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Brief Paper

Market-clearing Prices in Coupled, Dynamic Systems*

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Key Words—Hierarchical decision-making; decentralized control; economic systems; Stackelberg incentives.

Abstract—A standard, hierarchical control formulation is used for the economic problem of price coordination of subsystems connected by a general coupling system. With this formulation, where the coordinating price for a given input or output *a priori* may be different for each subsystem, we determine conditions on the coupling system whereby the coordinating prices are the same for all subsystems. We show that the coupling system does not need to be specialized to the constraint that the sum of the outputs minus the sum of the inputs (the excess supply) be zero. On the other hand, a coupling system with symmetrically appearing input–output vectors is not restrictive enough. Rather, there exists a common, coordinating price vector for all subsystems if the coupling system is a general *function* of excess supply.

1. *Introduction*

In classical economic theory, the market-clearing price (the price at which supply equals demand) is the same for all producers and consumers. In contrast, in hierarchical control theory, the coordinating dual variable is not so constrained and, in general, is different for different subsystems; see for example Cohen (1980), Findeisen *et al.* (1980) and Mesarovic *et al.* (1970). The contribution of this paper is to use a general formulation from hierarchical control theory and to determine conditions where the coordinating prices to all subsystems are the same for a given output, or the same for a given input and conditions where, in addition, the price vector for output equals the price vector for inputs (herein called market-clearing prices). One can view the market-clearing price herein as equilibrating supply and demand where the equilibration is the satisfaction of contraints that are more general than the excess supply being zero. It coincides with the standard definition, given the particular coupling system that constrains the sum of the outputs to be the sum of the inputs.

The conditions below only require knowledge of the coupling system, and thus, from the viewpoint of hierarchical control theory, the coordinator can know *a priori* whether there exist market-clearing prices. If the conditions are satisfied and if the coordinator uses an iteration to determine the coordinating, dual variables, then the iteration can be over a common vector for all subsystems. From an economic viewpoint, the coordinator of the hierarchical control formulation is an artifice to determine the existence of market-clearing prices. If the solution to the coordinator's optimization is such that all price vectors are equal, then there exists market-clearing prices for the competitive market of the price-taking subsystems.

The following section, Section 2, presents the problem formulation. Section 3 gives the results, and Section 4 illustrates the results with four examples.

2. *Problem formulation*

We consider a dynamic system consisting of n subsystems connected via a coupling system. The ith subsystem is modeled as:

$$\max_{u_i, v_i} \int_{t_o}^{t_f} \{l_i(x_i, u_i, v_i) + \rho'_{y_i} y_i - \rho'_{v_i} v_i\}\, dt + \phi_i(x_i(t_f)) \quad (1a)$$

$$\dot{x}_i = f_i(x_i, u_i, v_i) \quad x_i(t_o) \text{ given} \quad (1b)$$

$$y_i = c_i(x_i, u_i, v_i) \quad (1c)$$

where $u_i(t) \in R^{k_i}$ is the control vector for subsystem i, and $x_i(t) \in R^{n_i}$ is the state vector of subsystem i. $v_i(t) \in R^m$ is the vector of inputs from the coupling system to the subsystem i and is called the interaction input or factor input or simply the input. $y_i(t) \in R^m$ is the vector of outputs from subsystem i to the coupling system. $\rho_{y_i} \in R^m$ is the price vector for the outputs of subsystem i, and $\rho_{v_i} \in R^m$ is the price vector for the interaction inputs for subsystem i. m, n_i, $k_i \in$ positive integers, $i = 1, \ldots, n$. Furthermore, let y denote the vector of all of the y_is, i.e. $y' \triangleq (y'_1, \ldots, y'_n)$ where $'$ denotes transpose. Similarly, $v' \triangleq (v'_1, \ldots, v'_n)$, and $u' \triangleq (u'_1, \ldots, u'_n)$. The subsystem decision maker (agent, controller) optimizes over the control variable and interaction input and considers the prices as given, exogenous inputs.

