# 6

# Image Processing: Continuous Images

It is often useful to transform an image in some way, producing a new one that is more amenable to further manipulation. Image processing involves the search for methods to accomplish such transformations. Most of the methods examined so far are linear and shift-invariant. Methods with these properties allow us to apply powerful analytic tools. We show in this chapter that linear, shift-invariant systems can be characterized by convolution, an operation introduced in its one-dimensional form when we discussed the probability distribution of the sum of two random variables.

We also demonstrate the utility of the concept of spatial frequency and of transformations between the spatial and the frequency domains. Image-processing systems, whether optical or digital, can be characterized either in the spatial domain, by their point-spread function, or in the frequency domain, by their modulation-transfer function. The tools discussed in this chapter will be applied to the analysis of partial differential operators used in edge detection, and to the analysis of optimal filtering methods for the suppression of noise.

Most of the methods discussed here are simple extensions to two dimensions of linear systems techniques for one-dimensional signals, but we shall not assume that the reader is familiar with these techniques. We treat the continuous case in this chapter and the discrete case in the next.

## 6.1 Linear, Shift-Invariant Systems

An image can be thought of as a two-dimensional signal. We can develop an approach to image processing based on this observation. Consider an out-of-focus imaging system (figure 6-1). We can think of the image $g(x, y)$ produced by the defocused system as a processed version of the ideal image, $f(x, y)$, that one would obtain in a correctly focused imaging system. Now, if the lighting is changed so as to double the brightness of the ideal image, the brightness of the out-of-focus image is also doubled. Further, if the imaging system is moved slightly, so that the ideal image is shifted in the image plane, the out-of-focus image is similarly shifted. The transformation from the ideal image to that in the out-of-focus system is said to be a linear, shift-invariant operation. In fact, incoherent optical image-processing systems that are more complicated are typically also linear and shift-invariant. These terms will now be defined more precisely.

Consider a two-dimensional system that produces outputs $g_1(x, y)$ and $g_2(x, y)$ when given inputs $f_1(x, y)$ and $f_2(x, y)$, respectively:

$$f_1 \;\to\; \boxed{\phantom{xxxx}} \;\to\; g_1$$

$$f_2 \;\to\; \boxed{\phantom{xxxx}} \;\to\; g_2$$

The system is called *linear* if the output $\alpha g_1(x, y) + \beta g_2(x, y)$ is produced when the input is $\alpha f_1(x, y) + \beta f_2(x, y)$, for arbitrary $\alpha$ and $\beta$:

$$\alpha\, f_1 + \beta\, f_2 \quad \rightarrow \quad -\boxed{\phantom{xxxxxx}}- \quad \rightarrow \quad \alpha\, g_1 + \beta\, g_2$$

Most real systems are limited in their maximum response and thus cannot be strictly linear. Moreover, brightness, which is power per unit area, cannot be negative. The original input, an image, is thus restricted to nonnegative values. Intermediate results of our computations can, however, have arbitrary values.

Now consider a system that produces output $g(x, y)$ when given input $f(x, y)$:

$$f(x, y) \quad \rightarrow \quad -\boxed{\phantom{xxxxxx}}- \quad \rightarrow \quad g(x, y)$$

The system is called *shift-invariant* if it produces the shifted output $g(x - a, y - b)$ when given the shifted input $f(x - a, y - b)$, for arbitrary $a$ and $b$:

$$f(x - a, y - b) \quad \rightarrow \quad -\boxed{\phantom{xxxxxx}}- \quad \rightarrow \quad g(x - a, y - b)$$

In practice, images are limited in area, so that shift invariance only holds for limited displacements. Moreover, aberrations in optical imaging systems vary with the distance from the optical axis; such systems are therefore only approximately shift-invariant.

Methods for analyzing linear, shift-invariant systems are important for understanding the properties of image-forming systems. System shortcomings can often be discussed in terms of the linear, shift-invariant system that would transform the ideal image into the one actually observed. More importantly for us, a study of linear, shift-invariant systems leads to useful algorithms for processing images using either optical or digital methods.

A simple example of a linear, shift-invariant system is one that produces the derivative of its input with respect to $x$ or $y$. Linearity follows from the rules for differentiating the product of a constant and some function and the rule for differentiating the sum of two functions. Shift invariance is equally easy to prove. Systems taking derivatives will prove useful as preprocessing stages in edge-detection systems.

We start by considering continuous images in order to lay the groundwork for the discrete operations. Linear, shift-invariant systems for processing images are extensions to two dimensions of one-dimensional linear, shift-invariant systems, such as simple passive electrical circuits. Not surprisingly, most of the results presented here can be derived using simple extensions of methods used to prove similar results applying to the one-dimensional case. To simplify matters, we shall factor functions of two variables into products of two functions of one variable whenever possible. This will allow us to split the two-dimensional integrals that arise into products of one-dimensional integrals.

In the analysis of one-dimensional systems, functions of time are typically used as both inputs and outputs. No system can anticipate its input. This places a severe restriction on systems for processing one-dimensional signals: They have to be causal. Only those that obey this restriction can be physically realized. There is no such problem in the synthesis of two-dimensional systems.

While we inherit the powerful methods of signal processing from one-dimensional systems, we must also point out the shortcomings. The constraints of linearity and shift invariance are severe and greatly limit the kinds of things we can do with an image. Still, it is hard to make progress without some guiding theory.

## 6.2 Convolution and the Point-Spread Function

Consider a system that, given an input $f(x, y)$, produces as its output

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - \xi, y - \eta) h(\xi, \eta) \, d\xi \, d\eta.$$

Here $g$ is said to be the convolution of $f$ and $h$. It is easy to show that such a system is linear by applying it to $\alpha f_1(x, y) + \beta f_2(x, y)$ and noting that the output is $\alpha g_1(x, y) + \beta g_2(x, y)$. Here again $g_1(x, y)$ is the output produced when $f_1(x, y)$ is the input and $g_2(x, y)$ is the output produced when $f_2(x, y)$ is the input. The result follows from the rule for integrating the product of a constant and a function and from the rule for integrating the sum of two functions. It is also easy to show that the system is shift-invariant by applying it to $f(x - a, y - b)$ and noting that the output is $g(x - a, y - b)$. Thus a system whose response can be described by a convolution is linear and shift-invariant. We shall soon show the converse:

- Any linear, shift-invariant system performs a convolution.

