SIAM J. APPL. MATH.
Vol. 60, No. 4, pp. 1330–1339

DISTRIBUTION OF PROCESSOR-SHARING CUSTOMERS FOR A LARGE CLOSED SYSTEM WITH MULTIPLE CLASSES*

ARTHUR BERGER† AND YAAKOV KOGAN‡

Abstract. A closed processor-sharing (PS) system with multiple customer classes is considered. The system consists of one infinite server (IS) station and one PS station. For a system with a large number of customers, a saturated PS station, and an arbitrary number of customer classes, asymptotic approximations to the stationary distribution of the total number of customers at the PS station are derived. The asymptotics for the probability mass function is described by a quasi-potential function, which defines the exponential decay for the distribution, and a state-dependent preexponential factor. Both functions have an explicit expression in terms of the solution at each point x of a polynomial equation whose order equals the number of classes and whose coefficients are explicit functions of x. The quasi-potential function at its minimum point provides the logarithmic asymptotics for the normalization constant, and the asymptotic approximation for the variance is inversely proportional to the second derivative of the quasi-potential function at its minimum point. The complementary probability distribution is computed using the normal approximation and its refinements, which do not require repeated solution of polynomial equations. Numerical results demonstrate the range of applicability of the approximations. The results can be applied to the problem of dimensioning bandwidth and of admission control for different data sources in packet-switched communication networks.

Key words. heavy traffic approximation, queueing theory

AMS subject classifications. 60K30, 34E20

PII. S0036139998346063

1. Introduction.

This paper is motivated by a new application of closed queueing networks (CQN) with a large number of customers. The application is the dimensioning of bandwidth and of admission control for different data sources subject to feedback control in packet-switched communication networks when available bandwidth at the network nodes is shared between all active sources. Data sources are modeled by an infinte server (IS) station, network nodes are modeled by processor-sharing (PS) stations, and a "customer" in the PS station represents an active data source. We consider a CQN that consists of one IS station with multiple customer classes and one PS station. The distinguishing property of the new application is that this CQN model is valid only if the PS station is saturated; see [3] for further details. The saturated station is defined asymptotically as the station, where the number of customers grows proportionally to the total number of customers in the network as the latter increases with service rates at the PS station.

The application includes the performance metric that the bandwidth received by an active data source at a given network node is greater than a target value with probability $1-\alpha$, where α is in the range of 0.001 to 0.1. As the network nodes of interest have a packet-based implementation of PS, the performance metric can be restated as the number of active data sessions at a network node (the total number of customers at the PS station in the CQN model) is less than a target value with the given probability $1-\alpha$. As the above probability will be calculated in the context

*Received by the editors October 27, 1998; accepted for publication (in revised form) June 18, 1999; published electronically March 23, 2000.
http://www.siam.org/journals/siap/60-4/34606.html

†Bell Labs, Lucent Technologies, Holmdel, NJ 07733 (awberger@lucent.com).
‡AT&T Labs, Middletown, NJ 07748 (yakov@buckaroo.mt.att.com).

of network planning and of network operations, the calculation will need to be done often and quickly.

The steady-state per-customer-class queue length distribution in our CQN has a product form which is defined explicitly up to the normalization constant. However, the total number of PS customers is not a Markov process, which complicates computation of its steady-state distribution, especially when the total number of customers in the network is large. In this paper, we derive the asymptotics for the probability mass function of the total number of customers at the PS station. These asymptotics are described by a quasi-potential function [8] $F(x)$, which defines the exponential decay for the distribution, and a state-dependent preexponential factor $f(x)$. Both functions have an explicit expression in terms of the solution at each point x of a polynomial equation whose order equals the number of classes and whose coefficients are explicit functions of x. The logarithmic asymptotics for the normalization constant is given by $-F(x^*)$, where x^* is the minimum point of $F(x)$. In [11, 12] similar results were derived for $M/M/1$ and $M/G/1$ state-dependent queues with a single customer class. With multiple customer classes, the quasi-potential function is easily derived only for Markov processes with a product-form stationary distribution [22] [1, Chapter 4] .

