Grangeat's "trick"

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Grangeat derives a fundamental relationship between a derivative of a line integral of projection data and the first derivative of the Radon transform. This equivalence is crucial to "exact" reconstruction methods, since the line integral of density in a projection does not give the integral over a plane needed for the Radon transform.

The cleverness — or "trick" — of Grangeat is to have two weighting effects cancel each other:

(1) If we integrate along a line in the projection we do not get the integral over the plane of the absorption. Instead we get a weighted integral of absorption with weight proportional to $1/r$, where r is the distance from the X-ray source along the ray. The reason is that the rays are closer together near the source.

(2) If we differentiate the line integral w.r.t. to the "slice angle" we do not get the radial derivative. Instead we get a result weighted by $r \cos\theta$. The reason for the weighting is that the change in the radial direction equals the change in the slice angle times the distance from the axis about which the slice rotates.

The "slice angle' is the angle between the plane formed by the line in the projection and the source, and the central ray which falls perpendicularly on the detector.

The weighting by r in (2) can be exploited to cancel the weighting by $1/r$ in (1) above. The integral of density over the plane can be written

$$\iint f(l, r, \theta)\, r\, dr\, d\theta$$

where l is the radial distance of the plane of integration from the origin, and r and θ are polar coordinates in that plane, with the X-ray source at the origin.

In the X-ray projection each individual measurement is of the form $\int f(l, r, \theta) dr$ from which we can compute

$$\iint f(l, r, \theta)\, dr\, d\theta$$

or

$$\iint (1/r) f(l, r, \theta)\, r\, dr\, d\theta$$

which is the integral over the plane of $(1/r)f$ instead of the integral of f itself (which appears in the Radon transform).

We want the radial derivative

$$\frac{\partial}{\partial l}$$

Given the projection data, we can instead compute the derivative with respect to slice angle

$$\frac{\partial}{\partial \alpha}$$

Now $l = r \cos\theta\, \alpha$ so

$$\frac{\partial}{\partial \alpha} = r \cos\theta \frac{\partial}{\partial l}$$

If we compute a derivative this way we get a result weighted by $r \cos\theta$. We can combine the two "faulty" methods to cancel the weighting in r.

For the radial derivative of the Radon transform, we actually want

$$\frac{\partial}{\partial l} \iint f(l, r, \theta)\, r\, dr\, d\theta$$

Instead we compute

$$\frac{\partial}{\partial \alpha} \iint \frac{1}{\cos\theta} f(l, r, \theta)\, dr\, d\theta$$

Interchanging the order of differentiation and integration:

$$\iint \frac{1}{\cos\theta} \frac{\partial}{\partial \alpha} f(l, r, \theta)\, dr\, d\theta$$

or

$$\iint \frac{1}{\cos\theta} (r \cos\theta) \frac{\partial}{\partial l} f(l, r, \theta)\, dr\, d\theta$$

or

$$\iint \frac{\partial}{\partial l} f(l, r, \theta)\, r\, dr\, d\theta$$

or

$$\frac{\partial}{\partial l} \iint f(l, r, \theta)\, r\, dr\, d\theta$$