Convolution is usually denoted by the symbol $\otimes$. So the above formula can be abbreviated

$$g = f \otimes h.$$

It would be useful to relate the function $h(x, y)$ to some observable property of the system. Given an arbitrary function $h(x, y)$, can we always find an input $f(x, y)$ that causes the system to produce $h(x, y)$ as output? That is, can we find an $f(x, y)$ such that

$$h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - \xi, y - \eta) h(\xi, \eta) \, d\xi \, d\eta \, ?$$

Cursory inspection suggests that if this is to be true for arbitrary $h(x, y)$, then $f(x, y)$ needs to be zero at all points away from the origin and "infinite" at the origin. The "function" we are looking for is called the *unit impulse*, denoted $\delta(x, y)$. It is also sometimes referred to as the *Dirac delta function.*

Loosely speaking, $\delta(x, y)$ is zero everywhere except at the origin, where it is "infinite." The integral of $\delta(x, y)$ over any region including the origin is one. (If we think of a function of $x$ and $y$ as a surface, then this integral is the volume under that surface.) The impulse $\delta(x, y)$ is not a function in the classical sense (that is, it is not defined by giving its value for all arguments). It is a *generalized function* that can be thought of as the "limit" as $\epsilon \to 0$ of a series of square pulses of width $2\epsilon$ in $x$ and $y$ and of height $1/(4\epsilon^2)$. We shall have more to say about this later, but for now we simply note the *sifting property*,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) h(x, y) \, dx \, dy = h(0, 0),$$

by which the impulse can be defined. It follows that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - \xi, y - \eta) h(\xi, \eta) \, d\xi \, d\eta = h(x, y),$$

as can be seen by a simple change of variables. By comparing this with our original equation for the output of the system, we see that $h(x, y)$ is the response of the system when presented with the unit impulse as input.

Considered as an image, $\delta(x, y)$ is black everywhere except at the origin, where there is a point of bright light. Thus $h(x, y)$ tells us how the system blurs or spreads out a point of light. In the case of a two-dimensional system it is called the *point-spread function.* It is the response of the two-dimensional system to an impulse and is thus analogous to the familiar impulse response of a one-dimensional system.

We now want to show that the output of any linear, shift-invariant system is related to its input by convolution. The point-spread function of the system, $h(x, y)$, can be determined by applying the test input $\delta(x, y)$. Given that the response to an impulse is now known, it is convenient to think of the input, $f(x, y)$, as made up of an infinite collection of shifted, scaled impulses,

$$k(\xi, \eta)\delta(x - \xi, y - \eta).$$

A simple geometric construction will help show how this can be done. Divide the $xy$-plane into squares of width $\epsilon$. On each such elementary square erect a pulse of height equal to the average of $f(x, y)$ in the square. Figure 6-2 shows a cross section through such a two-dimensional array of square pulses. The function $f(x, y)$ is approximated by the piecewise-constant function that is the sum of all these pulses.

We can go one step further and replace each rectangular pulse by an impulse at the center of its square base. The volume under the impulse can be made equal to the volume of the rectangular pulse, that is, the integral of the function $f(x, y)$ over the elementary square. If the function $f(x, y)$ is continuous, and $\epsilon$ is small enough, one can approximate this integral by the product of the value of $f(x, y)$ at the center of the square and the area of the square. The desired result is obtained in the limit as we let $\epsilon \to 0$.

The same decomposition of $f(x, y)$ in terms of impulses can be obtained by appealing to the sifting property of the unit impulse function. Either way, we find that

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta)\, \delta(x-\xi, y-\eta)\, d\xi\, d\eta.$$

Having decomposed the function in terms of impulses, we can determine the overall output, $g(x, y)$, when $f(x, y)$ is the input, by adding the responses of the system to the shifted, scaled impulses. This is so because the system is linear.

The response to $k\,\delta(x-\xi, y-\eta)$ is $k\,h(x-\xi, y-\eta)$, since the system is shift-invariant. Thus, since $k$ is just $f(\xi, \eta)$,

$$g(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\,h(x-\xi, y-\eta)\,d\xi\,d\eta.$$

This can be written in the form $h \otimes f$. We show below that convolution is commutative, so that $h \otimes f = f \otimes h$, and so

$$g(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)\,h(\xi, \eta)\,d\xi\,d\eta.$$

Linear, shift-invariant systems can always be described by a suitable point-spread function $h(x, y)$. Using this function we can compute the output $g(x, y)$, given an arbitrary input $f(x, y)$. The point-spread function is a complete characterization of a linear, shift-invariant system. We have thus shown that a linear, shift-invariant system performs a convolution.

We now show that convolution is commutative, that is, that

$$b \otimes a = a \otimes b.$$

Let $c = a \otimes b$, or

$$c(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x-\xi, y-\eta)\,b(\xi, \eta)\,d\xi\,d\eta.$$

Now let $x - \xi = \alpha$ and $y - \eta = \beta$, so that

$$c(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\alpha, \beta)\,b(x-\alpha, y-\beta)\,d\alpha\,d\beta.$$

Since $\alpha$ and $\beta$ are arbitrary dummy variables, we can substitute $\xi$ and $\eta$ for them without changing the value of the integral. We obtain

$$c(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b(x-\xi, y-\eta)\,a(\xi, \eta)\,d\xi\,d\eta,$$

which is $b \otimes a$. Convolution is also associative; that is,

$$(a \otimes b) \otimes c = a \otimes (b \otimes c).$$

This allows us to consider the cascade of two systems with point-spread functions $h_1(x, y)$ and $h_2(x, y)$:

$$f \quad \rightarrow \quad \boxed{\quad h_1 \quad} \text{———} \boxed{\quad h_2 \quad} \quad \rightarrow \quad g$$

If the input is $f(x, y)$, then the output of the first system is $f \otimes h_1$. This new signal is the input of the second system, and so the output of the second system is $(f \otimes h_1) \otimes h_2$. This can be written in the form $f \otimes (h_1 \otimes h_2)$, that is, the output produced when the input $f$ is applied to a system with point-spread function $h_1 \otimes h_2$:

$$f \quad \rightarrow \quad \boxed{\; h_1 \otimes h_2 \;} \quad \rightarrow \quad g$$

## 6.3 The Modulation-Transfer Function

It is harder to visualize the effect of convolution than it is the multiplication of two functions. Because convolution in the spatial domain becomes multiplication in the frequency domain, a transformation to the frequency domain is attractive in the case of linear, shift-invariant systems. Before we can explore these ideas, however, we must understand what frequency means for two-dimensional systems.

