New Evidence for the AND- and OR-Conjectures

Andrew Drucker
MIT

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Basic concepts

- Given: an instance x of a decision problem L.
- Is $x \in L$?

- **Instance Compression:** an algorithm $A(x)$ that outputs a **shorter** string x', such that:

x' is in some **target language** L' iff $x \in L$.

[Harnik, Naor '06; Downey, Fellows; earlier works]

Self-compression
(“kernelization”)

		2	A			D	3				6	F			
		G		7	4	E	F								
		6					5		2				1	C	4
1					A	6		9	3	G					7
2					C			3		6				9	
			G	F	2	3				4	D	C		6	
		9	B			G	4	2	F	7		A		5	1
			D	B		1			C				F	8	2
C	G	E				5			9		2	1			
B	D		F		1	C	E	4	8			5	9		
	1		2	4	9				5	D	C	E			
	6				F		8			3					B
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D					5	7	2		1	9					F
F	9	C				4		5					G		
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7		8				3		6
			9		3	4		
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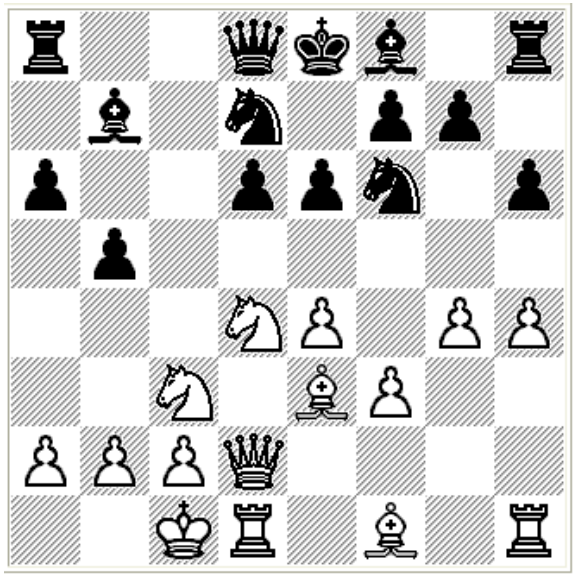

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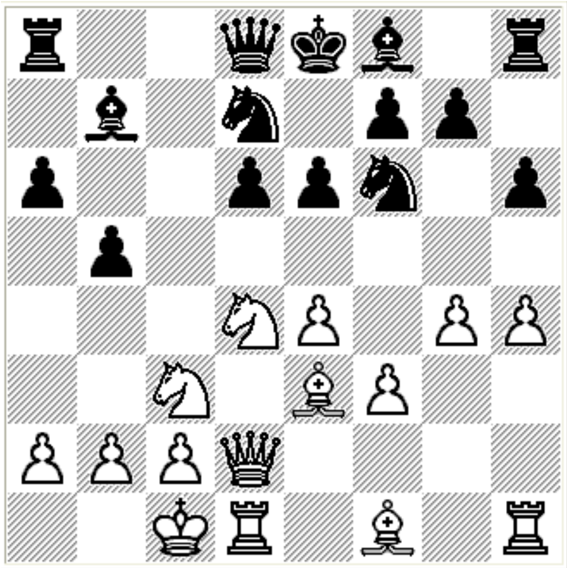

Target problem could be harder!

Why study instance compression?

Why study general compression?

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- Of course, complexity of target language matters....

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O		N	H					J	F	A			E		B
H					C				E			F	I		
L				B					P	F	N		J	K	
B		C	N		F		G				O	D		M	
F		D	A			K	L	G				C			
	E		O	P		C	F						N		D
	B	G				N		A	M		E				J
		K		G	I			P	N			L			F
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O		N	H					J	F	A			E		B
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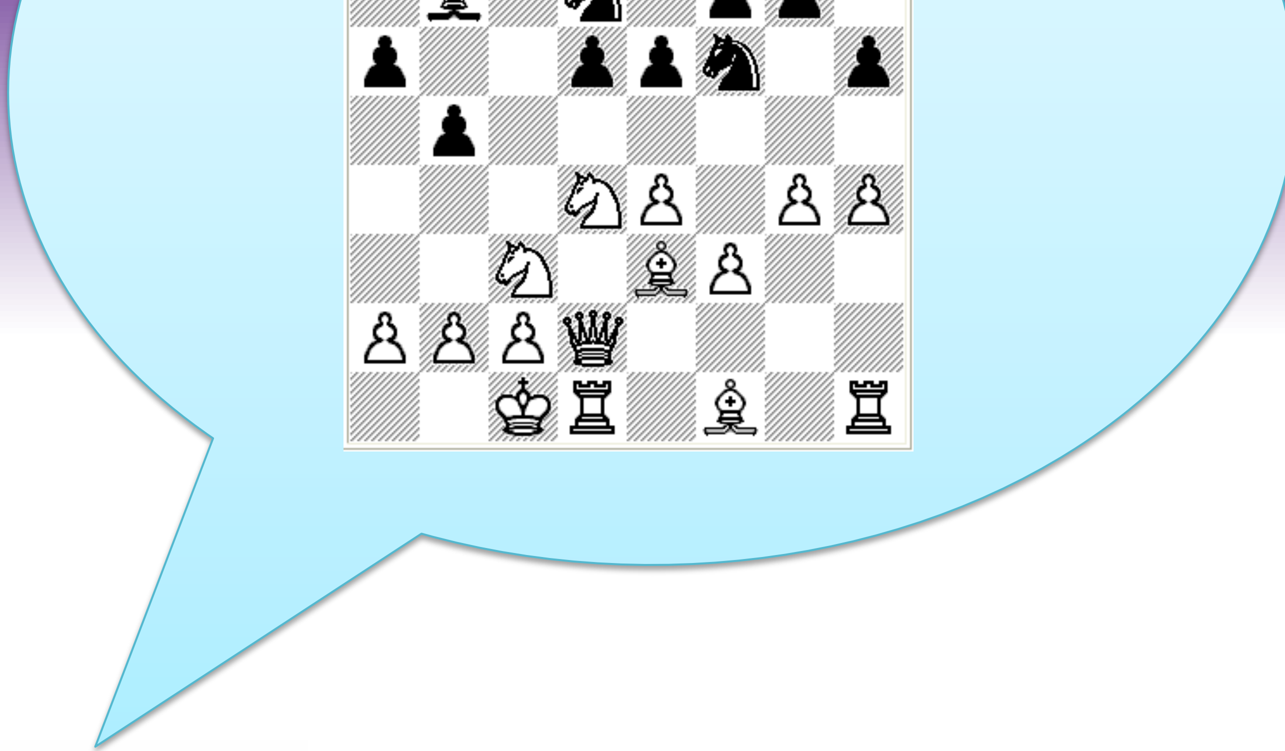