Using the asymptotics of the probability mass function one can easily derive for the complementary probability distribution the normal approximation and its refinements. The advantage of these approximations is that they do not require repeated solution of polynomial equations as in the case of direct summation of probabilities approximated by their asymptotic expressions. In particular, the mean and the variance of the normal approximation are proportional and inversely proportional to the minimum point x^* of $F(x)$ and to the second derivative $F''(x^*)$, respectively. Their calculation requires only the solution of the polynomial equation that defines the asymptotics of the normalization constant [16]. The parameters of the normal approximation can be also derived from the multidimensional normal approximation [22], [1, Chapter 4] for the number of customers of different classes at the PS station. However, our expression for the variance is much simpler.

There are two main approaches to the asymptotic expansion of the steady-state distributions in CQNs with large numbers of customers. One is based on the Wentzel–Kramers–Brilbuin (WKB) method applied to the forward Kolmogorov equation for the probability distribution [11, 12]. Its application in our case would require consideration of an auxiliary multidimensional Markov process and complicate the analysis. Therefore we follow another approach, which is based on generating functions and integral representations [5, 15, 16, 17, 18, 19, 4, 13, 14].

In section 2, we start with the exponential generating function for the probability distribution of the total number of customers at the PS station. This generating function is one-dimensional, and it has an explicit form up to the normalization constant. This allows us to derive the asymptotics of the probability mass function in a direct way similar to [13, 14, 4] by evaluating the Cauchy integral by the saddle-point method. The final result is derived by proving that $-F(x^*)$ gives the logarithmic asymptotics for the normalization constant. We conclude section 2 with remarks on simplification and generalizations of our approximations. In section 3, we refine the normal approximation for the complementary probability distribution and provide numerical results illustrating the accuracy of our approximations.

2. Asymptotics of the probability mass function. We consider a CQN which consists of one IS station and one PS station. There are K customer classes with N_k customers in class k, $k = 1, \dots, K$, and $N = \sum_{k=1}^{K} N_k$. Let λ_k^{-1} be the mean think time of a customer of class k and let μ_k^{-1} be the mean service time for a class-k customer at the PS node. Let Q_k be the random variable for the steady-state number of class-k customers at the PS node. It is known (see, e.g., [9]) that the steady-state probability distribution $\Pr\{Q_1 = n_1, \dots, Q_K = n_K\}$ has a product form

$$\Pr\{Q_1 = n_1, \dots, Q_K = n_K\} = \frac{1}{G} \prod_{k=1}^{K} \frac{N_k!}{(N_k - n_k)!} n! \frac{r_k^{n_k}}{n_k!}, \tag{2.1}$$

where $n \equiv \sum_{k=1}^{K} n_k$, $r_k \equiv \lambda_k/\mu_k$ and G is the normalization constant.

Denote by $Q \equiv \sum_{k=1}^{K} Q_k$ the total number of customers at the PS node, and let

$$P(n) \equiv \Pr\{Q = n\} = \sum_{n_1 + \cdots + n_K = n} \Pr\{Q_1 = n_1, \dots, Q_K = n_K\}$$

be its probability mass function. In general, the above sum does not seem to be reduced to a product of functions depending only on n and/or network parameters. However, the exponential generating function $\mathcal{P}(z)$ for the sequence $P(n)$ has the following simple expression:

$$\mathcal{P}(z) = \sum_{n=0}^{N} P(n) \frac{z^n}{n!} = G^{-1} \prod_{k=1}^{K} (1 + r_k z)^{N_k}, \tag{2.2}$$

which is easily derived from (2.1) and definitions of $P(n)$ and $\mathcal{P}(z)$. Using the Cauchy formula, we obtain for $P(n)$ the following integral representation in complex space:

$$P(n) = \frac{1}{G} n! \frac{1}{2\pi i} \oint_C \frac{\prod_{k=1}^{K} (1 + r_k z)^{N_k}}{z^{n+1}} dz, \tag{2.3}$$

where C is any circular contour around $z = 0$.

We study the asymptotics of $P(n)$ under the following two assumptions.

(1) The total number of customers in the network $N = \sum_{k=1}^{K} N_k$ is large, i.e., $N \gg 1$, and moreover

$$\rho_k = N r_k \quad \text{and} \quad \alpha_k = N_k/N, \tag{2.4}$$

where ρ_k and α_k, $k = 1, \dots, K$, remain bounded as $N \to \infty$.