In the case of one-dimensional linear, shift-invariant systems we find that $e^{i\omega t}$ is an *eigenfunction* of convolution. An eigenfunction of a system is a function that is reproduced with at most a change in amplitude:

$$e^{i\omega t} \quad \rightarrow \quad \boxed{\qquad\quad} \quad \rightarrow \quad A(\omega)\, e^{i\omega t}$$

Here $A(\omega)$ is the (possibly complex) factor by which the input signal is multiplied. That is, if we apply a complex exponential to a linear, shift-invariant system, we obtain a similar complex exponential waveform at the output, just scaled and shifted in phase. We call $\omega$ the frequency of the eigenfunction. In practice, we use real waveforms like $\cos \omega t$ and $\sin \omega t$, corresponding to the real and imaginary parts of $e^{i\omega t}$. The relationship between the two forms is, of course, just

$$e^{iwt} = \cos \omega t + i \sin \omega t.$$

The complex exponential form is used in deriving results because it makes the expressions more compact and helps avoid the need to treat cosines and sines separately.

In a two-dimensional linear, shift-invariant system, the input $f(x,y) = e^{+i(ux+vy)}$ gives rise to the output

$$g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(u(x-\xi)+v(y-\eta))} h(\xi,\eta)\, d\xi\, d\eta,$$

or

$$g(x,y) = e^{+i(ux+vy)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(u\xi+v\eta)} h(\xi,\eta)\, d\xi\, d\eta.$$

The double integral on the right is a function of $u$ and $v$ only, and the output $g(x,y)$ is therefore just a scaled, possibly shifted, version of the input $f(x,y)$. Thus $e^{+i(ux+vy)}$ is an eigenfunction of convolution in two dimensions:

$$e^{i(ux+vy)} \quad \rightarrow \quad -\boxed{\phantom{xxxxxx}}- \quad \rightarrow \quad A(u,v)\, e^{i(ux+vy)}$$

Note that frequency now has two components, $u$ and $v$. We refer to the $uv$-plane as the *frequency domain*, in contrast to the $xy$-plane, which is referred to as the *spatial domain*.

The real waveforms $\cos(ux+vy)$ and $\sin(ux+vy)$ correspond to waves in two dimensions. The maxima and minima of $\cos(ux+vy)$ lie on parallel equidistant ridges along the lines

$$ux + vy = k\pi$$

for integer $k$ (figure 6-3). Taking a cut through the surface at right angles to these lines, that is, in the direction $(u,v)$, gives us sinusoidal waves with wavelength

$$\lambda = \frac{2\pi}{\sqrt{u^2+v^2}}.$$

Such waves cannot occur on their own in an imaging system since brightness cannot be negative. There must be an added constant offset.

If we let

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(u\xi+v\eta)} h(\xi,\eta)\, d\xi\, d\eta,$$

then, in the special case treated so far,

$$g(x,y) = H(u,v)\, f(x,y),$$

as can be seen from the integral given previously. Thus $H(u,v)$ characterizes the system for sinusoidal waveforms, just as $h(x,y)$ does for impulsive waveforms. For each frequency, it tells us the response of the system in amplitude and phase. In the case of a two-dimensional system it is called the *modulation-transfer function*. It is the frequency response of the two-dimensional system and so is analogous to the familiar frequency response of a one-dimensional system. (Note that $H(u,v)$ need not be real-valued.)

Just as we can learn much about the quality of an audio amplifier from its frequency response curve, so we can compare camera lenses, for example, by looking at their modulation-transfer function plots.

## 6.4 Fourier Transform and Filtering

An input $f(x,y)$ can be considered to be the sum of an infinite number of sinusoidal waves, just as earlier we thought of it as the sum of an infinite number of impulses. This is another convenient way to decompose the input, since we once again already know the system's response to each component, provided we are given the modulation-transfer function $H(u,v)$. If we decompose $f(x,y)$ as

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{+i(ux+vy)}\, du\, dv,$$

then

$$g(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(u,v)\, F(u,v) e^{+i(ux+vy)}\, du\, dv.$$

(The $1/4\pi^2$ occurs here for consistency with formulae introduced later on.) The only problem is that the decomposition into sinusoidal waves is not quite as trivial as the decomposition into impulses. How do we find $F(u,v)$ given $f(x,y)$? As we shall demonstrate in a moment, the answer turns out to be

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-i(ux+vy)}\, dx\, dy$$

provided that this integral exists. We can see that this might be so by changing variables,

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)e^{-i(u\alpha+v\beta)}\,d\alpha\,d\beta,$$

and substituting into the expression for $f(x,y)$ to obtain

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\left(u(x-\alpha)+v(y-\beta)\right)}\,du\,dv\right]\,d\alpha\,d\beta.$$

The inner integral does not converge. We show later, using so-called convergence factors, that it can be considered to equal $4\pi^2\delta(x-\alpha, y-\beta)$. We therefore have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\delta(x-\alpha,y-\beta)\,d\alpha\,d\beta = f(x,y),$$

so that

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{+i(ux+vy)}\,du\,dv = f(x,y).$$

$F(u,v)$ is called the *Fourier transform* of $f(x,y)$. Similarly, we can define the Fourier transform $G(u,v)$ of the output $g(x,y)$. Finally,

$$G(u,v) = H(u,v)\,F(u,v),$$

which is simpler than

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi,y-\eta)h(\xi,\eta)\,d\xi\,d\eta.$$

Thus convolution has been transformed into multiplication!