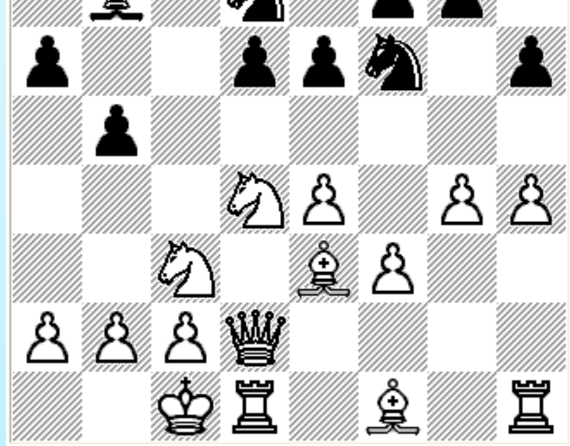


BLACK WINS!

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- Much of math can be viewed in this way…

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4) Even general compression for hard problems would have interesting applications in cryptography…

[Harnik, Naor '06]

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- Might as well give strongest possible impossibility statements…

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- This talk:

  -new, strong limits to compression for many NP-hard problems.

  -a notion of quantum compression to which our methods apply.

Parametrized compression

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- Here $k = \mathbf{k}(x)$.

Strong compression

- Say that A is a strong instance compression reduction for $(L, \mathbf{k})$, with target language L', if, for all x:

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2. A runs in time $\mathrm{poly}(|x|)$;
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Kernelization: $L = L'$

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- A general theory of limits to compression **for problems in FPT**?

Limits of compression

- Yes! **[Bodlaender, Downey, Fellows, Hermelin '08]; [Harnik, Naor '06]**

- Uses <u>reducibility</u> between compression tasks.

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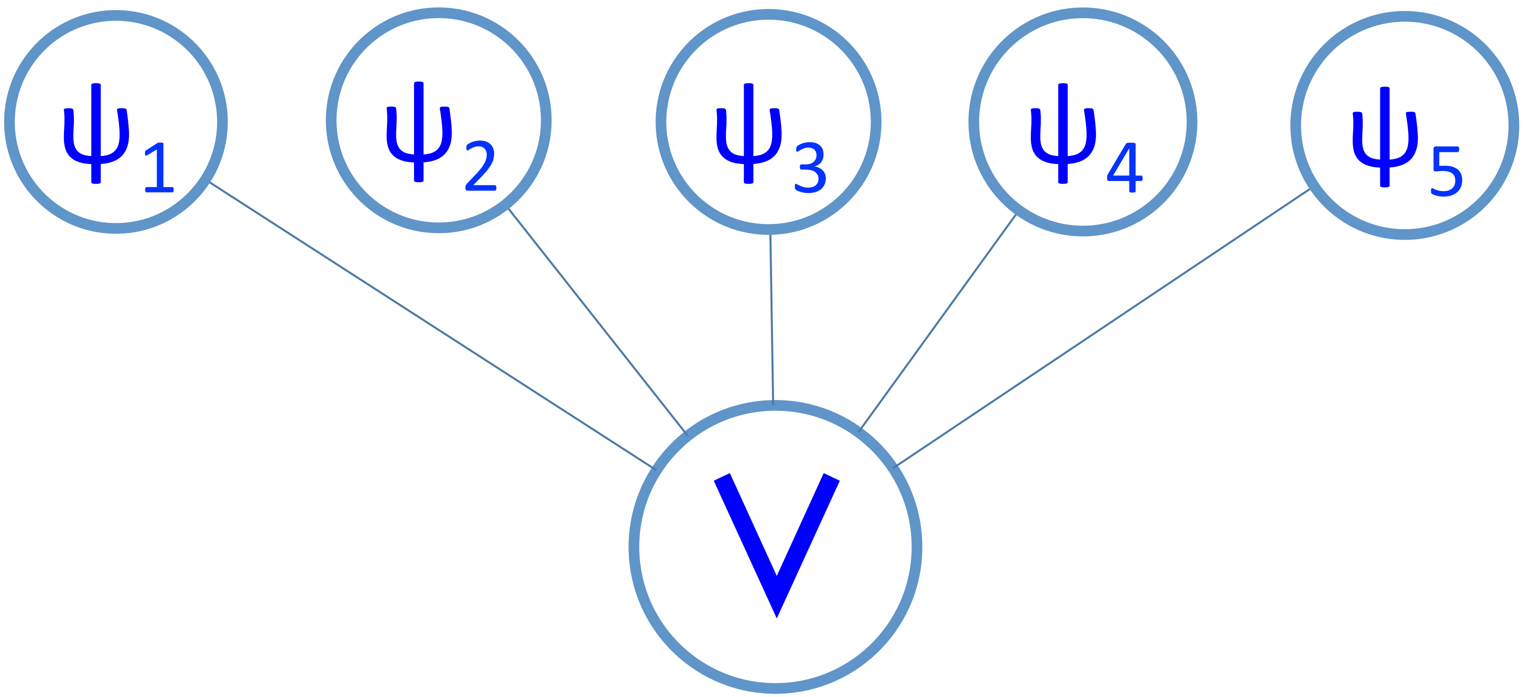
OPEN