(2) The PS station is saturated, which is expressed by the following very heavy usage condition [16, 19]:

$$\sum_{k=1}^{K} \alpha_k \rho_k > 1. \tag{2.5}$$

These assumptions appropriately capture the region of interest for the application of data transport in packet-switched communication networks. A current trend of higher speed transmission facilities implies faster service, i.e., large μ_k. Correspondingly, the number of data sources N is roughly growing in proportion to the increasing transmission speed. Thus, the natural asymptotic regime to examine is large N and small r_k, where the product is constant, as in (2.4). In addition, for determining session admission control policies or for dimensioning of link capacity, the traffic scenarios of relevance, represented in the model by ρ_k, are ones of high load, where the transmission link is a constraining resource. In the context of the model, this corresponds to the PS station being saturated as in (2.5).

Changing the variables $u = z/N$ in (2.3) and using (2.4) we get

$$\begin{aligned} P(n) &= \frac{1}{G}\frac{n!}{N^n}\frac{1}{2\pi i}\oint_{|u|=1}\frac{\prod_{k=1}^{K}(1+\rho_k u)^{N_k}}{u^{n+1}}du \\ &= \frac{1}{G}\frac{n!}{N^n}\frac{1}{2\pi i}\oint_{|u|=1}(1/u)\exp\{NS(u)\}du, \end{aligned} \tag{2.6}$$

where

$$S(u) = \sum_{k=1}^{K}\alpha_k \ln(1+\rho_k u) - x\ln u \tag{2.7}$$

and $x = n/N$. We derive an asymptotic approximation for $P(n)$ as $N \to \infty$, while $n = xN$, where $x \in (0,1)$ is a constant. Using Stirling's formula, we can express the second factor in $P(n)$ as

$$\frac{n!}{N^n} = \sqrt{2\pi n}\exp\{N[x\ln x - x]\}[1+O(1/n)]. \tag{2.8}$$

The asymptotic approximation to the contour integral in (2.6) is obtained by the saddle-point method. For each $x \in (0,1)$ there is a unique positive solution $u_o(x)$ of equation

$$S'(u) = \sum_{k=1}^{K}\frac{\alpha_k\rho_k}{1+\rho_k u} - \frac{x}{u} = 0 \tag{2.9}$$

on the real axis, where the prime denotes derivative. The uniqueness is implied by $\sum_{k=1}^{K}\alpha_k = 1$. Note that (2.9) can be transformed to a polynomial equation of order K whose minimal positive root is $u_o(x)$. Rewriting (2.9) into the form

$$\sum_{k=1}^{K}\alpha_k\frac{\rho_k u_o(x)}{1+\rho_k u_o(x)} = x, \tag{2.10}$$

one can see that

$$S''(u_o(x)) = \frac{1}{u_o^2(x)}\left(x - \sum_{k=1}^{K}\alpha_k\left(\frac{\rho_k u_o(x)}{1+\rho_k u_o(x)}\right)^2\right) > 0. \tag{2.11}$$

Hence, $S(u)$ has a minimum on the real axis at $u = u_o(x)$.

From (2.7) with $u = re^{i\theta}$, $0 \le \theta < 2\pi$,

$$\begin{aligned} \text{Re } S(u) &= \sum_{k=1}^{K}\alpha_k \ln|(1+\rho_k u)| - x\ln r \\ &\le \sum_{k=1}^{K}\alpha_k\ln(1+\rho_k r) - x\ln r, \end{aligned} \tag{2.12}$$

with equality only for $\theta = 0$. Hence,

$$\max_{|u|=r} \mathrm{Re}\ S(u) = S(r).$$

Since $S(u)$ has a minimum on the real axis at $u = u_o(x)$, it follows that

$$\min_r \max_{|u|=r} \mathrm{Re}\ S(u) = S(u_o(x)) = \max_{|u|=u_o(x)} \mathrm{Re}\ S(u).$$

This implies [7] that $|u| = u_o(x)$ is a saddle-point contour, and $u = u_o(x)$ is the only saddle point on it. By the saddle-point method [7]

$$\frac{1}{2\pi i} \oint_{|u|=1} \frac{\exp\{NS(u)\}}{u} du = \frac{\exp\{NS(u_o(x))\}}{\sqrt{2\pi N S''(u_o(x))}} [1/u_o(x) + O(1/N)]. \tag{2.13}$$