We also see once again that the modulation-transfer function $H(u,v)$ specifies how the system attenuates or amplifies each component $F(u,v)$ of the input. A linear, shift-invariant system thus acts as a filter that selectively attenuates or amplifies various parts of the spectrum of possible frequencies. It can also shift their phase, but this is all it does. We might conclude that restricting ourselves to linear, shift-invariant systems seriously limits what we can accomplish, but at the same time it allows us to derive a lot of useful results, because the mathematics is manageable.

Note the minor asymmetry in the expressions for the forward Fourier transform

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(ux+vy)}\,dx\,dy$$

and the inverse transform

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{+i(ux+vy)}\, du\, dv.$$

The constant multipliers are split up in this way to be consistent with other textbooks. The fact that the transforms are almost symmetric makes it possible to deduce properties that apply to the inverse transform, given properties that apply to the forward transform. Observe, however, that $F(u,v)$ is generally complex, whereas $f(x,y)$ is always real. Note also that $H(u,v)$ is the Fourier transform of $h(x,y)$.

Not all functions have a Fourier transform. Functions in certain simple classes are equal to the Fourier integrals of their Fourier transforms. But it is hard to characterize exactly which functions do, and which do not, have a transform.

A different kind of difficulty is that the integrals are taken over the whole $xy$-plane, whereas imaging devices only produce usable images over a finite part of the image plane. Moreover, computers only use discrete samples of these images. These two issues will be discussed in more detail in the next chapter.

## 6.5 The Fourier Transform of Convolution

Let $c = a \otimes b$; then the Fourier transform $C(u,v)$ of $c(x,y)$ is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(x-\xi, y-\eta) b(\xi,\eta)\, d\xi\, d\eta \right] e^{-i(ux+vy)}\, dx\, dy$$

or

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(x-\xi, y-\eta) e^{-i(ux+vy)}\, dx\, dy \right] b(\xi,\eta)\, d\xi\, d\eta.$$

That is,

$$C(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(u,v) e^{-i(u\xi+v\eta)} b(\xi,\eta)\, d\xi\, d\eta = A(u,v)B(u,v).$$

Convolution in the spatial domain becomes multiplication in the frequency domain. This it is the ultimate justification for the introduction of the complex machinery of the frequency domain. The commutativity and associativity of convolution follow directly from the corresponding properties of multiplication.

Noting the near-symmetry between forward and inverse transforms, we can show that the transform of the product $d = ab$ is

$$D(u,v) = \frac{1}{4\pi^2} A(u,v) \otimes B(u,v).$$

The argument is similar to the one used above.

Next, consider the convolution $c = a \otimes b$ at $(x,y) = (0,0)$ :

$$c(0,0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(-\xi,-\eta) b(\xi,\eta)\, d\xi\, d\eta.$$

We also have

$$c(0,0) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(u,v)\, du\, dv,$$

by taking the inverse transform of $C(u,v)$. Since

$$C(u,v) = A(u,v)B(u,v),$$

we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(-\xi,-\eta) b(\xi,\eta)\, d\xi\, d\eta = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(u,v)B(u,v)\, du\, dv.$$

If we reflect $a(x,y)$ and repeat the above argument for $a(-x,-y)$, we obtain instead

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(\xi,\eta) b(\xi,\eta)\, d\xi\, d\eta = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A^*(u,v)B(u,v)\, du\, dv,$$

since the transform of $a(-x,-y)$ is $A^*(u,v)$, the complex conjugate of $A(u,v)$. In particular, we see that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^2(\xi,\eta)\, d\xi\, d\eta = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A(u,v)|^2\, du\, dv,$$

assuming that $a(x,y)$ is real. Here $|A(u,v)|^2 = A^*(u,v)A(u,v)$. This result, equating power in the spatial domain with power in the frequency domain, is known as *Raleigh's theorem.* The discrete equivalent is *Parseval's theorem.*

## 6.6 Generalized Functions and Unit Impulses

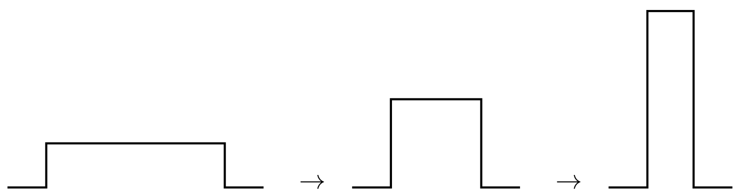
The unit impulse $\delta(x,y)$ is not a function in the traditional sense, because we cannot define its value for all $x$ and $y$. A consistent interpretation is possible, though, if we think of $\delta(x,y)$ as the limit of a sequence of functions.

We need a function that depends on a parameter in such a way that its properties approach those defined for the unit impulse as the parameter tends to a specified limit. This sequence is said to define a *generalized function.* An example will help clarify this idea.

Consider the sequence of square pulses of unit volume:

$$\delta_\epsilon(x,y) = \begin{cases} 1/(4\epsilon^2), & \text{for } |x| \le \epsilon \text{ and } |y| \le \epsilon; \\ 0, & \text{for } |x| > \epsilon \text{ or } |y| > \epsilon. \end{cases}$$

Cross sections through three functions in this sequence look like this:



Clearly

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_\epsilon(x,y)\, dx\, dy = 1,$$

and further, if $f(x,y)$ is sufficiently well behaved,

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_\epsilon(x,y) f(x,y)\, dx\, dy = \lim_{\epsilon\to 0} \frac{1}{4\epsilon^2} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} f(x,y)\, dx\, dy.$$

This is just $f(0,0)$, as can be seen by expanding $f(x,y)$ in a Taylor series about the point $(0,0)$. Also

$$\lim_{\epsilon\to 0} \delta_\epsilon(x,y) = 0 \qquad \text{for any } (x,y) \ne (0,0).$$

Thus the sequence of functions $\{\delta_\epsilon(x,y)\}$ can be thought of as defining the unit impulse. When evaluating an integral involving $\delta(x,y)$, we can use $\delta_\epsilon(x,y)$ instead and then take the limit of the result as $\epsilon \to 0$.