One approach: sparsification



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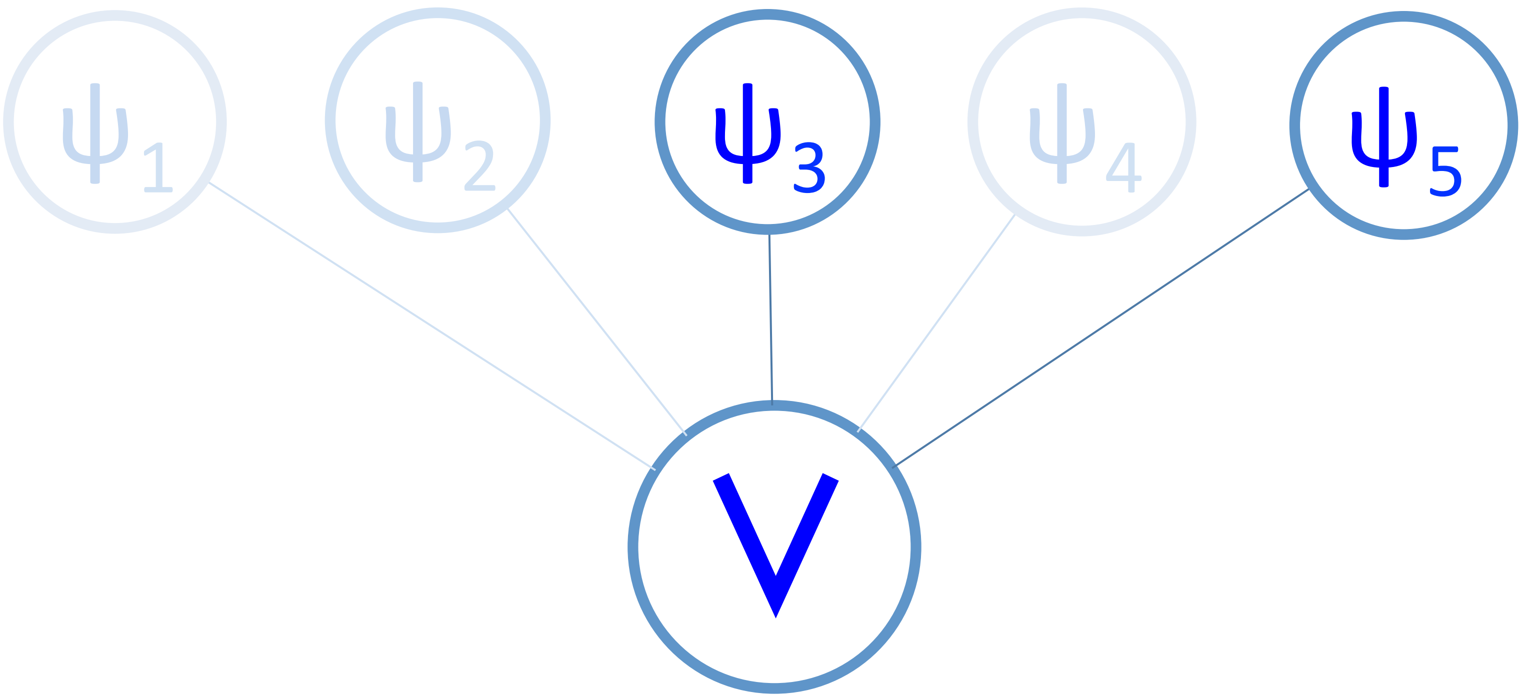

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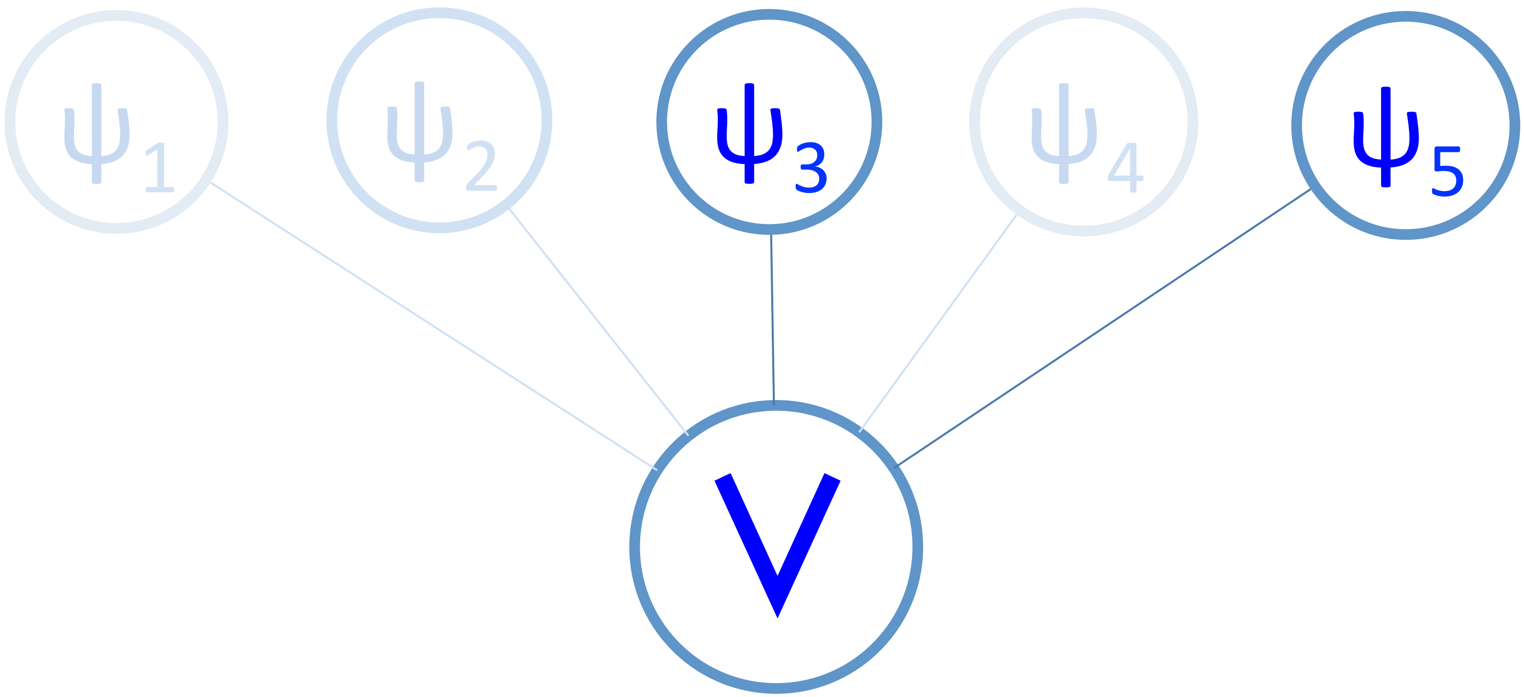

- Just one possible way to compress an OR of SAT instances…

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  - k-Path, k-Cycle, k-Short Cheap Tour
  - k-Graph Minor Order Test, k-Bounded Treewidth Subgraph Test, k-Planar Subgraph Test
  - w-Independent Set, w-Dominating Set
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- Many other examples in subsequent works…
- (Same implication holds for general strong compression!)

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These two hardness assumptions:

the "**OR-** and **AND-Conjectures**"

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- Applies to deterministic compression schemes, and randomized w/o false negatives.
- Compressibility of AND-SAT (and its relatives) remained unclear.

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- Applies to two-sided error compression schemes, even with success probability quite close to 1/2.

