Automatica, Vol. 25, No. 5, pp. 781–784, 1989
Printed in Great Britain.

0005-1098/89 $3.00 + 0.00
Pergamon Press plc

Brief Paper

Existence of Coordinating Prices in Dynamic Systems*

ARTHUR W. BERGER†‡ and FRED C. SCHWEPPE§

Key Words—Hierarchical decision making; decentralized control; economic systems.

Abstract—Necessary and sufficient conditions are presented for the existence of coordinating prices in coupled dynamic systems. The formulation includes a coordinator with a non-separable payoff, and a coupling system with dynamics. New insights are obtained by examining three entities-to-price—the subsystem control, the subsystem output, and the coupling system state.

1. *Introduction*

IN MOST price coordination methods, the entities priced are the input/output variables of the subsystems (Mesarovic *et al.*, 1970; Findeisen *et al.*, 1980). An exception, though, is the work by Wierzbicki (1972) where the entity priced is the subsystem control. Another exception is the work by Calvet and Titli (1980), who considered a coupling system with dynamics and a coordination method that iterates on the coupling system state, amongst other variables. The above papers make the traditional assumption that the coordinator's payoff is separable, and in particular, is the sum of the subsystem payoffs. Cohen (1980), however, considered non-separable payoffs, and, assuming the coordinating price exists, presented algorithms that converge to this price, where the entity priced is a general function of the subsystem control. Thus, price coordination schemes are known for different entities priced, but are derived under different assumptions and problem formulations.

The contribution of this paper is to study, under a single problem formulation, the existence of pricing mechanisms for different entities priced. For non-separable payoffs, we present necessary and sufficient conditions for the existence of coordinating prices for three entities to price: (1) the subsystem output, (2) the subsystem control, and (3) the coupling system state.

2. *Formulation of the problem*

Consider a dynamic system consisting of n subsystems connected via a coupling system. The subsystems are modeled as

$$\dot{x}_i = f_i(x_i, u_i, v_i), \quad x_i(t_0) \text{ given} \quad i = 1, \ldots, n \tag{1a}$$

$$y_i = c_i(x_i, u_i, v_i), \quad i = 1, \ldots, n \tag{1b}$$

and the coupling system is modeled as

$$\dot{x}_0 = f_0(x_0, y), \quad x_0(t_0) \text{ given} \tag{1c}$$

$$0 = c_0(x_0, y, v) \tag{1d}$$

where $u_i(t) \in R^{m_i}$ is the control vector for subsystem i; $x_i(t) \in R^{n_i}$ is the state vector of subsystem i, or of the coupling system for $i = 0$; $v_i(t) \in R^{k_i}$ is the vector of inputs from the coupling system to subsystem i; and $y_i(t) \in R^{r_i}$ is the vector of outputs from subsystem i to the coupling system; m_i, n_i, k_i, r_i, are $\in$ positive integers, $i = 1, \ldots, n$. Furthermore, let "u" denote the vector of all the u_i's, i.e. $u' \equiv (u_1', \ldots, u_n')$ where $'$ denotes transpose. Similarly, $x' \equiv (x_0', x_1', \ldots, x_n')$, $y' \equiv (y_1', \ldots, y_n')$, $v' \equiv (v_1', \ldots, v_n')$.

Associated with each subsystem there exists a decision maker called a controller (infimal, follower, agent) with a criterion function

$$\max \int_{t_0}^{t_f} l_i(x_i, u_i, v_i) + \rho_i' q_i \, dt \tag{2a}$$

where $l_i(\cdot)$ represents individual i's monetarily measured satisfaction index or the negative of a cost function. The term

$$\int_{t_0}^{t_f} \rho_i' q_i \, dt$$

constitutes the pricing mechanism (side payment, inducement). ρ_i and q_i respectively the price vector and the entity priced, and are chosen by another decision maker called a coordinator (suprimal, leader, market maker). Herein, q_i is either u_i, y_i or x_0. The coordinator, in turn, wishes to maximize the criterion

$$\max \int_{t_0}^{t_f} L(x, u, y, v) \, dt \tag{2b}$$

subject to the constraints of the dynamic system, equation (1). Thus, for $q_i \in \{u_i, y_i, x_0\}$, the problem of the coordinator is to choose a corresponding ρ_i, $i = 1, \ldots, n$ such that the resulting optimal control of the controllers will coincide with the optimal control of the coordinator. When such a choice exists for ρ_i, we say "the pricing mechanism exists" when the entity priced is q_i.

