WP10 - 5:30

Proceedings of 23rd Conference on Decision and Control
Las Vegas, NV, December 1984

PRICING MECHANISMS IN THE CONTROL OF LINEAR DYNAMIC SYSTEMS

Arthur W. Berger

AT&T Bell Laboratories
Room WB 1A-356
Holmdel, N.J. 07733

Abstract

A necessary and sufficient condition is given for the existence of prices that induce coordination in coupled linear dynamic systems. The condition is shown to be equivalent to the servomechanism controllability of an adjoint system.

1. Introduction

A characteristic of today's modern technological society is the growth of complex interconnected systems whose operation is influenced by the independent actions of many people. Much studied examples are electric power systems, telecommunications systems, computer networks and economic systems. Furthermore, prices are commonly used to assist in the operations of these systems. Recent work by F. Schweppe, R. Bohn and M. Caramanis [1,2] develops the theory of pricing in electric power systems in a framework that includes the various time-of-day and peak-load-pricing schemes currently in use, as well as the extension to time scales as fast as five minutes.

The following note reports theoretical results on the existence of prices as a means of coordination in general dynamic systems. The results extend the work of Schweppe et al. to the case of nonlinear social welfare functions and to the explicit modeling of the system's dynamics. The application of these results to the use of pricing mechanisms in the control of average system frequency in electric power systems is contained in [3] and will be reported elsewhere.

2. Problem Formulation

Consider a deterministic linear dynamic system consisting of a number of coupled subsystems each under the control of an independent decision maker, with his/her own prior payoff. The question of interest is: does there exist a pricing mechanism (a side payment to the individual's payoff) that induces the optimal control of the independent decision maker to equal the optimal control of a coordinator who, in turn, has his/her own criterion?

Specifically, the i^{th} subsystem is modeled as:

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) \qquad x_i(t_o) \text{ given} \qquad i=1,\dots,n \tag{1a}$$

$$y_i(t) = C_i x_i(t) + D_i u_i(t) \qquad i=1,\dots,n \tag{1b}$$

and the coupling system is modeled as:

$$\dot{x}_o(t) = A_o x_o + \sum_{i=1}^{n} B_{oi} y_i(t) \qquad x_o(t_o) \text{ given} \tag{1c}$$

$$0 = C_o x_o + D_o y + e(t) \tag{1d}$$

where $u_i \in R^{m_i}$ is the control vector for subsystem i, $x_i \in R^{n_i}$ is the state vector of subsystem i, or of the coupling system for i=0, y_i is the vector of outputs from subsystem i to the coupling system, and e(t) is given, i=1,...,n; m_i, n_i, r_i, $\in$ positive integers. Furthermore, let "u" denote the vector of all of the u_i's, i.e. $u' \triangleq (u_1',\dots,u_n')$ where ' denotes transpose. Similarly, $x' \triangleq (x_o', x_1',\dots,x_n')$, $y' \triangleq (y_1',\dots,y_n')$.

Associated with each subsystem there exists a decision maker with a criterion function

$$\max \int_{t_o}^{t_f} l_i(x_i,u_i) + \rho_i' y_i \, dt \tag{2a}$$

where $l_i(\cdot)$ might represent individual i's cost function or monetarily measured satisfaction index. The term: $\int_{t_o}^{t_f} \rho_i' y_i \, dt$ constitutes the pricing mechanism (side payment, inducement). ρ_i is the price vector and is chosen by another decision maker called a coordinator. This coordinator, in turn, wishes to maximize the criterion:

$$\max \int_{t_o}^{t_f} L(x,u) \, dt \tag{2b}$$

subject to the constraints of the dynamic system, equation (1). The coordinator's criterion could be a social welfare function, in which case, $L(\cdot)$ would be a function of the $l_i(\cdot)s$ i=1,...,n.

Thus the problem of the coordinator is to choose the prices ρ_i i=1,...,n such that the resulting optimal control of the subsystems will in fact maximize the criterion of the coordinator. In which case, the coordinator has been able to induce the optimal control of the subsystems to coincide with the optimal control of the coordinator - the value of u that maximizes (2b) subject to (1).