- Much more modest compression amounts also imply **NP** $\subseteq$ **SZK/poly**, if compression is more reliable.

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For proof sketch:

- Assume that compression reduction **R** for AND-SAT is perfectly reliable:

$$\mathbf{R}(\psi_1, \psi_2, \ldots, \psi_T) \in L' \quad \text{iff} \quad \psi_1, \ldots, \psi_T \in SAT$$

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- **Goal:** use **R** to build a non-uniform, interactive proof system for UNSAT.

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$\in L'$ $\notin L'$

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- **Basic observation'**: suppose

$$(\psi_1 , \ldots, \psi_T) \sim \mathcal{D} ,$$

where $\mathcal{D}$ is a distribution over $(SAT_n)^T$, and let $j \in [T]$.

- Then, $\mathbf{R}(\psi_1 , \ldots, \psi_T)$, $\mathbf{R}(\psi_1, \psi_2 , \ldots, \varphi, \ldots, \psi_T)$

j^{th} ind.

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A distinguishing task

- **Idea:** to prove $\varphi \in$ UNSAT, Prover will "show off" ability to *distinguish* between dist'ns

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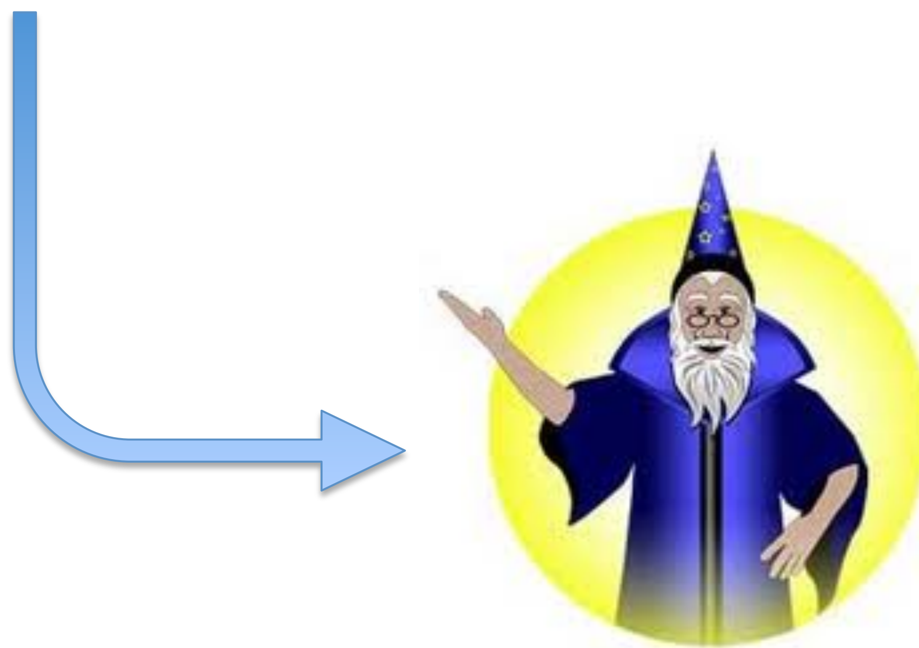

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A distinguishing task

- **Main Question:** how to choose our $\mathcal{D}$, j ?

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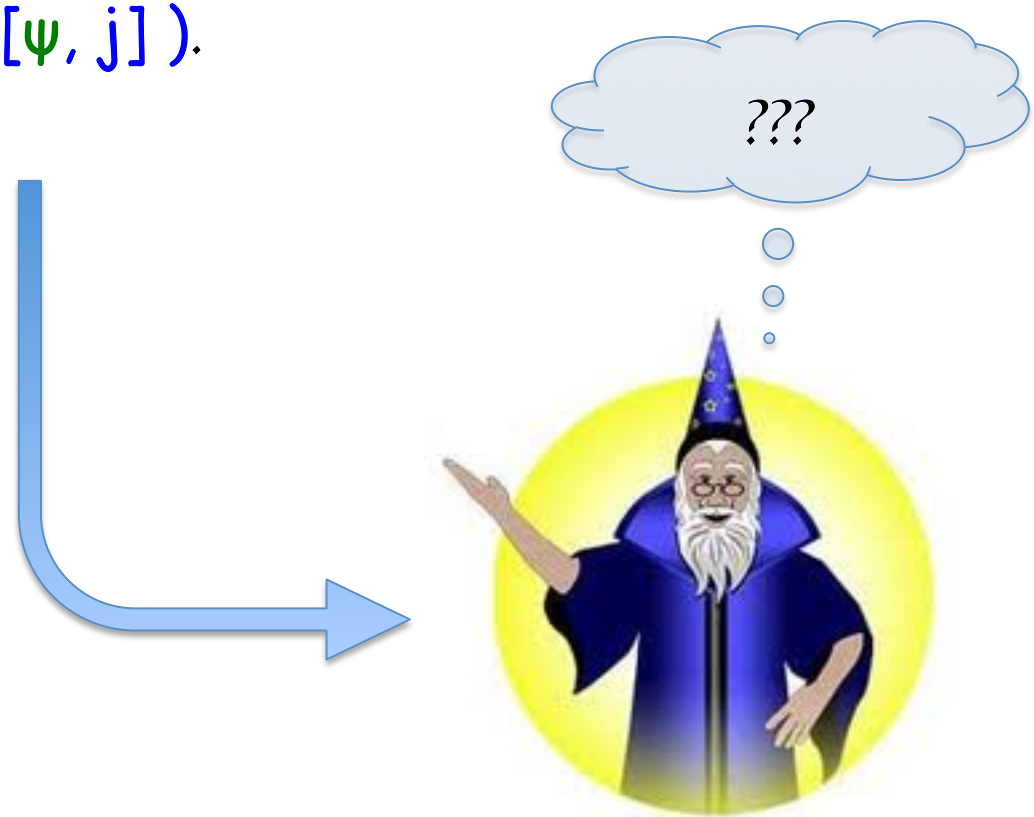

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$\mathcal{D}$: a "disguising distribution" for **R** on SAT_n .

Efficient Sampleability

- **Also want:** $\mathcal{D}$ sampleable in poly(n) time, with poly(n) bits of non-uniform advice.

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Main lemma: Such a $\mathcal{D}$ can be found!

The upshot

- Then, distinguishing task for

$$\mathbf{R}(\mathcal{D}) \quad , \quad \mathbf{R}(\mathcal{D}[\psi, j])$$

gives a <u>non-uniform, 2-message, private-coin proof system</u> for membership of ψ in UNSAT.