The problem of the coordinator (market maker) is to choose ρ_{y_i} and ρ_{v_i} so that the individual agent's optimal value for v_i, denoted v_i^{ind}, and the optimal control, u_i^{ind}, match those from the coordinator's optimization:

$$\max_{u, v} \int_{t_o}^{t_f} \left\{l_o(x_o, y, v) + \sum_{i=1}^{n} l_i(x_i, u_i, v_i)\right\} dt + \sum_{i=0}^{n} \phi_i(x_i(t_f)) \quad (2a)$$

$$\dot{x}_i = f_i(x_i, u_i, v_i) \quad x_i(t_o) \text{ given } i = 1, \ldots, n \quad (2b)$$

$$\dot{x}_o = f_o(x_o, y, v) \quad x_o(t_o) \text{ given} \quad (2c)$$

$$y_i = c_i(x_i, u_i, v_i) \quad i = 1, \ldots, n \quad (2d)$$

$$0 = c_o(x_o, y, v) \quad (2e)$$

where $x_o(t) \in R^{n_o}$ is the state vector of the coupling system. The coupling system is characterized by the terms:

$$l_o(x_o, y, v),\ f_o(x_o, y, v) \text{ and } c_o(x_o, y, v) \quad (3)$$

2.1. *Economic interpretation of the problem.* The subsystem decision makers are price takers, since they consider the prices as given, exogenous inputs. The common dimension of the vectors y_i, v_i, ρ_{y_i} and ρ_{v_i} equals the number of goods in the model, m. Consider for the moment, a model of one good, $m = 1$, and subsystem i being either a firm producing the good at level y_i, or a consumer consuming the good in the amount v_i. (If subsystem i is a firm, then v_i

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would be set equal to zero, and likewise, if subsystem i is a consumer, then $y_i = 0$.) For the case of one good, the firm's factor inputs are modeled implicitly by the firm's cost function. In particular, if subsystem i is a firm, then let $l_i(\cdot)$ be the negative of the firm's cost function. If we further suppose that the firm can directly control the level of output (i.e. suppose $f_i(\cdot) \equiv 0$ and $y_i = c_i(\cdot) \equiv u_i$), then $l_i(\cdot)$ can be written as $-\text{cost}_i\,(y_i)$ and the model for the subsystem, (1), reduces to the static optimization:

$$\max_{y_i} \; -\text{cost}_i\,(y_i) + \rho_{y_i} y_i, \quad y_i \geq 0, \tag{4}$$

which is the classic profit-maximization criterion for a price-taking firm. Likewise, if subsystem i is a consumer, then let $l_i(\cdot)$ be a monetarily measure satisfaction index, and given that the consumer directly controls the level of consumption, v_i, then the subsystem optimization (1) reduces to the analogous static optimization:

$$\max_{v_i} l_i(v_i) - \rho_{v_i} v_i, \quad v_i \geq 0. \tag{5}$$

Thus, for given prices ρ_{y_i} and ρ_{v_i}, the total amount supplied is $\sum_{i=1}^{n} y_i^{\text{ind}}$ and the total amount demanded is $\sum_{i=1}^{n} v_i^{\text{ind}}$, where the superscript "ind" means the individual agent's optimal value. By varying the prices, the above sums become supply and demand curves, respectively.

The above description easily generalizes to a vector of goods, $m > 1$. Also, the product of one firm could be a factor input of another firm, in which case, (4) becomes:

$$\max_{y_i, v_i} - \text{cost}_i\,(y_i, v_i) + \rho_{y_i} y_i - \rho_{v_i} v_i, \quad y_i \geq 0, \quad v_i \geq 0, \tag{6}$$

where $\text{cost}_i\,(y_i, v_i)$ represents the cost to produce level y_i, which depends on the firm's production possibilities set and possible additional factor inputs besides v_i. Likewise, one could reintroduce the dynamic constraints (1b) and the firm would control the level of output through (1b) and (1c).