From the form given for $\delta_\epsilon(x,y)$ we see that $\delta(x,y)$ can be thought of as the product of two one-dimensional unit impulses,

$$\delta(x,y) = \delta(x)\delta(y),$$

where the one-dimensional impulse is defined by the *sifting property*,

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0) \quad \text{for arbitrary } f(x).$$

The integral of the one-dimensional unit impulse is the *unit step function*,

$$\int_{-\infty}^{x} \delta(t)\,dt = u(x),$$

where

$$u(x) = \begin{cases} 1, & \text{for } x > 0; \\ 1/2, & \text{for } x = 0; \\ 0, & \text{for } x < 0. \end{cases}$$

Conversely, we can think of the unit impulse as the derivative of the unit step function. This can be seen by considering the step function as the limit of a sequence $\{u_\epsilon(x)\}$, where

$$u_\epsilon(x) = \begin{cases} 1, & \text{for } x > +\epsilon. \\ (1/2)\big(1 + (x/\epsilon)\big), & \text{for } |x| \le \epsilon; \\ 0, & \text{for } x < -\epsilon. \end{cases}$$

Then clearly

$$\frac{d}{dx} u_\epsilon(x) = \begin{cases} 1/(2\epsilon), & \text{for } |x| \le \epsilon; \\ 0, & \text{for } |x| > \epsilon. \end{cases}$$

It must be pointed out that different sequences may define the same generalized function. We can, for example, consider the sequence of Gaussians,

$$\delta_\sigma(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}},$$

as $\sigma \to 0$. Functions in this sequence have unit volume, and $\delta_\sigma(x, y)$ tends to zero for all points $(x, y) \ne (0, 0)$ as $\sigma \to 0$. The sequence $\delta_\sigma(x, y)$ has the advantage over $\delta_\epsilon(x, y)$ of being infinitely differentiable.

What is the Fourier transform of the unit impulse? We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) e^{-i(ux+vy)}\,dx\,dy = 1,$$

as can be seen by substituting $x = 0$ and $y = 0$ into $e^{-i(ux+vy)}$, using the sifting property of the unit impulse. Alternatively, we can use

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_\epsilon(x, y) e^{-i(ux+vy)}\,dx\,dy,$$

or

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} e^{-iux}\,dx\; \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} e^{-ivy}\,dy,$$

that is,

$$\lim_{\epsilon \to 0} \frac{\sin u\epsilon}{u\epsilon} \frac{\sin v\epsilon}{v\epsilon} = 1.$$

We conclude that a system whose point-spread function is the unit impulse is the identity system, since it does not modify anything in the signal. All frequencies are passed through with unit gain and no phase shift, since the modulation-transfer function $H(u, v)$ is unity: The output is equal to the input.

## 6.7 Convergence Factors and the Unit Impulse

The integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(ua+vb)} \, du \, dv$$

does not converge. The problem is that the oscillations in the integrand do not die away as $u$ and $v$ become large. One way to assign a meaning to the integral, despite this problem, is to multiply the integrand by a *convergence factor* that forces it to be small when $u$ and $v$ are large (figure 6-4). The convergence factor has to depend on a parameter in such a way that the modified integral approaches the original one when the parameter approaches a specified limit. The value assigned to the integral is the limit of the modified integral as the parameter approaches this limit. The method will become clear as we apply the notion of convergence factor to the integral given above.

A convenient convergence factor, in this case, is the Gaussian,

$$c_\sigma(u, v) = e^{-\frac{1}{2}\frac{u^2+v^2}{\sigma^2}},$$

where $\sigma$ is the parameter that will be varied. Note that

$$\lim_{\sigma\to\infty} c_\sigma(u, v) \to 1, \qquad \text{for any finite } (u, v).$$

The integral we have to evaluate is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{u^2+v^2}{\sigma^2}} e^{i(ua+vb)}\, du\, dv,$$

or

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{u}{\sigma}\right)^2+iua}\, du \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^2+ivb}\, dv.$$

Now

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{u}{\sigma}\right)^2} \cos(ua)\, du = \sqrt{2\pi}\sigma e^{-\frac{1}{2}a^2\sigma^2},$$

while

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{u}{\sigma}\right)^2} \sin(ua)\, du = 0,$$

since $\sin(ua)$ is an odd function of $u$. The overall integral is thus

$$2\pi\sigma^2 e^{-\frac{1}{2}(a^2+b^2)\sigma^2},$$

which tends to zero as $\sigma \to \infty$ as long as $a^2 + b^2 \neq 0$. When $a = b = 0$, however, the result does not tend to a finite limit as $\sigma \to \infty$. The integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(ua+vb)}\, du\, dv$$

must therefore be a scaled version of the impulse function $\delta(a, b)$. But what is the scale factor? Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(a, b)\, da\, db = 1,$$

we can determine the scale factor by considering

$$\lim_{\sigma\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi\sigma^2 e^{-\frac{1}{2}(a^2+b^2)\sigma^2}\, da\, db.$$

The double integral can be split into the product of two single integrals as follows:

$$2\pi\sigma^2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}a^2\sigma^2}\, da \int_{-\infty}^{\infty} e^{-\frac{1}{2}b^2\sigma^2}\, db = 4\pi^2.$$

The product is independent of $\sigma$, and we finally have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(ua+vb)}\, du\, dv = 4\pi^2\delta(a,b).$$

This result was used earlier in our discussion of the Fourier transform.