Thus, from (2.8) and (2.13),

$$P(n) = \frac{1}{G} \exp\{-NF(n/N)\}(f(n/N) + O(1/N)), \tag{2.14}$$

where

$$F(x) = x - x \ln x - S(u_o(x)) \tag{2.15}$$

and

$$f(x) = \frac{1}{u_o(x)} \sqrt{\frac{x}{S''(u_o(x))}}. \tag{2.16}$$

To complete the asymptotic approximation of $P(n)$ we need an asymptotic approximation for the normalization constant G. It is proved in [16] that under conditions (2.4) and (2.5)

$$G = G(N) = Ne^{-Nh(x^*)} \sqrt{\frac{2\pi}{N\Delta}} \left(1 + O\left(\frac{1}{N}\right)\right), \tag{2.17}$$

where

$$h(\tau) = \tau - \sum_{k=1}^{K} \alpha_k \ln(1 + \rho_k \tau), \tag{2.18}$$

x^* is the single positive root of equation $h'(\tau) = 0$ and

$$\Delta = h''(x^*) = \sum_{k=1}^{K} \frac{\alpha_k \rho_k^2}{(1 + \rho_k x^*)^2}. \tag{2.19}$$

Finally, we prove that x^* minimizes $F(x)$ and

$$F(x^*) = h(x^*). \tag{2.20}$$

Denote the minimum point of $F(x)$ by x^o and consider the first-order condition $F'(x) = 0$. Using (2.9), we have that

$$F'(x) = -\ln x + \ln u_o(x) \tag{2.21}$$

and therefore $x^o = u_o(x^o)$. Thus, from (2.9) we have the following equation for x^o:

$$1 - \sum_{k=1}^{K} \frac{\alpha_k \rho_k}{1 + \rho_k x^o} = 0. \tag{2.22}$$

Hence, $x^o = x^*$ since the left-hand side of (2.22) is $h'(x)$ and (2.22) has a single positive root. The equality (2.20) is now obtained by substituting $u_o(x^*) = x^*$ into (2.15).

Substituting (2.17) with $h(x^*) = F(x^*)$ in (2.14) we arrive at the following proposition.

PROPOSITION 2.1. *Let conditions* (2.4) *and* (2.5) *be satisfied and $N \to \infty$ while $n = Nx$, where both x and $1 - x$ are $O(1)$. Then the probability distribution of the total number of customers at the PS station has the following asymptotic expansion:*

$$\Pr\{Q = n\} = \sqrt{\frac{\Delta}{2\pi N}} f(n/N) \exp\{-N(F(n/N) - F(x^*))\}(1 + O(1/N)), \tag{2.23}$$

where x^* is the single positive root of (2.22) *while Δ and functions $F(\cdot)$ and $f(\cdot)$ are defined by* (2.19), (2.15), *and* (2.16), *respectively.*

COROLLARY 2.2. *The function $F(x)$ defines the logarithmic asymptotics of the probability distribution $P(n) = \Pr\{Q = n\}$ in the following sense:*

$$\lim_{N \to \infty} \frac{\ln P(n)}{N} = -(F(x) - F(x^*)).$$

Moreover, $F(x^*)$ defines the logarithmic asymptotics of the normalization constant $G = G(N)$:

$$\lim_{N \to \infty} \frac{\ln G(N)}{N} = -F(x^*).$$

Following [8] the function $F(x)$ is referred to as the quasi potential for the distribution $P(n)$. For a general birth and death process with fast transition rates, the quasi potential satisfies a nonlinear differential equation (see [8]), which has an explicit solution only in some particular cases, e.g., for product-form CQNs. Proposition 2.1 provides an example of a non-Markov process, where the quasi potential either can be found explicitly or easily computed.

We conclude this section with remarks on simplification and generalizations of our approximations.