A question of economic interest is whether there exists a market-clearing price: a price ρ^o such that if $\rho_{y_i} = \rho_{v_i} = \rho^o$ for all $i = 1, \ldots, n$, then the total supply equals the total demand: $\sum_{i=1}^{n} y_i^{\text{ind}} = \sum_{i=1}^{n} v_i^{\text{ind}}$. This question is answered by the solution to the coordinator's optimization, (2). One can think of the coordinator's optimization as an artifice to determine the existence of market-clearing prices. When the solution to (2) is such that all the price vectors are equal, $\rho_{y_i} = \rho_{v_j}$ for all $i, j = 1, \ldots, n$, then there exists market-clearing prices for the competitive market of the n price-taking subsystems. In particular, a special case of the coupling system is the equality of supply and demand:

$$l_o(\cdot) = 0, \; f_o(\cdot) = 0 \text{ and } c_o(\cdot) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} v_i.$$

For this coupling system, suppose the coordinator solves (2), and let μ_o^{coor} be the Lagrange multiplier associated with (2e) evaluated at the coordinator's optimal control. If the coordinator chooses the price vectors to be $\rho_{y_i} = \rho_{v_i} = \mu_o^{\text{coor}}$ for all $i = 1, \ldots, n$, then the agent i's optimal values, y_i^{ind} and v_i^{ind}, will coincide with the coordinator's, and $\sum_{i=1}^{n} y_i^{\text{ind}} - \sum_{i=1}^{n} v_i^{\text{ind}}$ will equal 0 (see Lemma 1 below). Thus, there exists a price vector, namely μ_o^{coor} such that the total supply equals the total demand.

For the general coupling system, (3), and under reasonable assumptions (assumptions 1–3 below), the coordinator indeed can choose ρ_{y_i} and ρ_{v_i} such that $u_i^{\text{ind}} = u_i^{\text{coor}}$ and $v_i^{\text{ind}} = v_i^{\text{coor}}$. However, in general, the prices will not be the same for all subsystems. The interest of this paper is to determine those coupling systems where the coordinating prices are equal for all subsystems. This corresponds to the existence of market-clearing prices in the competitive market of the n price-taking subsystems.

2.2. *Game theory interpretation of the problem.* From the viewpoint of game theory, the above problem formulation is a simple Stackelberg incentive problem, a topic that has received much attention; see, for example Başar and Olsder (1980), Ho *et al.* (1982), Zheng *et al.* (1984). In the vocabulary of game theory, the coordinator is the leader. The leader's information set is empty, and the leader acts once and first by announcing the price vectors ρ_{y_i} and ρ_{v_i}, $i = 1, \ldots, n$. The subsystem decision makers are the followers. The information set of follower i is $\{\rho_{y_i}, \rho_{v_i}\}$, and follower i's action is the open loop control law that solves (1). An optimal choice for the leader's action would induce the followers' actions to solve (2) also. Thus, from the viewpoint of game theory, the interest of this paper is the form of the leader's optimal action; namely, whether it is the same for all followers.

3. *Conditions for coordinating, uniform and market-clearing prices*

Definition. Price vectors are called coordinating prices if they induce v_i^{ind} and u_i^{ind} to be equal to the optimal values, u_i^{coor} and v_i^{coor}, of the coordinator's optimization (2).

With coordinating prices, the constraints of the coupling system are satisfied; however, this is accomplished with prices that for a given good may be different for each subsystem. The following two definitions place additional requirements on the choice of the price vectors.

Definition. Particular choices for ρ_{y_i} and ρ_{v_i}, $i = 1, \ldots, n$ are called uniform prices if:

1. they are coordinating prices, and
2. $\rho_{y_i} = \rho_{y_j}$ for all $i, j = 1, \ldots, n$, and
3. $\rho_{v_i} = \rho_{v_j}$ for all $i, j = 1, \ldots, n$.