- Implies UNSAT $\in$ **NP/poly** by standard techniques.

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- No clear good choice for j… so, choose j uniformly!

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- Focus on <u>indistinguishability requirement:</u>

For all $\psi \in SAT_n$,

$$E_j\left[\ \left| R(\mathcal{D}) - R(\mathcal{D}[\psi, j]) \right|_{stat}\ \right] <= .9$$

- No clear good choice for j... so, choose j uniformly!

$$\boxed{\psi_1 \quad \ldots \quad \psi_T} \xrightarrow{\ \mathbf{R}\ } \boxed{\psi_2\ \psi_7}$$

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The game perspective

- Consider this **2-player, simul-move** game:

P2 “Breaker”

P1 “Maker”

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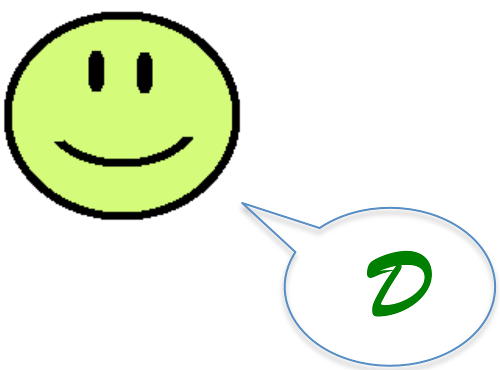

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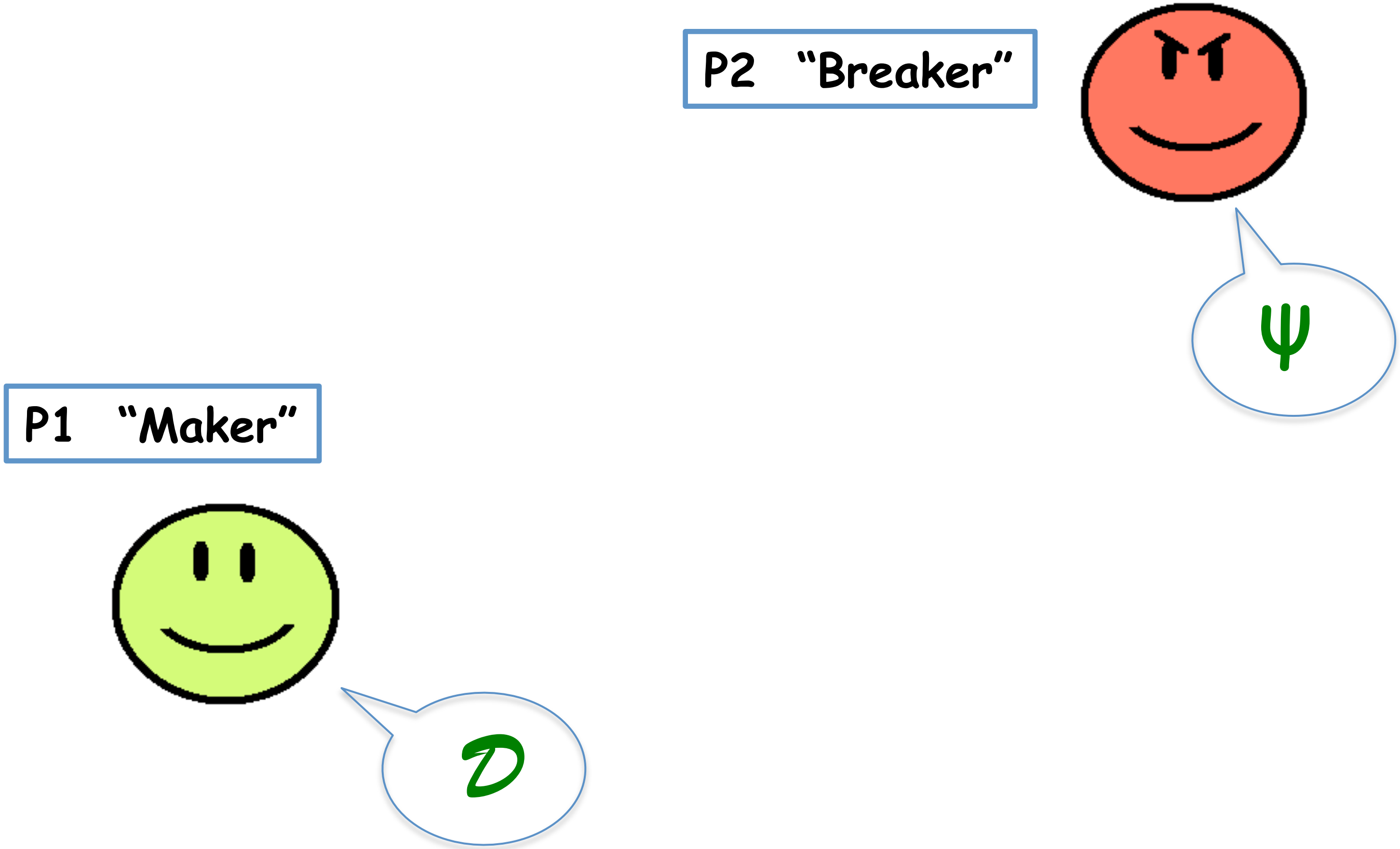

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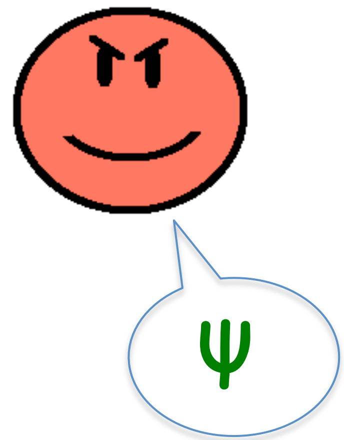

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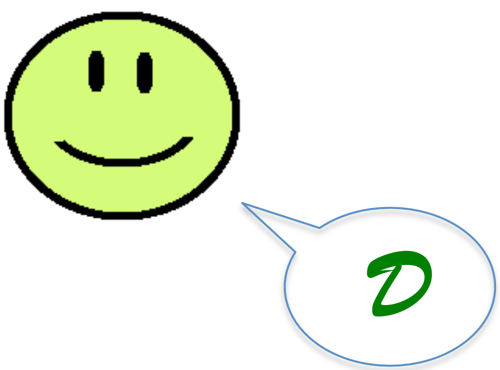