Remark 2.1. Representation (2.23) implies (see [22]) that

$$\frac{Q - Nx^*}{\sqrt{N}}$$

is asymptotically normal with mean 0 and variance

$$\sigma^2 = \frac{1}{F''(x^*)},$$

where

$$F''(x^*) = \Delta(1 - x^*\Delta)^{-1}$$

was calculated using (2.21). Note that σ^2 can be also calculated from the multidimensional normal approximation for the K classes at the PS station given in [22, 1]. However, such a calculation produces a very cumbersome expression.

Remark 2.2. There are two cases $K = 1$ and $K = 2$, where the equation for $u_o(x)$, (2.9), and x^*, (2.22), are, respectively, linear and quadratic, which leads to explicit expressions for the quasi-potential $F(x)$ and function $f(x)$. Moreover, for $K = 1$ the distribution function for the number of customers at the PS station is expressed as the ratio of two partition functions, and its asymptotics is derived in a much simpler way in our recent paper [3].

Remark 2.3. The results of this paper can be generalized. First, Proposition 2.1 can be generalized for a bottleneck PS station [2] in CQNs with two or more PS stations. Second, similar results can be obtained in the context of loss networks [10] for the distribution of the number of busy circuits for a link with normal load. We are pursuing this work and hope to report on the results subsequently. A more difficult problem is to derive multidimensional asymptotic expansions that are pertinent to the case of several bottlenecks.

3. Asymptotics of the distribution function. In applications we need to evaluate the complementary distribution function

$$\Psi(m) \equiv \Pr\{Q > m\} = \sum_{n=m+1}^{N} P(n) \tag{3.1}$$

in the range of $m > Nx^*$, where $\Psi(m) \geq 10^{-3}$. Computation of $\Psi(m)$ using approximation (2.23) for $P(n)$ would require repeated solution of (2.9) whose root has a simple explicit form only for $K \leq 2$. Therefore in this section, we give numerical examples that demonstrate the accuracy of the normal approximation and show the gain in accuracy obtained by refining the normal approximation. We refine the normal approximation by approximating the sum in (3.1) by an integral using the Euler–Maclaurin formula [21] applied to function

$$g(n) = g(n; N) = f(n/N)\exp\{-Nh(n/N)\} \tag{3.2}$$

on the interval $[m, M]$, where $h(x) = F(x) - F(x^*)$ and $0 < M - m = O(N)$, $0 < N - M = O(N)$. Let $a = m/N$ be sufficiently close to x^* so that $F'(a) < c < 1$. Then by the Euler–Maclaurin formula, the sum $\sum_{n=m}^{M} g(n)$ is asymptotic to

$$\int_m^M g(t)dt + 0.5g(m) - \sum_{l=1}^{\infty} g^{(2l-1)}(m)\frac{B_{2l}}{(2l)!}, \tag{3.3}$$

where B_{2l} are the even Bernoulli numbers. To leading order as $N \to \infty$, we have

$$g^{(2l-1)}(m) \sim -g(m)[F'(a)]^{2l-1}. \tag{3.4}$$

Using (3.4) and noting that

$$\sum_{l=1}^{\infty} x^{2l-1}\frac{B_{2l}}{(2l)!} = \frac{1}{e^x - 1} + \frac{1}{2} - \frac{1}{x}$$

we have

$$\sum_{n=m}^{M} g(n) \sim N\int_a^b f(x)e^{-Nh(x)}dx + g(a)\left[1 + \frac{1}{\exp\{F'(a)\} - 1} - \frac{1}{F'(a)}\right]. \tag{3.5}$$

Using for the integral in (3.5) the uniform asymptotic expansion [6, 7], we finally obtain

$$\begin{aligned}\Pr\{Q>m\} \sim & \frac{1}{2}\operatorname{erfc}\left\{\sqrt{N(F(a)-F(x^*))}\right\} \\ & -\sqrt{\frac{\Delta}{2\pi N}}\exp\{N(F(x^*)-F(a))\} \\ & \times\left(f(a)H(a)+\frac{(2\Delta)^{-1/2}-f(a)[F(a)-F(x^*)]^{1/2}[F'(a)]^{-1}}{[F(a)-F(x^*)]^{1/2}}\right),\end{aligned} \tag{3.6}$$

where

$$H(a)=\frac{1}{F'(a)}-\frac{1}{\exp\{F'(a)\}-1}=\frac{0.5+\sum_{l=1}^{\infty}\frac{[F'(a)]^l}{(l+2)!}}{1+\sum_{l=1}^{\infty}\frac{[F'(a)]^l}{(l+1)!}},$$