The above formulation, when viewed in terms of game theory, is a simple, deterministic, single stage, Stackelberg incentive problem. In the vocabulary of game theory, and for a given entity priced, the information set of the coordinator is empty and the coordinator acts once by first announcing the price vectors ρ_i, $i = 1, \ldots, n$ and the values of exogenous parameters of the controllers' optimizations, see below. The information set of controller i is the relevant portion of the act of the coordinator, and the controller acts once by announcing its optimal, open loop control law. The strategy of controller i, which is a mapping from its information set to its optimal control law, implicitly solves the subsystem optimization. Thus, we have three simple Stackelberg incentive problems, one for each of the possible entities priced: v_i, y_i and x_0. However, it turns out that the game theory formulation is not convenient for the present note, because to compare the entities priced, it will be advantageous to consider the model equations (1) explicitly, as opposed to implicitly as part of the players' payoffs. By deviating from the standard game theory formulation, we are able to obtain new conceptual insights. For further discussion

* Received 9 July 1988; received in final form 20 January 1989. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Editor A. P. Sage.

† AT&T Bell Laboratories, Room 3H-611, Holmdel, NJ 07732, U.S.A.

‡ Author to whom all correspondence should be addressed.

§ Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A. Prof. Schweppe passed away on 8 July 1988.

on the relationship of the present problem to Stackelberg incentives, see Berger (1983), and for further information on Stackelberg incentives, see, e.g. Başar and Olsder (1980), Ho *et al.* (1982) and Zheng *et al.* (1984).

The optimization of the non-linear criterions (2a) and (2b) subject to the non-linear systems (1) is in general a difficult problem. Since the interest of this paper is not to solve these optimization problems, per se, but rather is, *given* the existence of the optimal control, to induce it via a pricing mechanism, we make the following simplifying assumptions.

A1. The input/output relations (1b) and (1d), are well defined; i.e. given x and u, then y and v are uniquely determined.

A2. The functions $f_i(\cdot)$, $l_i(\cdot)$, and $c_i(\cdot)$, $i = 0, \ldots, n$, are differentiable with respect to their arguments.

A3. The optimal control exists, is unique and is determined by Pontryagin's maximum principle.

Sufficient conditions for the validity of assumption A1 are given in Berger (1983). Assumption A3 holds, of course, for linear systems with quadratic payoffs where the resulting Riccati equation has a solution. Moreover, on p. 567 of Hsu and Meyer (1968) it is shown that the necessary conditions of Pontryagin's maximum principle are sufficient conditions for a class of systems with non-quadratic payoffs and with dynamics nonlinear in the control.

Likewise, since the concern of this paper is the existence of the pricing mechanism, as opposed to its computation, we assume the hypothetical coordinator has perfect knowledge.

A4. The coordinator knows the subsystem models, as well as the coupling system, and hence has perfect knowledge.

Lastly, we assume that:

A5. For any choice of ρ_i and q_i, controller i solves the resulting optimization, as opposed to not-playing-the-game, or shutting-down.

Hence, the pricing mechanisms presented here are relevant for shaping the controllers' marginal behavior.

3. *When the entity priced is the output of the subsystem*

This section examines the case where the entity priced is the output of the subsystem: $q_i \equiv y_i$, $i = 1, \ldots, n$, and where the controllers consider the input to the subsystem from the coupling system, v_i, to be exogenous.† It is assumed that the perfect-knowledge coordinator tells the subsystems the value of the exogenous input v_i in addition to the exogenous input ρ_i. Furthermore, the value for v_i that the coordinator transmits is the coordinator's optimal value. Thus, controller i's optimization becomes

$$\max \int_{t_0}^{t_f} l_i(x_i, u_i, v_i^*) + \rho_i' y_i \, dt \tag{3a}$$

such that

$$\dot{x}_i = f_i(x_i, u_i, v_i^*), \quad x_i(t_0) \text{ given} \tag{3b}$$

$$y_i = c_i(x_i, y_i, v_i^*) \tag{3c}$$

where $\rho_i(t)$ and $v_i^*(t)$ have been specified, $t \in [t_0, t_f]$, and "*" denotes the optimal value of the coordinator.

