Numerically Stable Method for Solving Quadratic Equations

The commonly used formula for the solutions of a quadratic does not provide for the most accurate computation of both roots when faced with the limitations of finite precision arithmetic. One of the two roots is found with lower precision than the other due to round-off when two quantities of the same sign and similar magnitude are subtracted from one another.

By multiplying

$$ax^2 + bx + c = 0 \tag{1}$$

by $4a$ and completing the square one gets

$$(2ax + b)^2 + (4ac - b^2) = 0 \tag{2}$$

and so finds

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

By instead completing the square in

$$a + b\frac{1}{x} + c\frac{1}{x^2} = 0 \tag{4}$$

one finds

$$\frac{1}{x} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2c} \tag{5}$$

or

$$x = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}} \tag{6}$$

Given limitations of computer arithmetic, one or the other of these (eq. (3) or eq. (6)) may provide more accuracy for a particular root. When quantities of the same sign are subtracted, some loss in precision may be expected. This is a particular concern here if ac is relatively small compared to b^2, in which case b has about the same magnitude as $\sqrt{b^2 - 4ac}$.

This suggests that one use one of the above equations for one root, and use the other equation for the other root:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_2 = \frac{2c}{-b - \sqrt{b^2 - 4ac}} \quad \text{when } b \geq 0 \tag{7}$$

and

$$x_1 = \frac{2c}{-b + \sqrt{b^2 - 4ac}} \quad \& \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{when } b < 0 \tag{8}$$

We can check these results by noting that $x_1 x_2 = c/a$ and $x_1 + x_2 = -b/a$.

Note that no more work is involved using eq. (7) or eq. (8) than blindly using either eq. (3) or eq. (6) for both solutions of the quadratic.