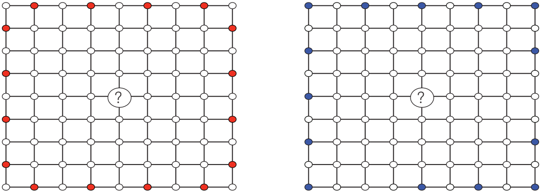
- Question: For restricted class of graphs (e.g. $\mathbb{Z}^2$), FPTAS exists beyond $\lambda_{reg}(\Delta)$?

Where does $\lambda_{reg}(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$ come from?

- For example, consider the hard-core model μ of square lattices $G = \mathbb{Z}_2$.



Let $\mathrm{p}_L^{\mathrm{even}} = \mathbf{Pr}\,[\text{Origin is occupied} \mid \text{even boundary vertices of } L \times L \text{ box are occupied}]$.

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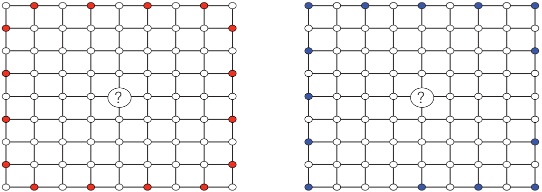

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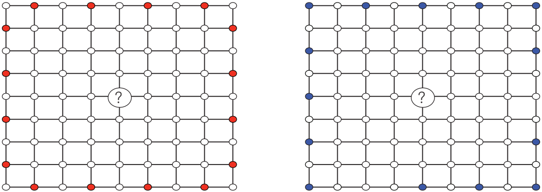

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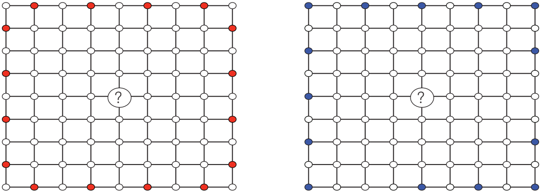

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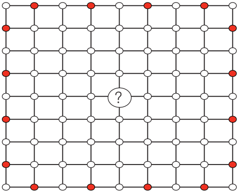

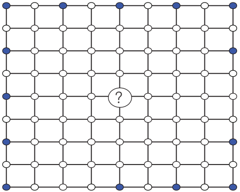

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  - Seems useful to approximate Pr[Origin occupied] and Z as well !

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Bounds for SM in Square Lattice $\mathbb{Z}^2$

- Weitz's result implies SM for $\lambda < \lambda_{reg}(4) = 27/16 = 1.6875$.
  - Previous best bound was 1.255 [vandenBerg-Steif 1994]
  - Conjectured bound is around 3.796 [Gaunt-Fisher 1965]

Our Result for Square Lattice $\mathbb{Z}^2$

Our Main Result

Theorem

If $\lambda < 2.3882$,

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Our High-level Idea

- Weitz (2006) studied general graphs.
- We refine his approach utilizing the structure of $\mathbb{Z}^2$ to get a better result
- Our method is generic & applicable to other structured graphs.

Our Proof Approach based on Weitz's Result (2006)

– DMS Condition

Our Proof Strategy

Weitz's self-avoiding-tree representation (2006)

- Given $G = (V, E)$ and $v \in V$, he constructs a tree T_{saw} with root v such that

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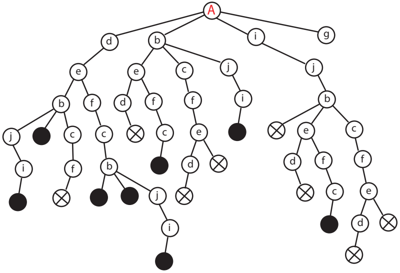
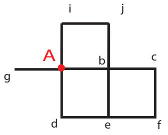


T_saw (G,A)



G

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Weitz's proof strategy for general G

SM & FTPAS for graph G of max degree $\Delta = O(1)$

$$\uparrow$$

SM for self-avoiding-tree T_{saw} of G

$$\uparrow$$

SM for regular trees of degree $\Delta \quad \overset{\text{Kelly (1991)}}{\longleftarrow} \quad \lambda < \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$

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Weitz's proof strategy for square lattice $\mathbb{Z}^2$

SM & FTPTAS for $\mathbb{Z}^2$

$\uparrow$

SM for self-avoiding-tree T_{saw} of $\mathbb{Z}^2$

$\uparrow$

SM for regular trees of degree $\Delta = 4$ $\quad \overset{\text{Kelly (1991)}}{\longleftarrow} \quad \lambda < \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} = 1.6875$