The above problem formulation, although it contains a simple dynamic system, captures several conceptually interesting features to be discussed below. Reference [3] studies the case where the entities priced are both the outputs of the subsystems as well as inputs to the subsystem from the coupling system.

3. Existence Theorem

The optimization of the nonlinear criterions (2a) and (2b) subject to (1) is in general a difficult problem. This paper will make the simplifying assumption that:

A1 the optimal control exists, is unique, and is determined by the necessary conditions of Pontryagin's maximum principle, and lastly, that the maximization of the Hamiltonian over u is obtained by setting the partial of the Hamiltonian with respect to u equal to zero.

Assumption A1 is made because the interest of this paper is not to solve these optimization problems, per se, but rather is, *given* the existence of the optimal control can it be obtained via a pricing mechanism?*

In the following we will say "the pricing mechanism exists for subsystem i" when there exists a choice for ρ_i such that the resulting optimal control of the subsystem matches the optimal control of the coordinator.

* Pontryagin's necessary conditions become sufficient in the case where the criterions $L(\cdot)$ and $l_i(\cdot)$ i=1,...,n are continuous, separable in x and u and convex in x. [4, page 567]

Reprinted from THE 23RD IEEE CONFERENCE ON DECISION AND CONTROL, December 12-14, 1984

Theorem 1: There exists a pricing mechanism for subsystem i if and only if there exists a function $\rho_i(t)$ that satisfies the condition:

$$B_i'g_i + D_i'(\rho_i - \mu_i^*) = \frac{\partial L'}{\partial u_i}\bigg|_* - \frac{\partial l_i'}{\partial u_i}\bigg|_* \qquad (3a)$$

where $\big|_*$ means the term is evaluated at the optimal value of the coordinator, and where μ_i is the Lagrange multiplier associated with subsystem i's output relation, equation 1b, and where $g_i(t)$ is determined by:

$$\dot{g}_i = -A_i'g_i - C_i'(\rho_i - \mu_i^*) + \frac{\partial L'}{\partial x_i}\bigg|_* - \frac{\partial l_i'}{\partial x_i}\bigg|_* \qquad (3b)$$

$$g_i(t_f) = 0$$

Furthermore if such a $\rho_i(t)$ exists it is the price.

Theorem 1 states a necessary and sufficient condition for the existence of a price that will induce the individual's optimal control to match the coordinator's. Its proof is based on the idea of choosing ρ_i so that the necessary conditions of individual i will match the corresponding condition of the coordinator.

4. *Discussion of Existence Theorem*

4.1 Terminal Condition

Note that since $g_i(t_f) = 0$, equation 3a at the terminal time becomes:

$$D_i'(\rho_i(t_f) - \mu_i^*(t_f)) = \left(\frac{\partial L'}{\partial u_i} - \frac{\partial l'}{\partial u_i}\right)\bigg|_{t_f}^{*}$$

and this holds iff the right hand side lies in the column space of D_i'.

Thus if the output relation is not a function of u_i then $D_i \equiv 0$. Hence equation 3a is violated and there exists no pricing mechanism. This result makes intuitive sense. If the output is not directly a function of the control then by pricing the output the coordinator can not influence $u_i(t)$ at the terminal time. In particular, the coordinator can not induce $u_i(t_f)$ to be $u_i^*(t_f)$.

4.2 Linear Social Welfare Functions

In the special case where the coordinator's payoff is a linear social welfare function:

$$L(x,u) = \sum_{i=1}^{n} l_i(x_i,u_i)$$

then the condition of theorem 1, equation (3), becomes:

$$B_i'g_i + D_i'(\rho_i - \mu_i^*) = 0 \qquad (4a)$$

$$\dot{g}_i = -A_i' - C_i'(\rho_i - \mu_i^*) \qquad g_i(t_f) = 0 \qquad (4b)$$

and is satisfied trivially by choosing the price to be the Lagrange multiplier of the subsystem output relation: $\rho_i = \mu_i^*$. Thus for linear social welfare functions we have obtained the well known result that there exists a pricing mechanism [5, page 46].

Note that μ_i^* need not be the only choice for ρ_i. For example, in the case where $D_i = 0$, the condition (4) is satisfied as long as g_i remains in the null space of B_i'. (Reference [3] discusses the conditions where this pertains.) But, out of the possibly infinitely many choices for ρ_i that satisfy (4), μ_i^* is the 'best' one - 'best' in the sense of the following heuristic reasoning.