## 6.8 Partial Derivatives and Convolution

We shall use differentiation to accentuate edges in images, and it will be useful to know how the Fourier transform of the derived image is related to the Fourier transform of the original image. That is, if $F(u,v)$ is the Fourier transform of $f(x,y)$, what are the Fourier transforms of $\partial f/\partial x$ and $\partial f/\partial y$? Consider the transform

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} e^{-i(ux+vy)}\, dx\, dy,$$

or

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} e^{-iux}\, dx\right] e^{-ivy}\, dy.$$

We can attack the inner integral using integration by parts:

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} e^{-iux}\, dx = \left[f(x,y)e^{-iux}\right]_{-\infty}^{\infty} + (iu)\int_{-\infty}^{\infty} f(x,y)e^{-iux}\, dx.$$

We cannot proceed, however, unless $f(x,y) \to 0$ as $x \to \pm\infty$. In that case the Fourier transform is just

$$\int_{-\infty}^{\infty} (iu) \int_{-\infty}^{\infty} f(x,y)e^{-i(ux+vy)}\, dx\, dy = iuF(u,v).$$

The integral does not converge if $f(x,y)$ does not tend to zero at infinity, but we can resort to convergence factors if this happens and obtain basically the same result. It is easy to show in a similar fashion that the Fourier transform of $\partial f/\partial y$ is just $ivF(u,v)$. We conclude that differentiation accentuates the high-frequency components and suppresses the low-frequency components. In fact, any constant offset or zero-frequency term is lost completely.

The Laplacian of the function $f(x, y)$ is defined as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

So the Fourier transform of the Laplacian is just

$$-(u^2 + v^2)\, F(u, v).$$

We can think of $-(u^2 + v^2)$ as the modulation-transfer function of the operator $\nabla^2$, in a sense to be made precise later. Note that this modulation-transfer function is rotationally symmetric, that is, it depends only on $(u^2 + v^2)$, not on $u$ and $v$ independently. This suggests that the Laplacian operator itself is rotationally symmetric.

It may seem a strange coincidence that taking derivatives in the spatial domain corresponds to multiplication in the frequency domain, since we saw earlier that convolution in the spatial domain corresponds to multiplication in the frequency domain. This becomes less surprising when we consider that differentiation is linear and shift-invariant! Is it possible that taking a derivative is just like convolution with some peculiar function? (It has to be a peculiar function, because it must be zero except at the origin, since the derivative operates locally.) Let us study this question in more detail.

The modulation-transfer function $H(u, v)$ corresponding to the first partial derivative with respect to $x$ is $iu$. We can find the point-spread function corresponding to the first partial derivative by finding the inverse Fourier transform of $iu$:

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} iu e^{+i(ux+vy)}\, du\, dv.$$

This integral does not converge. We could attack it using a convergence factor, but it is easier to note that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{+i(ux+vy)}\, du\, dv = 4\pi^2 \delta(x, y).$$

The integral is thus

$$\frac{\partial}{\partial x} \delta(x, y),$$

since multiplication of the transform with $iu$ corresponds to differentiation with respect to $x$. Now $\delta(x, y)$ is already somewhat pathological, so we cannot expect its derivative to be a function in the classic sense. It can, however, be defined as the limit of a sequence of functions, for example, by

$$\frac{\partial}{\partial x}\delta_\sigma(x,y) = -\frac{x}{2\pi\sigma^4}e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}}.$$

Alternatively, it can be thought of as the limit of the sequence

$$\delta_{x;\epsilon}(x,y) = \frac{1}{2\epsilon}\big(\delta(x+\epsilon,y) - \delta(x-\epsilon,y)\big),$$

where we have two closely spaced impulses of opposite polarity. The result, called a *doublet*, will be denoted $\delta_x(x,y)$. This definition corresponds to the usual way of defining a partial derivative as the limit of a difference, for

$$f(x,y)\otimes\delta_{x;\epsilon}(x,y) = \frac{f(x+\epsilon,y) - f(x-\epsilon,y)}{2\epsilon},$$

so that

$$\lim_{\epsilon\to 0} f(x,y)\otimes\delta_{x;\epsilon}(x,y) = f(x,y)\otimes\delta_x(x,y) = \frac{\partial f}{\partial x}.$$

The generalized function corresponding to the Laplacian can be considered as the limit of the sequence

$$L_\sigma(x,y) = \left(\frac{x^2+y^2-\sigma^2}{2\pi\sigma^6}\right)e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}},$$

obtained by differentiating $\delta_\sigma(x,y)$, for example (figure 6-5a). This function is circularly symmetric. It has a central depression of magnitude $1/(2\pi\sigma^4)$ and radius $\sigma$ surrounded by a circular wall of maximum height $e^{-3/2}/(\pi\sigma^4)$ and radius $\sqrt{3}\sigma$. The form of this function suggests another sequence (figure 6-5b):

$$L_\epsilon(x,y) = \begin{cases} -2/(\pi\epsilon^4), & \text{for } 0 \le x^2+y^2 \le \epsilon^2; \\ +2/(3\pi\epsilon^4), & \text{for } \epsilon^2 < x^2+y^2 \le 4\epsilon^2; \\ 0, & \text{for } 4\epsilon^2 < x^2+y^2. \end{cases}$$

We shall find this form useful later when we look for discrete analogs of these continuous operators.

## 6.9 Rotational Symmetry and Isotropic Operators

The Laplacian is the lowest-order linear combination of partial derivatives that is rotationally symmetric. That is, the Laplacian of a rotated image is the same as the rotated Laplacian of an image. Conversely, if we rotate an image, take the Laplacian, and rotate it back, we obtain the same result as if we had just applied the Laplacian.

Another second-order operator that is rotationally symmetric is the quadratic variation,

$$\left(\frac{\partial^2}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2}{\partial x \partial y}\right)\left(\frac{\partial^2}{\partial y \partial x}\right) + \left(\frac{\partial^2}{\partial y^2}\right)^2.$$

It is, however, not linear. If we allow nonlinearity, then the lowest-order rotationally symmetric differential operator is the squared gradient,

$$\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2.$$

The Laplacian, the squared gradient, and the quadratic variation are useful in detecting edges in images, as we shall see in chapter 8.

Rotationally symmetric operators are particularly attractive because they treat image features in the same way, no matter what their orientation is. Also, a rotationally symmetric function can be described by a simple profile rather than a surface. Finally, the Fourier transform of a rotationally symmetric function can be computed using a single integral instead of a double integral, as we show next.