Theorem 1. If the entity priced is the output of the subsystem $q_i = y_i$, then, under assumptions A1–A5, there exists a pricing mechanism for subsystem i if and only if there exists a function $\rho_i(t)$ that satisfies the condition

$$\frac{\partial f_i'}{\partial u_i}\bigg|_* g_i + \frac{\partial c_i'}{\partial u_i}\bigg|_* (\rho_i - \mu_i^*) = \frac{\partial L'}{\partial u_i}\bigg|_* - \frac{\partial l_i'}{\partial u_i}\bigg|_* \tag{4a}$$

where $\big|_*$ means the term is evaluated at the optimal value of the coordinator, and where μ_i^* is the Lagrange multiplier associated with subsystem i's output relation, equation (1b), and where $g_i(t)$ is determined by

$$\dot{g}_i = -\frac{\partial f_i'}{\partial x_i}\bigg|_* g_i - \frac{\partial c_i'}{\partial x_i}\bigg|_* (\rho_i - \mu_i^*) + \frac{\partial L'}{\partial x_i}\bigg|_* - \frac{\partial l_i'}{\partial x_i}\bigg|_*, \quad g_i(t_f) = 0. \tag{4b}$$

Furthermore, if such a $\rho_i(t)$ exists, it is the price.

The proofs of Theorem 1, and of Theorems 2 and 3 below, are based on finding the condition such that the Pontryagin necessary conditions of the controller match those of the coordinator. The proofs are straightforward, and have been omitted for the sake of brevity.

Theorem 1 states a necessary and sufficient condition for the existence of a price that will induce the individual's optimal control to match that of the coordinator. The condition can be veiwed as finding an input, ρ_i, for a dynamic system (4b) such that the output, the left-hand side of (4a), exactly tracts a given trajectory, the right-hand side of (4a). Such a condition is a type of controllability known as servomechanism controllability (see p. 75 of Brockett (1970)).

3.1. *Satisfaction of condition of Theorem 1 given no knowledge of subsystems.* Dropping assumption A4, consider a real coordinator who does not have perfect knowledge and who would like to know a priori whether the condition of Theorem 1 is satisfied. Typically, such a coordinator does know the coupling system model, and thus we can ask whether there are special cases for the coupling system where the condition of Theorem 1 is guaranteed to be satisfied. In general, $L(\cdot)$ is unrelated to the $l_i(\cdot)$'s, though, an important special case is the linear social welfare function, where $L(\cdot)$ is the sum of the $l_i(\cdot)$'s

$$L(x, u, y, v) = \sum_{i=1}^{n} l_i(x_i, u_i, v_i) + l_0(x_0) \tag{5}$$

where a separate term for x_0 can be included without complication.

Corollary 1. Special case‡ for linear social welfare, $L(\cdot)$ as in (5), the condition of Theorem 1 simplifies and is satisfied trivially when ρ_i is chosen to be μ_i^*.

Note: from the necessary conditions of social welfare

$$\mu_i^* = \left(\frac{\partial c_0'}{\partial y_i}\mu_0 + \frac{\partial f_0'}{\partial y_i}\lambda_0\right)\bigg|_*$$

where μ_0 is the Lagrange multiplier associated with the coupling system equality constraint (1d) and λ_0 is the costate vector associated with the coupling system dynamics (1c).

4. *When the entity priced is the subsystem control*

Returning to arbitrary $L(\cdot)$, when the entity priced is the subsystem control then no condition need be satisfied as there always exists the desired price.

Theorem 2. If the entity priced is the control of the subsystem, $q_i = u_i$, then, under assumptions A1–A5, there exists a pricing mechanism for subsystem i where the price, ρ_i, is chosen as

$$\rho_i = \left(\frac{\partial L}{\partial u_i} - \frac{\partial l_i}{\partial u_i}\right)'\bigg|_* + \frac{\partial c_i'}{\partial u_i}\bigg|_* \mu_i^* - \frac{\partial f_i'}{\partial u_i}\bigg|_* g_i \tag{6a}$$

and where g_i is given by

$$\dot{g}_i = -\frac{\partial f_i'}{\partial x_i}\bigg|_* g_i + \frac{\partial c_i'}{\partial x_i}\bigg|_* \mu_i^* + \left(\frac{\partial L}{\partial x_i} - \frac{\partial l_i}{\partial x_i}\right)'\bigg|_*, \quad g_i(t_f) = 0. \tag{6b}$$

† Berger (1983) discusses the case where the subsystem controller considers the value of v_i to be endogenous and where both y_i and v_i are priced.