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SM & FTPAS for $\mathbb{Z}^2$

$\uparrow$ (from Weitz's work)

SM for self-avoiding-tree T_{saw} of $\mathbb{Z}^2$

$\uparrow$ (new)

SM for branching trees with average-degree < 3.8 $\quad \overset{\text{(new)}}{\longleftarrow} \quad$ $\lambda < 2.3882$

Branching Trees (Multi-type Galton-Watson Trees)

Branching Tree T_M generated by $t \times t$ matrix M

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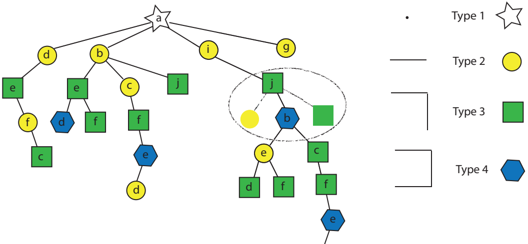

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How to prove SM for subtrees of T_M?

- Weitz's proof is only applicable to the regular case, i.e. 1×1 matrix $M = (\Delta - 1)$

Proof Idea for SM in Branching Trees

Want to prove

In the hard-core model for given branching tree T_M (or its subtree) of root r,

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$$\text{where} \quad \left.\begin{matrix} \alpha_r^L(+) \\ \alpha_r^L(-) \end{matrix}\right\} \text{is the probability } r \text{ is occupied given } L\text{-level leaves are} \left\{\begin{matrix} \text{occupied} \\ \text{unoccupied} \end{matrix}\right. .$$

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  We want to prove $\quad \|\nabla F(x)\|_\infty \;<\; \beta \quad$ for some function F.

  - Better inequality & easier to analyze than the previous Lipschitz inequality

Proof Idea for SM in Branching Trees

Summary of our ideas

- Study some contraction (or decaying) inequality for statistics ϕ.
  - Recently re-used in anti-ferromagnetic spin systems [Sinclair et al. 2011]
- Consider fractional boundary conditions at leaves.

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How to choose M and $\{\phi_i\}$ for $G \subset \mathbb{Z}^2$?

- Self-avoiding-walk tree of G should be contained in branching tree T_M.
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- We choose $\phi_i(x) = \frac{1}{s_i} \log \frac{x}{s_i - x}$ for some $s_i > 1$
  - Since we found that Weitz's result $\lambda < \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$ follows from $s_i = 1 + 1/\Delta$.
  - But, we do not know wether this is an optimal choice.

Main Theorem : DMS Condition

Theorem

(Strong) SM holds in the hard-core model of G with activity $\lambda < \lambda^*$ if there exist

- *$t \times t$ branching matrix M*
- *$s = [s_1, \ldots, s_t] > 1$ and $c = [c_1, \ldots, c_t] > 0$.*

so that every self-avoiding-walk tree is contained in branching tree generated by M and

$$(DMS)c \; < \; c$$

where D and S are diagonal matrices determined by M, s, c, λ^*.

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$$D_{jj} = \sup_{\alpha \in \left[\frac{1}{1+\lambda^*}, 1\right]} \frac{(1-\alpha)\left(1 - \theta_j \left(\frac{1-\alpha}{\lambda^* \alpha}\right)^{1/\Delta_j}\right)}{s_j - \alpha} \qquad S_{jj} = s_j,$$

$$\theta_j := \frac{\left(\prod_\ell c_\ell^{M_{j\ell}}\right)^{1/\Delta_j}}{\sum_\ell c_\ell s_\ell M_{j\ell}/\Delta_j} \qquad \Delta_j = \sum_\ell M_{j\ell}.$$

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- Remember we choose M considering $\{4, 6, 8, \ldots\}$-cycle avoiding-walk-trees
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- Given M, we find s, c, λ^* doing stochastic hill-climbing on a GPU machine.

Length of avoiding cycles	# of Types (size of M)	λ^*
≤ 4	3	1.8801
≤ 6	131	2.3335
≤ 8	921	2.3882

Reference

Improved Mixing Condition on the Grid for Counting and Sampling Independent Sets
Ricardo Restrepo, Jinwoo Shin, Prasad Tetali, Eric Vigoda, Linji Yang, **FOCS 2011**

Thank you !