Let us introduce polar coordinates in both the spatial and the frequency domains (figure 6-6):

$$x = r\cos\phi \quad \text{and} \quad y = r\sin\phi,$$
$$u = \rho\cos\alpha \quad \text{and} \quad v = \rho\sin\alpha,$$

so that $ux + vy = r\rho\cos(\phi - \alpha)$. Now, if $f(x, y) = \overline{f}(r)$, then the transform

$$F(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-i(ux+vy)}\, dx\, dy,$$

becomes just

$$\overline{F}(\rho) = \int_{-\pi}^{\pi}\int_{0}^{\infty} r\overline{f}(r)e^{-ir\rho\cos(\phi-\alpha)}\, dr\, d\phi.$$

(The $r$ in the integrand is of course just the determinant of the Jacobian of the transformation from Cartesian to polar coordinates.) If we change the order of integration, then a simple change of variables turns the inner integral into

$$\int_{-\pi}^{\pi} e^{-ir\rho\cos\phi}\, d\phi = 2\int_{0}^{\pi} \cos(r\rho\cos\phi)\, d\phi = 2\pi J_0(r\rho),$$

where $J_0(x)$ is the zeroth-order Bessel function. Thus if $F(u,v) = \overline{F}(\rho)$, then

$$\overline{F}(\rho) = 2\pi \int_0^{\infty} r\overline{f}(r) J_0(r\rho)\, dr.$$

Similarly, one can show that

$$\overline{f}(r) = \frac{1}{2\pi} \int_0^{\infty} \rho\overline{F}(\rho) J_0(r\rho)\, d\rho.$$

These two formulae define the *Hankel transforms*. (The asymmetry can be traced to our asymmetric definition of the Fourier transform.)

We conclude that the Fourier transform of a rotationally symmetric function is also rotationally symmetric. It is also real, which means that the phase shift is zero. We shall use these results in analyzing some simple imaging system defects. As an example, consider the point-spread function of a system that acts as a *lowpass filter* with cutoff frequency $B$, that is,

$$\overline{H}(\rho) = \begin{cases} 1, & \text{for } \rho \leq B; \\ 0, & \text{for } \rho > B. \end{cases}$$

Taking the inverse transform, we obtain

$$\overline{h}(r) = \frac{1}{2\pi} \int_0^{\infty} \rho H(\rho) J_0(r\rho)\, d\rho = \frac{1}{2\pi} \int_0^{B} \rho J_0(r\rho)\, d\rho.$$

Let $z = r\rho$; then

$$\overline{h}(r) = \frac{1}{2\pi} \frac{1}{r^2} \int_0^{rB} z J_0(z)\, dz.$$

Now

$$\frac{d}{dz} z J_1(z) = z J_0(z),$$

where $J_1(z)$ is the first-order Bessel function. Therefore

$$\overline{h}(r) = \frac{1}{2\pi} \frac{1}{r^2} (rB) J_1(rB) = \frac{1}{2\pi} B^2 \frac{J_1(rB)}{(rB)}.$$

It can be shown that

$$\lim_{z \to 0} \frac{J_1(z)}{z} = \frac{1}{2},$$

so $\overline{h}(r)$ has a maximum at the origin and then drops smoothly to zero at $r = 3.83171\ldots$ (figure 6-7). It is negative for a while and then oscillates about zero with decreasing amplitude. The amplitude decreases asymptotically as $z^{-3/2}$. The function $J_1(z)/z$ plays a role for two-dimensional systems that is similar to that played by $\sin(z)/z$ in the case of one-dimensional systems.

It should be apparent, by the way, that a filter with a sharp cutoff will produce oscillatory responses, or "ringing" effects in the spatial domain (sometimes referred to as *Gibbs's phenomena*). In many cases a filter with a more gradual rolloff is better, since it suffers less from these overshoot phenomena. A Gaussian filter, for example, has a very smooth rolloff that extends over a considerable frequency band. It does not introduce any spurious inflections into the filtered image.

## 6.10 Blurring, Defocusing, and Motion Smear

In a typical imaging system we find that the rays that would be focused at a single point in an ideal system are, in fact, slightly spread out. This blurring of the image can take various forms, but it can sometimes be modeled by a Gaussian point-spread function,

$$h(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}},$$

with unit volume. This is a rotationally symmetric point-spread function, since it depends only on $x^2+y^2$, not on $x$ or $y$ separately. We can compute its Fourier transform using the Hankel transform formula.

Note, however, that the Gaussian happens to be separable into the product of a function of $x$ and a function of $y$. So another approach may be easier:

$$\begin{aligned} H(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}} e^{-i(ux+vy)}\,dx\,dy \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} e^{-iux}\,dx\ \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2} e^{-ivy}\,dy. \end{aligned}$$

The first integral on the right-hand side equals

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\cos(ux)\,dx = \sigma e^{-\frac{1}{2}u^2\sigma^2}.$$

So finally,

$$H(u,v) = e^{-\frac{1}{2}(u^2+v^2)\sigma^2},$$

which is rotationally symmetric, as expected.

We note that low frequencies are passed unattenuated, while higher frequencies are reduced in amplitude, significantly so for frequencies above about $1/\sigma$. Now $\sigma$ is a measure of the size of the original point-spread function; therefore, the larger the blur, the lower the frequencies that are attenuated. This is an example of the inverse relationship between scale changes in the spatial domain and corresponding scale changes in the frequency domain. In fact, if $\overline{r}$ is a measure of the radius of a blur in the spatial domain, and $\overline{\rho}$ is a measure of the radius of its transform, then $\overline{r}\,\overline{\rho}$

is constant.