Since a pricing mechanism attempts to make an individual's viewpoint coincide with the coordinator's, one can say that one pricing mechanism is better than another if it induces a fuller coincidence of viewpoints. In particular, one pricing mechanism is better than another if, in addition to inducing the individuals optimal control to coincide with the coordinator's, it also induces the coincidence of the costate vector that corresponds to the subsystem dynamics. Furthermore, from the proof of theorem 1, the variable g_i can be shown to be equal to the difference of these costate vectors. Thus, in the sense above, the choice of $\rho_i = \mu_i^*$ is the 'best' one as it causes $g_i(t)$ to equal 0 for all $t \in [t_o, t_f]$.

4.3 Equivalence to Servomechanism Controllability

Returning to the arbitrary form of L(x,u), equation (3) can be rewritten as follows. Let $\tilde{\rho}_i \triangleq \rho_i - \mu_i^*$ and let $g_i = g_{1i} + g_{2i}$ where g_{1i} and g_{2i} are given by:

$$\dot{g}_{1i} = -A_i'g_{1i} - C_i'\tilde{\rho}_i \qquad g_{1i}(t_f) = 0$$

$$\dot{g}_{2i} = -A_i'g_{2i} + \frac{\partial L'}{\partial x_i}\bigg|_* - \frac{\partial l_i'}{\partial x_i}\bigg|_* \qquad g_{2i}(t_f) = 0$$

and equation (3) becomes:

$$\dot{g}_{1i} = -A_i'g_{1i} - C_i'\tilde{\rho}_i \qquad g_{1i}(t_f) = 0 \qquad (5a)$$

$$d(t) = B_i'g_{1i} + D_i'\tilde{\rho}_i \qquad (5b)$$

where $d(t) = \frac{\partial L'}{\partial u_i}\big|_* - \frac{\partial l_i'}{\partial u_i}\big|_* - B_i'g_{2i}$. Thus there will exist a pricing mechanism if and only if there exists a $\tilde{\rho}_i$ that satisfies equation (5). Satisfaction of (5), thinking of it as a system in its own right, means there exists a control $\tilde{\rho}_i(t)$ such that the output of the the system, $B_i'g_{1i} + D_i'\tilde{\rho}_i$, will equal a given time function, d(t), for all $t \in [t_o, t_f]$. To guarantee the existence of such a $\tilde{\rho}_i(t)$ requires more than simple controllability of the system as the output is required to follow a particular trajectory and not just to move from one point to another in finite time. The ability of a system to follow an arbitrary trajectory is known as servomechanism controllability [6]. Thus Theorem 1 can be rephrased as:

Theorem 1': For arbitrary payoffs $L(\cdot)$ and $l_i(\cdot)$ there exists a pricing mechanism for subsystem i if and only if the system (5) is servomechanism controllable.

The fact that equation 3 amounts to a servomechanism controllability condition makes sense. The coordinator, by twiddling a function in the individual's payoff, the price, is trying to cause another function, the individual's optimal control, to follow exactly a given trajectory - the coordinator's optimal control.

References

1. Caramanis, Michael; Roger Bohn, Fred Schweppe. "Optimal Spot Pricing; Practice and Theory." *IEEE Transactions on Power Apparatus and Systems* Volume 101, No. 9. September, 1982. pages 3234-3245
2. Bohn, Roger E. "Spot Pricing of Public Utilities Services." *M.I.T. Energy Laboratory Report* No. MIT-EL-81-033, May 1982
3. Berger, Arthur "Pricing Mechanisms in Coupled Dynamic Systems." Doctoral Thesis. Harvard University. May,1983
4. Hsu, Jay C. & Andrew U. Meyer. Modern Control Principles and Applications McGraw-Hill, New York.
5. Findeisen, W., F. N. Bailey, M. Brdys, K. Malinowski, P. Tatjewski & A. WozniaK Control and Coordination in Hierarchical Systems John Wiley and Sons, New York. 1980.
6. Brockett, Roger Finite Dimensional Linear Systems John Wiley and Sons, New York. 1970.