Payoff to Breaker:
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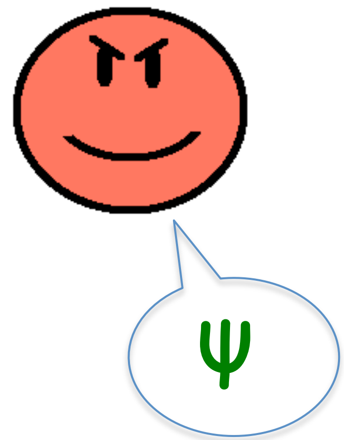

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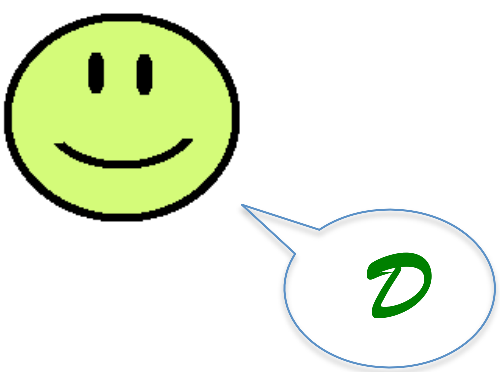

Payoff to Breaker:
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Want to show: ∃ a Maker strategy to force Breaker payoff <= .9.

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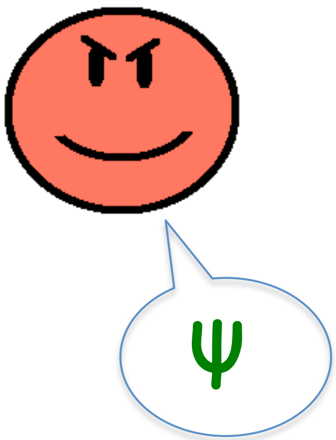

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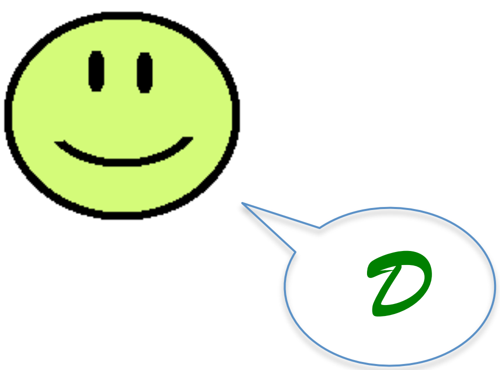

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Idea: Use Minimax Theorem!

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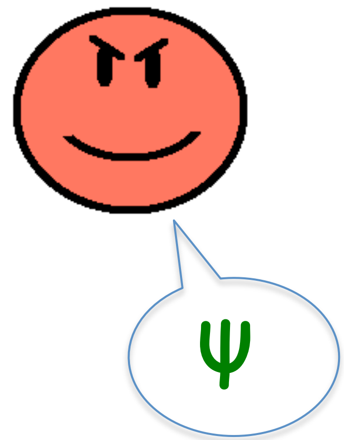

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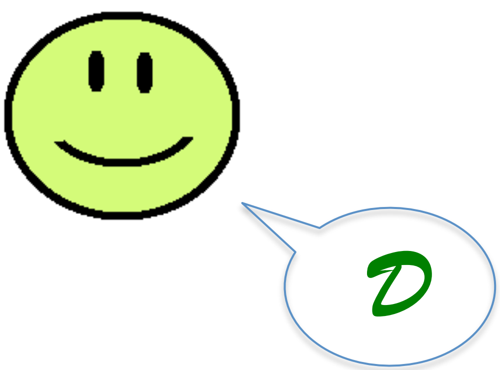

Minimax theorem says: it’s enough to Show that against any randomized (“mixed”) strategy for Breaker, ∃ a good strategy for Maker.

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- Just show: ∀ distributions Y over SAT_n ,
  ∃ a dist'n $\mathcal{D}_Y$ over $(SAT_n)^T$ such that:

$$\mathbf{E}_{j,\,\psi\sim Y}\ [\ \|\mathbf{R}(\mathcal{D}_Y) - \mathbf{R}(\mathcal{D}_Y[\psi, j])\|_{stat}\ ] \ \ <= \ \ .9$$

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- **Natural idea:** try $\mathcal{D}_Y \ := \ Y \otimes Y \otimes \ldots \otimes Y$.

Product distributions

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- Can prove this intuition.

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- Basic idea: **mutual information** between $\mathbf{R}(\mathcal{D}_Y)$ and a typical input coord. is small...

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$\psi \sim Y$

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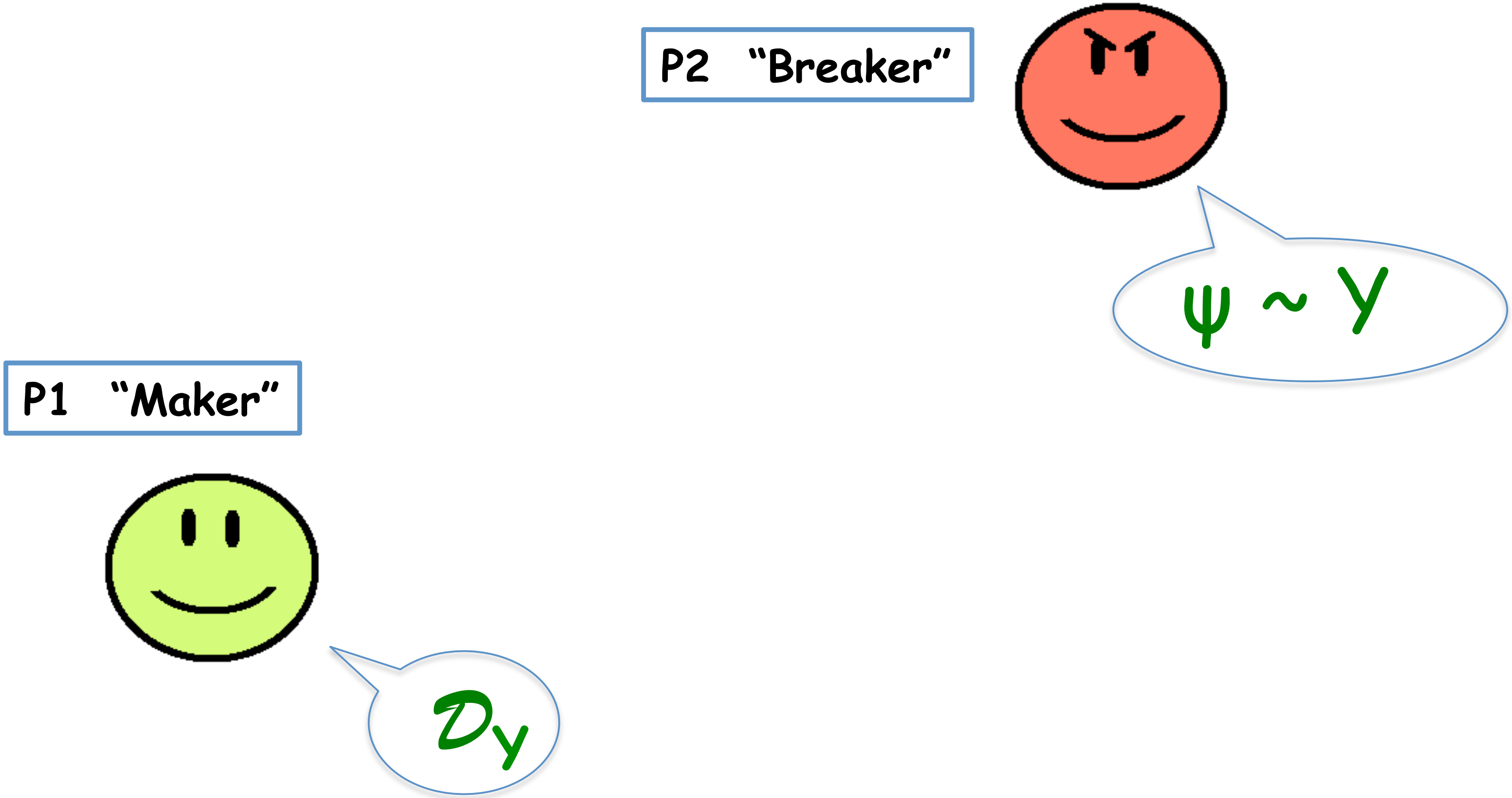