Definition. Particular choices for ρ_{y_i} and ρ_{v_i}, $i = 1, \ldots, n$ are called market-clearing prices if:

1. they are coordinating prices, and
2. $\rho_{y_i} = \rho_{v_j}$ for all $i, j = 1, \ldots, n$.

Note that market-clearing prices are also uniform prices; however, uniform prices are not necessarily market-clearing prices.

Although the optimizations (1) and (2) are stated for nonlinear dynamics and output relations, we are not interested here in the difficulties of solving such problems, but rather, given that the optimal control exists, what is the form of the coordinating prices? Thus, we make simplifying assumptions that guarantee the existence of the optimal control:

Assumption 1. the functions $f_i(\cdot)$, $l_i(\cdot)$ and $c_i(\cdot)$, $i = 0, \ldots, n$, are differentiable with respect to their arguments.

Assumption 2. The optimal control exist and is unique and is determined by Pontryagin's maximum principle.

Assumption 3. For any choice of ρ_{y_i} and ρ_{v_i}, agent i solves the resulting optimization, as opposed to not-playing-the-game, or shutting-down.

Assumption 2 holds of course for linear systems with quadratic payoffs where the resulting Riccati equation has a solution. Moreover, Hsu and Meyer (1968, p. 567) show that the necessary conditions of Pontryagin's maximum principle are sufficient conditions for a class system with nonquadratic payoffs and with dynamics nonlinear in the control. Assumption 3 implies that the pricing mechanisms presented here are relevant for shaping the agent's marginal behavior.

3.1. *Coordinating prices.* Given assumption 2 and that the $l_i(\cdot)$s appear as a sum in (2a), it is well known that there exist coordinating prices (Findeisen *et al.*, 1980). If the $l_i(\cdot)$s did not appear as a sum in (2a), then the coordinating prices do not necessarily exist. Berger and Schweppe (1989) consider a coordinator with an arbitrary objective function, which may or may not be a function of the $l_i(\cdot)$s, and they present necessary and sufficient conditions for the existence of coordinating prices. The interest there is a comparison of three potential *entities-to-be-priced*: the subsystem control, the subsystem output and the coupling system state. For the present model, we have the following lemma.

Lemma 1. Under assumptions 1–3, the individual subsystem's optimal controls, u_i^{ind}, v_i^{ind} equal the coordinator's optimal values, u_i^{coor}, v_i^{coor}, $i = 1, \ldots, n$, if the prices are chosen to be:

$$\rho_{y_i} = \left(\frac{\partial l_o}{\partial y_i} + \mu_o' \frac{\partial c_o}{\partial y_i} + \lambda_o' \frac{\partial f_o}{\partial y_i}\right)\bigg|_{\text{coor}} \tag{7a}$$

$$\rho_{v_i} = -\left(\frac{\partial l_o}{\partial v_i} + \mu_o' \frac{\partial c_o}{\partial v_i} + \lambda_o' \frac{\partial f_o}{\partial v_i}\right)\bigg|_{\text{coor}} \tag{7b}$$

where μ_o is the Lagrange multiplier associated with the coupling system output relation, (2e), and λ_o is the costate vector associated with the coupling system dynamics, (2c), and $|_{\text{coor}}$ means the expression is evaluated at the coordinator's optimal value.

The proof of Lemma 1 chooses the prices so that the necessary conditions of agent i's optimization match those of the coordinator's. The proof is straightforward and has been omitted for the sake of brevity.

Lemma 1 confirms the heuristic idea that the prices ought to reflect how the subsystem influences the outside world; in particular, the price is the marginal effect of the input and output variables on the coupling system. In the degenerate case where $l_o(\cdot)$, $c_o(\cdot)$ and $f_o(\cdot)$ are not functions of y_i or v_i for some i, the prices are zero; as they ought to be, since the subsystem is decoupled from the rest of the overall system.