$F'(a)$ is calculated from (2.21) and

$$\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}e^{-y^2}dy.$$

We conclude this section by illustrating the accuracy of our approximations with numerical examples. All examples are computed for two customer classes, and where $N_1 = N_2$ and for the load parameters $\rho_1 = 1$, $\rho_2 = 3$, and $\rho = \alpha_1\rho_1 + \alpha_2\rho_2 = 2$. The total number of customers $N = N_1 + N_2$ is varied ($N = 50,\ 100,\ 200$), while r_i is adjusted to keep ρ_i fixed, $i = 1, 2$ (see (2.4)). Table 3.1 reports four calculations for $\Psi(m) = \Pr\{Q > m\}$ for various values of m. The first calculation is the normal approximation given in Remark 2.1; the second is the first term (the erfc($\cdot$) term) of the asymptotic approximation (3.6); the third is the two terms, i.e., the entirety, of the asymptotic approximation (3.6); and the fourth is the exact value, numerically computed from (2.1). For each choice of N, the first value of m is the smallest integer that is greater than Nx^*, and thus is the smallest value of m for which the approximations could pertain. The subsequent three values of m are the integers where the true value of $\Pr\{Q > m\}$ is closest to 10^{-k} for $k = 1, 2, 3$, respectively. These probabilities cover the range of interest for the application of dimensioning bandwidth in packet-switched communication networks. The fifth value of m is the first value of m plus $0.3N$, which tests the range of the asymptotic approximation.

For the first listed value of m for each N, the normal and the first term approximations are accurate to one significant digit only. However, with the two terms the accuracy jumps to four significant digits. For larger values of m, we see that the first term in (3.6) becomes significantly more accurate than the normal approximation, and that the inclusion of the second term in (3.6) provides a further, significant improvement. For example, for the fourth value of m for each N, which corresponds to the most stressful case for the intended application, the error of the normal approximation ranges from 170% to 70% as N increases, while the error from the first term is significantly less and is in a more narrow range from 32% to 13%. The inclusion of the second term reduces the error significantly to only 0.06% to 0.006%. As one would expect, the accuracy of all of the approximations increases with N. In contexts where a rougher approximation is adequate, and particularly for the less strict performance criterion corresponding to $\Pr\{Q > m\} \approx 10^{-1}$, the normal approximation is suitable.

TABLE 3.1
Comparison of three approximations.

		$\Pr(Q > m)$			
N	m	Normal approx.	First term approx.	Two terms approx.	Exact
50	22	.4767	.4766	.45158	.45155
	28	.100	.092	.08117	.08115
	32	.0180	.0130	.010618	.010614
	35	.0034	.0016	.0012554	.0012546
	37	9.2e-4	3.0e-4	2.127e-4	2.125e-4
100	44	.46704	.46699	.449230	.44928
	52	.108	.103	.094720	.094712
	59	.012	.0095	.008283	.008282
	63	.0024	.0014	.0011863	.0011861
	74	5.3e-6	5.9e-7	4.318e-7	4.317e-7
200	87	.493973	.493972	.48146	.48145
	99	.108	.104	.098381	.098378
	109	.012	.010	.0091272	.0091268
	116	1.5-e3	9.9e-4	8.7684e-4	8.7679e-4
	147	4.5e-10	7.7e-12	5.715e-12	5.711e-12

However, when greater accuracy is needed, the two terms approximation provides three to four significant digits, which is easily sufficient for the intended application, and furthermore has the desirable attribute of being a closed-form expression that allows for easy computation.

Results for the last value of m for each N demonstrate the uniform accuracy of the two terms approximation (3.6). Here $a \equiv m/N$ is not close to x^*, as $a - x^* > 0.3$. Note that for these values of m the normal approximation can be in error by orders of magnitude (see $N = 200, m = 147$), while the first term of (3.6) has the correct power, but no significant digits. In contrast, the two term approximation retains an accuracy of three significant digits. Therefore we do not need further refinement similar to one given in [20].

Acknowledgment. We thank an anonymous referee for detailed comments, which have improved the presentation of the results, and for derivation of (3.5).