The game perspective

P2 "Breaker"

$\psi \sim Y$

P1 "Maker"

$\mathcal{D}_y$

This choice works for Maker!

The game perspective

P2 “Breaker”

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Applying Minimax Thm…
A <u>fixed choice</u> works for Maker!

The game perspective

P2 "Breaker"

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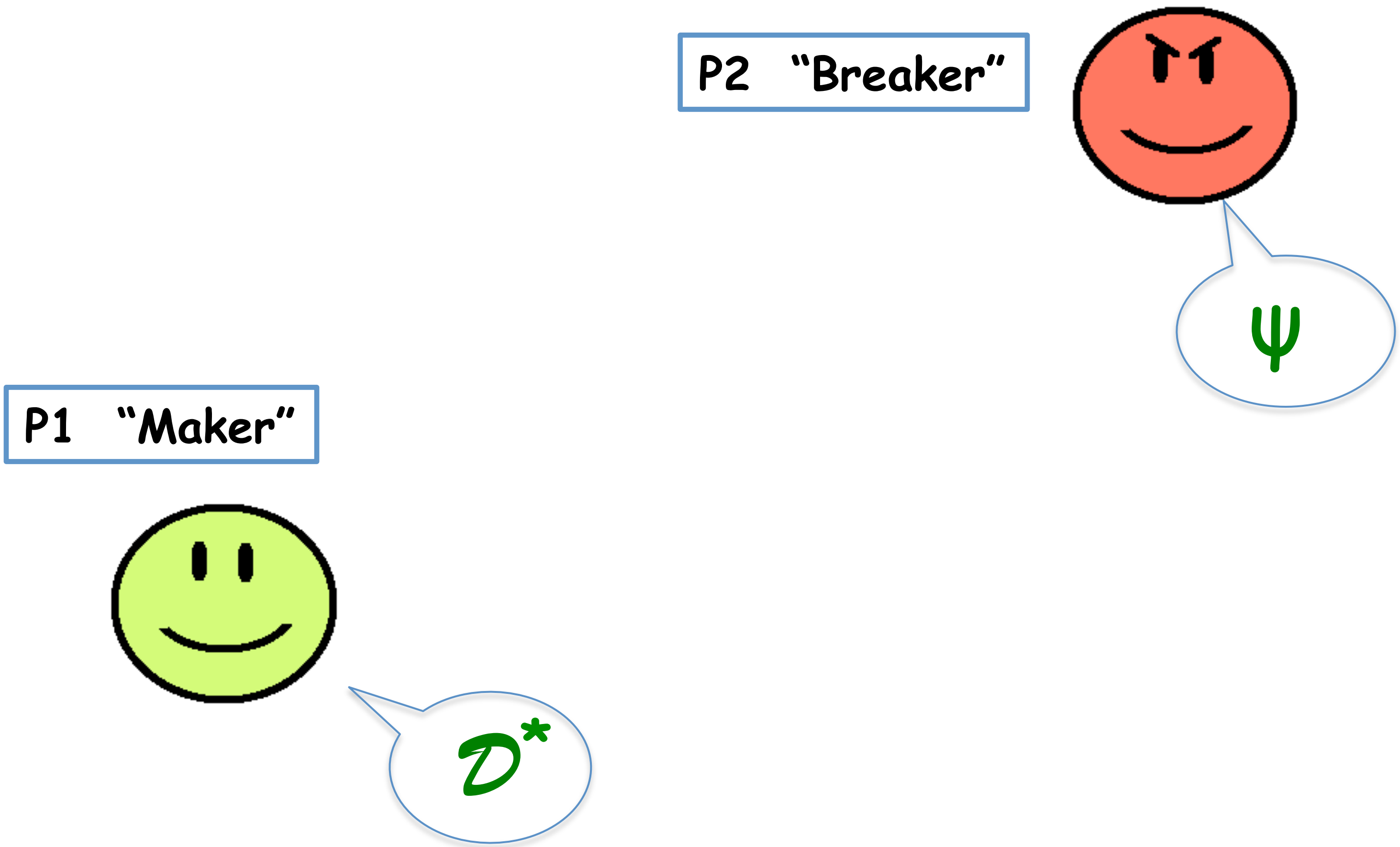
The game perspective



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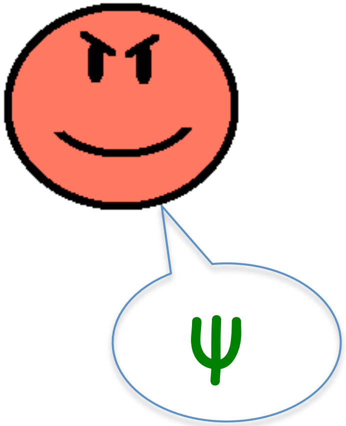

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The game perspective

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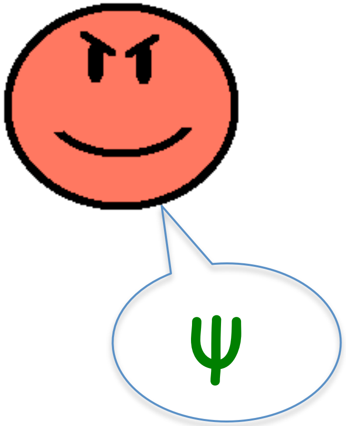

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Strictly, $\mathcal{D}^*$ is a <u>distribution</u> over distributions…

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Note: D'_y is easy to (<u>non-uniformly</u>) sample!