3.2. *Uniform prices.* In general, the coordinating prices of Lemma 1, equation (7), are different for different subsystems. However, in the special case when the functional form of the coupling system is such that:

$$\frac{\partial}{\partial y_i}(l_o(x_o, y, v) + \lambda_o' f_o(x_o, y, v) + \mu_o' x_o(x_o, y, v))$$
$$= \frac{\partial}{\partial y_j}(l_o(x_o, y, v) + \lambda_o' f_o(x_o, y, v) + \mu_o' c_o(x_o, y, v)) \tag{8a}$$

then the prices ρ_{y_i} and ρ_{y_j} are guaranteed to be equal when (8a) is evaluated at the coordinator's values. Furthermore, we have the following lemma.

Lemma 2.

$$\frac{\partial}{\partial y_i}(l_o(x_o, y, v) + \lambda_o' f_o(x_o, y, v) + \mu_o' c_o(x_o, y, v))$$
$$= \frac{\partial}{\partial y_j}(l_o(x_o, y, v) + \lambda_o' f_o(x_o, y, v) + \mu_o' c_o(x_o, y, v)) \tag{8a}$$

and

$$\frac{\partial}{\partial v_i}(l_o(x_o, y, v) + \lambda_o' f_o(x_o, y) + \mu_o' c_o(x_o, y, v))$$
$$= \frac{\partial}{\partial v_j}(l_o(x_o, y, v) + \lambda_o' f_o(x_o, y) + \mu_o' c_o(x_o, y, v)) \tag{8b}$$

for all $i, j = 1, \ldots, n$ if and only if there exist functions $\bar{l}_o(\cdot)$, $\bar{f}_o(\cdot)$ and $\bar{c}_o$ such that:

$$l_o(x_o, y, v) = \bar{l}_o(x_o, y_T, v_T) \tag{9a}$$
$$f_o(x_o, y, v) = \bar{f}_o(x_o, y_T, v_T) \tag{9b}$$
$$c_o(x_o, y, v) = \bar{c}_o(x_o, y_T, v_T) \tag{9c}$$

where $y_T \triangleq \sum_{i=1}^n y_i$ and $v_T \triangleq \sum_{i=1}^n v_i$.

Outline of proof of Lemma 2: given (9) then (8) holds by a simple application of the chain rule. The reverse direction is more involved. The idea is to show that (8) implies that the isoclines of

$$l_o(x_o, y, v) + \lambda_o' f_o(x_o, y) + \mu_o' c_o(x_o, y, v) \tag{10}$$

with respect to y are the surfaces $\sum_{i=1}^n y_i = \text{a constant}$. Hence, (10) can be expressed as terms where the y_is appear only as a sum. The details are fairly mechanical and are omitted for the sake of brevity. An analogous argument applies to the v_is.

Lemmas 1 and 2 imply the following theorem.

Theorem 1. If the output variables appear as a sum in the coupling system, (3), and the interaction inputs also appear as a sum in the coupling system, then there exist coordinating prices that are uniform; i.e. if there exist functions $\bar{l}_o(\cdot)$, $\bar{f}_o(\cdot)$ and $\bar{c}_o(\cdot)$ such that:

$$l_o(x_o, y, v) = \bar{l}_o(x_o, y_T, v_T) \tag{9a}$$
$$f_o(x_o, y, v) = \bar{f}_o(x_o, y_T, v_T) \tag{9b}$$
$$c_o(x_o, y, v) = \bar{c}_o(x_o, y_T, v_T) \tag{9c}$$

where $y_T \triangleq \sum_{i=1}^n y_i$ and $v_T \triangleq \sum_{i=1}^n v_i$, then there exist choices for ρ_{y_i} and ρ_{v_i} $i = 1, \ldots, n$ that are uniform prices. Moreover, the functions $\bar{l}_o(\cdot)$, $\bar{f}_o(\cdot)$ and $\bar{c}_o(\cdot)$ are the only functional forms that guarantee the existence of coordinating prices that are uniform.