REFERENCES

[1] G. P. BASHARIN, P. P. BOCHAROV, AND Y. A. KOGAN, *Queueing Analysis in Computer Networks*, Nauka, Moskow, 1989 (in Russian).

[2] A. BERGER, L. BREGMAN, AND Y. KOGAN, *Bottleneck analysis in multiclass closed queueing networks and its application*, Queueing Systems, 31 (1999), pp. 217–237.

[3] A. BERGER AND Y. KOGAN, *Dimensioning bandwidth for elastic traffic in high-speed data networks*, IEEE/ACM Trans. Networks, submitted.

[4] A. BIRMAN AND Y. KOGAN, *Asymptotic evaluation of closed queueing networks with many stations*, Comm. Statist. Stochastic Models, 8 (1992), pp. 543–564.

[5] A. BIRMAN AND Y. KOGAN, *Error bounds for asymptotic approximations of the partition function*, Queueing Systems, 23 (1996), pp. 217–234.

[6] N. BLEISTEIN AND R. A. HANDELSMAN, *Asymptotic Expansions of Integrals*, Dover, Toronto, 1986.

[7] M. V. FEDORYUK, *Asymptotics, Integrals and Series*, Nauka, Moscow, 1987 (in Russian).

[8] M. FREIDLIN AND A. WENTZELL, *Random Perturbation of Dynamical Systems*, Springer, New York, 1984.

[9] F. P. KELLY, *Reversibility and Stochastic Networks*, John Wiley, New York, 1979.

[10] F. P. KELLY, *Loss networks*, Ann. Appl. Probab., 1 (1991), pp. 319–378.

[11] C. KNESSL, B. J. MATKOWSKY, Z. SCHUSS, AND C. TIER, *On the performance of state-dependent single server queues*, SIAM J. Appl. Math., 46 (1986), pp. 657–697.
[12] C. KNESSL, B. J. MATKOWSKY, Z. SCHUSS, AND C. TIER, *Asymptotic expansions for a closed multiple access system*, SIAM J. Comput., 16 (1987), pp. 378–398.
[13] Y. KOGAN, *Another approach to asymptotic expansions for large closed queueing networks*, Oper. Res. Lett., 11 (1992), pp. 317–321.
[14] Y. KOGAN AND A. BIRMAN, *Asymptotic analysis of closed queueing networks with bottlenecks*, in IFIP Transactions C-5, Performance of Distributed Systems and Integrated Communication Networks, T. Hasegawa, H. Takagi, and Y. Takahashi, eds., North–Holland, Amsterdam, 1992, pp. 265–280.
[15] Y. KOGAN AND A. YAKOVLEV, *Asymptotic analysis for closed multichain queueing networks with bottlenecks*, Queueing Systems, 23 (1996), pp. 235–258.
[16] J. MCKENNA, D. MITRA, AND K. G. RAMAKRISHNAN, *A class of closed Markovian queueing networks: Integral representations, asymptotic expansions and generalizations*, Bell Syst. Tech. J., 60 (1981), pp. 599–641.
[17] D. MITRA AND J. A. MORRISON, *Asymptotic expansions of moments of the waiting time in open and closed processor-sharing systems with multiple job classes,* Adv. Appl. Probab., 15 (1983), pp. 813–839.
[18] J. A. MORRISON, *Conditioned response-time distribution for a large closed processor-sharing system in very heavy usage*, SIAM J. Appl. Math., 47 (1987), pp. 1117–1129.
[19] J. A. MORRISON, *Conditioned response-time distribution for a large closed processor-sharing system with multiple classes in very heavy usage*, SIAM J. Appl. Math., 48 (1988), pp. 1493–1509.
[20] J. A. MORRISON, K. G. RAMAKRISHNAN, AND D. MITRA,, *Refined asymptotic approximations to loss probabilities and their sensitivities in shared unbuffered resources*, SIAM J. Appl. Math., 59 (1998), pp. 494–513.
[21] F. W. J. OLVER, *Asymptotics and Special Functions*, Academic Press, New York, 1971.
[22] B. PITTEL, *Closed exponential networks of queues with saturation: The Jackson-type stationary distribution and its asymptotic analysis*, Math. Oper. Res., 6 (1979), pp. 357–378.