‡ The fact that there exists a pricing mechanism for linear social welfare is well known (Findeisen *et al.*, 1980). The interest here is that although pricing mechanisms exist for more general cases, it is linear social welfare that requires minimal knowledge by the coordinator.

5. *When the entity priced is the coupling system state*

In order that the controller's optimization problem be well defined when the entity priced is the coupling system state, the coordinator will inform the controller how the subsystem's output affects x_0. Thus, the ith controller's optimization becomes

$$\max \int_{t_0}^{t_f} l_i(x_i, u_i, v_i^*) + \rho_i' x_0 \, dt \tag{7a}$$

such that

$$\dot{x}_i = f_i(x_i, u_i, v_i'), \quad x_i(t_0) \text{ given} \tag{7b}$$

$$y_i = c_i(x_i, u_i, v_i^*) \tag{7c}$$

$$\dot{x}_0 = f_0(x_0, y_i, y_{-i}^*), \quad x_0(t_0) \text{ given} \tag{7d}$$

where the subscript "$-i$" denotes all the components of the vector except for i.

Controller i could interpret (7d) to mean that the entity priced is some, given function of y_i. Note that the coordinator knowing y_{-i}^* follows from the perfect knowledge assumption, A3, as does knowing v_i^*. (As an aside, if the coupling system were linear, $\dot{x}_0 = A_0 x_0 + \sum B_i y_i$ then the entity priced could be the subsystem's contribution to the state, x_0^i, defined by: $\dot{x}_0^i = A_0 x_0^i - B_i y_i$; in which case, the coordinator would not need to tell the subsystem the output from the other subsystems.)

Theorem 3. If the entity priced is the state vector of the coupling system, $q_i = x_0$, then, under assumptions A1–A5, there exists a pricing mechanism for subsystem i if and only if there exists a function $\rho_i(t)$ that satisfies the condition

$$\left.\frac{\partial f_i'}{\partial u_i}\right|_* g_i + \left.\frac{\partial c_i'}{\partial u_i}\right|_* \left.\frac{\partial f_0'}{\partial y_i}\right|_* g_0 = \left.\left(\frac{\partial c_i'}{\partial u_i}\left(\frac{\partial L'}{\partial y_i} + \frac{\partial c_0'}{\partial y_i}\mu_0\right) + \frac{\partial L'}{\partial u_i} - \frac{\partial l_i'}{\partial u_i}\right)\right|_* \tag{8a}$$

where $g_i(t)$ and $g_0(t)$ are determined by

$$\dot{g}_i = -\left.\frac{\partial f_i'}{\partial x_i}\right|_* g_i - \left.\frac{\partial c_i'}{\partial x_i}\right|_* \left.\frac{\partial f_0'}{\partial y_i}\right|_* g_0 + \left.\left(\frac{\partial c_i'}{\partial x_i}\left(\frac{\partial L'}{\partial y_i} + \frac{\partial c_0'}{\partial y_i}\mu_0\right) + \frac{\partial L'}{\partial x_i} - \frac{\partial l_i'}{\partial x_i}\right)\right|_*, \quad g_i(t_f) = 0 \tag{8b}$$

$$\dot{g}_0 = -\left.\frac{\partial f_0'}{\partial x_0}\right|_* g_0 + \left.\left(\frac{\partial L'}{\partial x_0} + \frac{\partial c_0'}{\partial x_0}\mu_0\right)\right|_* - \rho_i, \quad g_0(t_f) = 0. \tag{8c}$$

Furthermore, if such a ρ_i exists, it is the price.

Theorem 3, like Theorem 1, states a necessary and sufficient condition for the existence of a price that will induce the individual's optimal control to match that of the coordinator. However, unlike Theorem 1, the condition does not trivialize when $L(\cdot)$ is a linear social welfare function. Thus, the condition for existence of the price is more restrictive when the entity priced is x_0 as opposed to y_i.