Theorem 1 shows that to guarantee uniform prices it is not restrictive enough for the input and output variables to appear symmetrically in the coupling system, rather they ought to appear as a sum. The converse of Theorem 1 does not hold. It is possible that although (8) and (9) are violated, when the coordinator's optimal values are plugged in, the prices turn out to be the same. One can construct simple examples where this is the case. However, these examples are highly dependent on the subsystem models; with a slight change in parameter values, the prices become unequal. Thus, although strictly speaking Theorem 1 is only a sufficient condition for uniform prices, if one allows for unknown variations in the subsystem models, then, as a practical matter, Theorem 1 can be treated as a necessary condition as well. Thus, in a practical situation where the coordinator does not know the subsystem models and is iteratively searching for coordinating prices, Theorem 1 can be used as both a necessary and sufficient condition for uniform prices. Note that the condition for uniform prices, equation (9) depends only on the functional form of the coupling system, information typically known by a real coordinator.

The proof of Lemma 2 easily generalizes to cases where only a subset of the input and output vectors appear as a sum in the coupling system. Hence, we have the following generalization of Theorem 1.

Theorem 2. If for a subset of the subsystems $I \subseteq \{1, \ldots, n\}$ the output vectors appear as a sum in coupling system, then there exist coordinating prices such that $\rho_{y_i} = \rho_{y_j}$ for all $i, j \in I$.

The analogous statement applies for interaction inputs.

3.3. *Market-clearing prices.* In analogy to Lemma 2, we have:

Lemma 3.

$$\frac{\partial}{\partial y_i}(l_o(x_o, y, v) + \lambda_o' f_o(x_o, y, v) + \mu_o' c_o(x_o, y, v))$$
$$= -\frac{\partial}{\partial v_j}(l_o(x_o, y, v) + \lambda_o' f_o(x_o, y, v) + \mu_o' x_o(x_o, y, v))$$

for all $i, j = 1, \ldots, n$ if and only if there exist functions $\hat{l}_o(\cdot)$, $\hat{f}_o(\cdot)$ and $\hat{c}_o(\cdot)$ such that:

$$l_o(x_o, y, v) = \hat{l}_o(x_o, z)$$
$$f_o(x_o, y, v) = \hat{f}_o(x_o, z)$$
$$c_o(x_o, y, v) = \hat{c}_o(x_o, z)$$

where $z \triangleq \sum_{i=1}^n y_i - v_i$.

From an economic viewpoint, z is known as the excess supply. The proof of Lemma 3 is analogous to that of Lemma 2.

Lemmas 1 and 3 imply:

Theorem 3. If the coupling system (3) is a function of excess supply (if output variables and interaction inputs appear as a difference of sums in the coupling system), then there exist coordinating prices that are market clearing, i.e. if there exist functions $\hat{l}_o(\cdot)$, $\hat{f}_o(\cdot)$ and $\hat{c}_o(\cdot)$ such that:

$$l_o(x_o, y, v) = \hat{l}_o(x_o, z) \tag{11a}$$

$$f_o(x_o, y, v) = \hat{f}_o(x_o, z) \tag{11b}$$

$$c_o(x_o, y, v) = \hat{c}_o(x_o, z) \tag{11c}$$

where $z \triangleq \sum_{i=1}^{n} y_i - v_i$, then there exist choices for ρ_{y_i} and ρ_{v_i} $i = 1, \ldots, n$ that are market-clearing prices. Moreover, the functions $\hat{l}_o(\cdot)$, $\hat{f}_o(\cdot)$ and $\hat{c}_o(\cdot)$ are the only functional forms that guarantee the existence of coordinating prices that are market clearing.

Theorem 3 shows that the classical supply-demand equality constraint (excess supply = 0) is not a requirement for the existence of a common, coordinating, price vector. Rather, the market mechanism (coupling system) could be more intricate, and as long as it is a *function* of excess supply, then a common, coordinating, price vector exists. Also, note that the condition for market-clearing prices, (11), satisfies that for uniform prices, (9).