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P1 “Maker”

$\mathcal{D}'_y$

P2 “Breaker”

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P1 “Maker”

$\mathcal{D}'_Y$

This choice works (almost as well) for Maker!

Wrapping up

- Minimax Thm. now implies: a <u>fixed</u> distribution $\mathcal{D}^{**}$ over <u>easy-to-sample</u> distributions D'_y, that works against all Breaker strategies.
- Obtain our final Maker strategy $\mathcal{D}^{***}$ as a dist'n over poly(n) samples drawn from $\mathcal{D}^{**}$.

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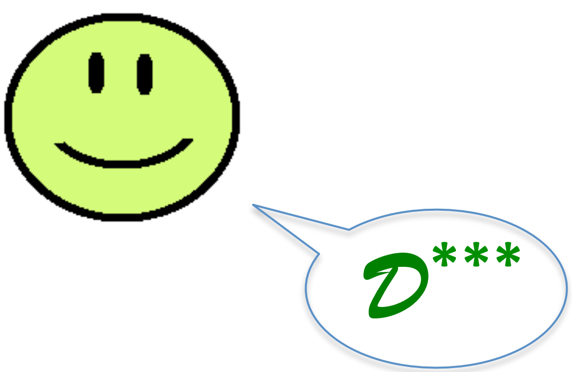

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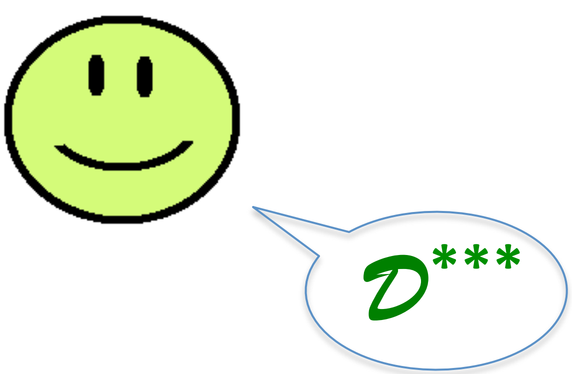

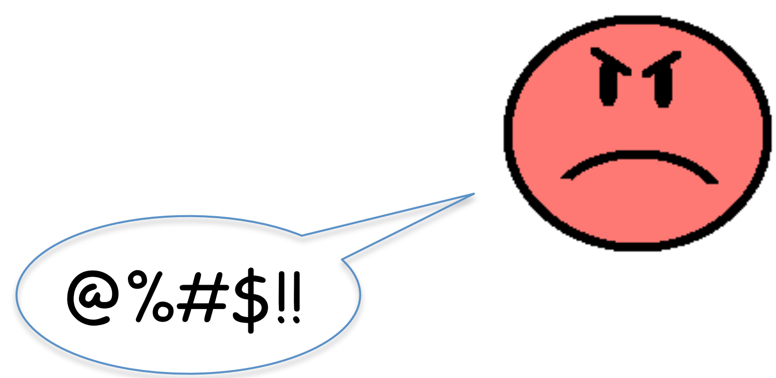

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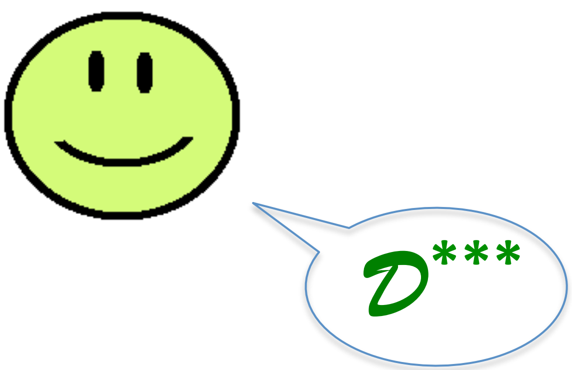

This is the "Disguising Distribution" Arthur will use in our protocol.

Quantum to the rescue?

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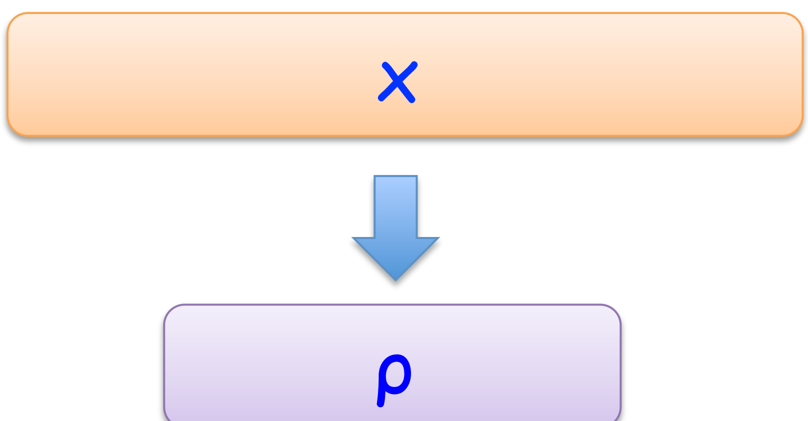

Compression to quantum states

- Input: an instance (x, k) of parametrized decision problem L.

- Output of a quantum compression scheme: a quantum state ρ on $c = c(|x|, k)$ qubits, such that ρ "contains the answer" to $L(x)$:

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- $\mathbf{M}$ need not be efficiently performable!

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- Strong compression: $c(|x|, k) = k^{O(1)}$.

Compression to quantum states

- Quantum compression could share some of the uses of classical compression.
- Might be the basis for interesting new quantum algorithms…

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- Limits to compression are as quantitatively strong as for our classical results.

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Thanks!