5.1. *Satisfaction of condition of Theorem* 3 *given no knowledge of subsystems.* As in Section 3.1, consider a real coordinator who does not know the subsystems and who would like to know whether the condition of Theorem 3 can be guaranteed to be satisfied based on information of the coupling system model. Indeed it can, though under more restrictive assumptions than in Theorem 1.

Corollary 3. Special case: for linear social welfare, $L(\cdot)$ as in (5), and for $c_0(\cdot)$ not dependent on y_i, the condition of Theorem 1 simplifies and is satisfied trivially when ρ_i is chosen to be

$$\left.\left(\frac{\partial c_0'}{\partial x_0}\mu_0 + \frac{\partial l_0'}{\partial x_0}\right)\right|_*.$$

6. *Comparison of entities priced*

We compare the entities priced from two perspectives: (1) the range of systems for which the pricing mechanism exists and (2) the coincidence of costate vectors and Lagrange multipliers.

6.1. *Range of systems.* When the entity priced is the subsystem control, there always exists a pricing mechanism, Theorem 2 (for the class of systems under discussion: equations (1) and (2) subject to assumptions A1–A5). In contrast, when the entity priced is the subsystem output or the coupling system state, then the existence of the pricing mechanism is conditional. Furthermore, the condition is more restrictive for the coupling system state than the subsystem output. Thus, from the viewpoint of the range of systems for which the pricing mechanism exists, the subsystem control is a better entity to price than the subsystem output, which in turn is a better entity than the coupling system state.

6.2. *Coincidence of costate vectors and Lagrange multipliers.* The range of systems is only one perspective from which to compare the entities. It has the drawback that it says nothing about particular systems where the pricing mechanism exists for multiple entities. We obtain another perspective by noting that the basic idea of a pricing mechanism is to induce the individual's view of the overall system to coincide with that of the coordinator. A deeper coincidence of world views is attained if, in addition to the optimal control, the pricing mechanism also induces the controller's costate and Lagrange multipliers to be equal to those of the coordinator.

From the perspective of matching costates, x_0 would seem to be a better entity to price than y_i, since the subsystem's optimization contains more costate vectors. However, it is not a priori clear whether the subsystem's optimal value for the costate equals that of the coordinator. In this regard, the state variables g_i and g_0 in the pricing mechanism theorems, equations (4b), (6b), (8b), and (8c) have particular importance. In all cases, $g_i(t) = \lambda_i^{\text{ind}} - \lambda_i^*$ and $g_0(t) = \lambda_0^{\text{ind}} - \lambda_0^*$, where superscript "ind" denotes the optimal value of the individual controller. Moreover, $g_i(t)$ and $g_0(t)$ are identically zero for the special cases in Corollaries 1 and 3 above. In Corollary 3, where the entity priced is x_0, the controller's optimal value for λ_i, μ_i and λ_0 coincide with that of the coordinator; while in Corollary 1, where the entity priced is y_i, just λ_i and μ_i coincide. Lastly, when the entity priced is μ_i, costate and Lagrange multipliers coincide only for degenerate cases.

7. *Conclusion*

For the classic case of pricing the subsystem output and of the coordinator's payoff being linear social welfare, we show that the pricing mechanism, in addition to inducing a coincidence of optimal controls, also induces the optimal costate vector of the subsystem to match that of the coordinator. The matching of costate vectors, in addition to the control, represents a fuller coincidence of world views than matching the control alone. When the classic case is generalized to arbitrary coordinator payoffs, then the pricing mechanism continues to exist for cases where a servomechanism controllability condition is satisfied, but at the expense of the coincidence of costate vectors.

When the entity priced is the subsystem control, the pricing mechanism exists for a larger class of systems than for the subsystem output, for which in turn, the class of systems is larger than for the coupling system state. On the other hand, for special cases where the conditions for existence of the pricing mechanism are guaranteed to be satisfied based solely on the form of the coupling system model, pricing the coupling system state induces a coincidence between the coordinator and the subsystem of both the costate vectors of the subsystem dynamics and the coupling system dynamics. Pricing the subsystem output, however, only induces a coincidence of the costate vector of the subsystem dynamics; and pricing the subsystem control induces no coincidence at all.

Acknowledgement—This work was supported by MIT Energy Lab. Contract #EX-76-A-012295 and the Office of Naval Research Contract # N00014-75-C-0648.

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