In analogy to Lemma 2, Lemma 3 also generalizes to a subset of the input–output vectors. Thus, in analogy to Theorem 2, we have:

Theorem 4. If for a subset of the subsystems $I \subseteq \{1, \ldots, n\}$ the coupling system is a function of excess supply (the output vectors and interaction inputs appear as a difference of sums in the coupling system), then there exist coordinating prices such that $\rho_{y_i} = \rho_{v_j}$ for all $i, j \in I$.

4. *Illustration of results*

Example 1. *The interaction matrix*. In hierarchical control, the classical coupling system is the interaction matrix given by: $v = H \cdot y$. The components of H are either one or zero, depending on whether a given component of an output vector equals a given component of an interaction input vector. For the interaction matrix, the general coupling system (3) specializes to:

$$l_o(\cdot) = 0, \quad f_o(\cdot) = 0, \quad c_o(\cdot) = H \cdot y - v.$$

Note that for this coupling system the y_is do not appear as a sum, nor do the v_is. Applying Theorem 1, we see that, in general, the prices are not uniform prices.

As an aside, there are properties that are known about these prices (Findeisen *et al.*, 1980). Partitioning H into submatrices and μ_o into subvectors, we can write $\mu_o' c_o(\cdot)$ as:

$$\mu_o'(H \cdot y - v) = \sum_{j=1}^{n} \sum_{i=1}^{n} \mu_{oj}' H_{ji} y_i - \sum_{j=1}^{n} \mu_{oj}' v_j.$$

Applying Lemma 1, we obtain the prices for agent i to be: $\rho_{y_i} = \sum_{j=1}^{n} H_{ji}' \mu_{oj}$ and $\rho_{v_i} = \mu_{oi}$. Hence, $\rho_{y_i} = \sum_{j=1}^{n} H_{ji}' \rho_{v_i}$. Thus, if ρ_{v_i} $i = 1, \ldots, n$ are known then so are ρ_{y_i} $i = 1, \ldots, n$. Another property is that the sidepayments sum to zero:

$$\begin{aligned}
\text{sum of all sidepayments} &= \sum_{i=1}^{n} \rho_{y_i}' y_i - \rho_{v_i}' v_i \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \mu_{oj}' H_{ji} \right) y_i - \mu_{oi}' v_i \\
&= \mu_o' H y - \mu_o' v = \mu_o'(Hy - v) = 0.
\end{aligned}$$

The interaction matrix is quite general; all of the following examples can be packaged into the form $v = Hy$ by considering the coupling system as another subsystem. The interaction inputs to this additional "subsystem," denoted v_o, are y, and the outputs, denoted y_o, are v. From this viewpoint, the expanded interaction matrix is:

$$\begin{bmatrix} v_o \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \cdot \begin{bmatrix} y_o \\ y \end{bmatrix}.$$

However, we have obscured the relevant structure. All that has been done is to introduce two dummy composite vectors, y_o and v_o, and the question whether there exist uniform or market-clearing prices for subsystems 1 to n still hinges on the form of $l_o(\cdot)$, $f_o(\cdot)$ and $c_o(\cdot)$, which have been packaged as a 0th subsystem.

Example 2. *Classical market equilibrium with transaction costs*. Consider the classical supply-demand constraint where the total supply equals the total demand: $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} v_i$. Suppose the market maker incurs linear transaction costs, then the coupling system is:

$$l_o(\cdot) = -\alpha' y - \beta' v = \sum_{i=1}^{n} -\alpha_i' y_i - \beta_i' v_i, \quad f_o(\cdot) = 0,$$

$$c_o(\cdot) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} v_i$$

where α and β are non-negative vectors equal to the unit transaction costs.

Since $l_o(\cdot)$ is not a function of y_T and v_T, then by Theorem 1, the coordinating prices are not, in general, uniform prices. However, if the unit costs are such that for all i, α_i equals a given vector, say α_c, and β_i equals a given vector β_c, then the coupling system *is* a function of y_T and v_T and there exist coordinating prices that are uniform. If, in addition, $\alpha_v = \beta_c = 0$, then the transaction costs are zero, the coupling system is a function of excess supply and there exist market-clearing prices.

Example 3. *Subsystems feed a downstream process*. Suppose the outputs of the subsystems feed into a downstream process. A possible coupling system that models the downstream process is:

$$l_o(x_o), \dot{x}_o = f_o(x_o, y).$$

Applying Theorem 1, if the subsystem outputs y_i enter $f_o(\cdot)$ as a sum then there exist prices that are the same for all subsystems.

As an illustration, suppose the subsystems are manufacturing plants, and are charged dumping fees for discharging waste, y_i, into a river. Let x_o be a measure of the status of the river, such as oxygen concentration, or bacteria counts, and $-l_o(x_o)$ be a measure of social cost. If x_o is affected by just the total discharge, then $f_o(\cdot)$ would be a function of y_T. If, however, the spatial location of the plant along the river mattered, then $f_o(\cdot)$ would not be a function of y_T and the unit dumping fees should not be uniform.

Example 4. *Electric power systems*. This example is based on work of Bohn, Caramanis, Schweppe and Tabor (1984, 1988) who develop a theory of electricity pricing that accurately reflects the underlying physical and engineering properties of electric power systems. Let subsystem i be either a generator producing power y_i or a load consuming power v_i. The key, coupling-system equation is the power balance equation: sum of generation = sum of demand + losses in the transmission network. Thus, the coupling is:

$$l_o(\cdot) = 0 \quad f_o(\cdot) = 0 \quad c_o(\cdot) = \sum_{\text{generators}} y_i - \sum_{\text{loads}} v_i - \text{Loss}(y, v).$$

There are other constraints, which for simplicity are ignored here; for details, see Bohn *et al.* (1984), Schweppe *et al.* (1988).

For those generators and loads that are connected to a common bus of the transmission network, the generation and demand appear as a difference of sums in the Loss function. Thus, for a transmission network of N buses, the Loss function has the form:

$$\text{Loss}(y, v) = \tilde{\text{L}}\text{oss}\left(\sum_{\substack{\text{generators} \\ \text{\& loads} \\ \text{at bus 1}}} y_i - v_i, \ldots, \sum_{\substack{\text{generators} \\ \text{\& loads} \\ \text{at bus } N}} y_i - v_i \right).$$

($\tilde{\text{L}}$oss is a nonlinear function of its arguments, and is frequently approximated as a quadratic form.) Thus, y and v appear in the coupling system as a difference of *partial* sums. Thus, from Theorem 4, the price for electricity is the same for all generators and loads connected to a common bus, but the price differs between buses.

5. *Conclusions*

We have considered a standard, hierarchical control formulation for the economic problem of price coordination of subystems connected by a general, coupling system. We have determined conditions where the coordinating prices to all agents turn out to be the same for a given output, or the same for a given interaction input (termed uniform prices), and conditions where, in addition, the price vector for outputs equals the price vector for inputs (called market-clearing prices herein). One can view the market-clearing price herein as equilibrating supply and demand where the equilibration is the satisfaction of constraints that are more general than the excess supply being zero. It coincides with the standard economic definition, given the particular coupling system that constrains the sum of the outputs to be the sum of the inputs.

To guarantee the existence of market-clearing prices, we have shown that the general coupling system does not need to be specialized to the constraint that the excess supply be zero. On the other hand, a coupling system with symmetrically appearing input–output vectors is not restrictive enough. Rather, there exist uniform prices if the coupling system is a function of the sum of the output vectors and the sum of the interaction inputs, and there exist market-clearing prices if the coupling system is a general *function* of excess supply. This result expands the classical excess-supply-equals-0 market to a larger class of systems where the market maker can know *a priori* whether there exist uniform or market-clearing prices.

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