One way to blur an image is to defocus it (figure 6-1). In this case the point-spread function is a little *pillbox*, as can be seen by considering the cone of light emanating from the lens with its vertex at the focal point. (This point does not lie on the image plane, but slightly in front of or behind it.) The image plane cuts this cone in a circle. Within the circle, brightness is uniform (figure 6-8), so we have

$$h(x,y) = \begin{cases} 1/(\pi R^2), & \text{for } x^2+y^2 \le R^2; \\ 0, & \text{for } x^2+y^2 > R^2. \end{cases}$$

Here

$$R = \frac{1}{2}\frac{d}{f'}e,$$

where $d$ is the diameter of the lens, $f'$ the distance from the lens to the correctly focused spot, and $e$ the displacement of the image plane. We can apply the Hankel transform formula to obtain

$$\overline{H}(\rho) = \frac{2}{R^2}\int_0^R r J_0(r\rho)\,dr = 2\frac{J_1(R\rho)}{(R\rho)},$$

using the fact that

$$\frac{d}{dz} z J_1(z) = z J_0(z),$$

as noted before. Again, low frequencies are passed unattenuated, while higher frequencies are reduced in amplitude, and some are not passed at all. Some are even inverted, since $J_1(z)$ oscillates about zero. For frequencies for which $J_1(R\rho) < 0$ we find that the brightest parts of the defocused image coincide with the darkest parts of the ideal image, and vice versa. Components of the waveform with frequencies for which $J_1(R\rho) = 0$ are removed completely. Such components cannot be recovered from the defocused image. As mentioned before, the first zero of the function $J_1(z)$ occurs at $z = 3.83171\ldots$ .We observe again the inverse scaling in the spatial and frequency domains, since in our case $z = R\rho$. That is, the larger the defocus radius $R$, the lower the frequency $\rho$ for which $J_1(R\rho) = 0$.

Another form of image degradation is due to image motion. This can result from motion of either the imaging system or the objects being imaged. In either case an image point is smeared into a line. For convenience, suppose the motion is along the $x$-axis and the length of the line is $2l$. Then the point-spread function can be described by the product

$$h_x(x,y) = \frac{1}{2l}\big(u(x+l) - u(x-l)\big)\,\delta(y),$$

where $u(z)$ is the unit step function, as before. In this case the point-spread function is not rotationally symmetric. Its Fourier transform can be found as follows:

$$H(u,v) = \int_{-\infty}^{\infty} \frac{1}{2l}\big(u(x-l) - u(l-x)\big)e^{-iux}\,dx \int_{-\infty}^{\infty} \delta(y)e^{-ivy}\,dy,$$

or

$$H(u,v) = \frac{1}{2l}\int_{-l}^{l} e^{-iux}\,dx,$$

so that

$$H(u,v) = \frac{\sin(ul)}{ul}.$$

The argument can easily be extended to motion in any direction. Once again, low frequencies are hardly affected, while higher ones are attenuated. Waves at some frequencies are inverted, and those for which $ul = \pi k$, where $k$ is an integer, are completely suppressed. Waves with crests parallel to the direction of motion are not affected at all, of course.

## 6.11 Restoration and Enhancement

To undo the effects of image blur we can pass the image through a system with a modulation-transfer function $H'(u,v)$ that is the algebraic inverse of the modulation-transfer function $H(u,v)$ of the system that introduced the blur. That is,

$$H(u,v)H'(u,v) = 1.$$

Equivalently, we need a system with point-spread function $h'(x,y)$ such that the convolution of $h'(x,y)$ with $h(x,y)$ is the unit impulse. That is, $h'(x,y) \otimes h(x,y) = \delta(x,y)$:

$$f(x,y) \quad \rightarrow \quad \boxed{\;h(x,y)\;} \text{———} \boxed{\;h'(x,y)\;} \quad \rightarrow \quad f(x,y)$$

The cascade of the two systems is the identity system.

An immediate problem is that we cannot recover frequencies that have been totally suppressed, for which $H(u,v) = 0$. A second problem occurs when we try to compute the inverse Fourier transform of $H'(u,v)$ in order to obtain $h'(x,y)$. It is likely that the needed integral will not converge, although we might be able to obtain a result by introducing a convergence factor. Such a result will not be a function in the classical sense, however.

The most serious problem is noise. Real image measurements are inexact, and we can usually model this defect as additive noise. The noise at one image point is typically independent of, and thus uncorrelated with, the noise at all other image points. It can be shown that this implies that the noise has a flat spectrum: The noise power in any given region of the frequency domain is as large as that in any other region with same area.

Unfortunately, the noise we are concerned with here is introduced after the blurring. The effect is that strongly attenuated frequencies tend to become submerged in the noise, and when we try to recover them by amplification, we also amplify the noise. This is the basic limitation of image restoration, and it is due to the fact that, at any given frequency, we cannot distinguish between signal and noise.

One approach to restoration is heuristic. We can design a system that has a modulation-transfer function approximately equal to the inverse of the modulation-transfer function of the blurring system. We place an upper limit, however, on the amplification. For example,

$$|H'(u,v)| = \min\left(\frac{1}{|H(u,v)|}, A\right),$$

where $A$ is the maximum gain. Or more elegantly, we can use something like

$$H'(u,v) = \frac{H(u,v)}{H(u,v)^2 + B^2},$$

where $1/(2B)$ is the maximum gain, if $H(u,v)$ is real (figure 6-9).

## 6.12 Correlation and the Power Spectrum

When images are processed, it is at times useful to correlate them. In this way we can tell, for example, how similar two brightness patterns are (figure 6-10). The *crosscorrelation* of $a(x,y)$ and $b(x,y)$ is defined by

$$a \star b = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\xi - x, \eta - y)\, b(\xi, \eta)\, d\xi\, d\eta.$$

We shall use the notation $\phi_{ab}(x,y)$ for this integral. Note the similarity to the definition of convolution. The only difference lies in the arguments of the first function in the integrand. Here $a(\xi,\eta)$ is simply shifted by $(x,y)$ before being multiplied by $b(\xi,\eta)$. In convolution the first function is also "flipped over" in $x$ and $y$:

$$a \otimes b = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(x - \xi, y - \eta)\, b(\xi, \eta)\, d\xi\, d\eta.$$

If $b(x,y) = a(x,y)$, the result is called the *autocorrelation*. The autocorrelation of a function is symmetric, that is, $\phi_{aa}(-x,-y) = \phi_{aa}(x,y)$. It can be shown that the autocorrelation of any function has a maximum at $(x,y) = (0,0)$, so that

$$\phi_{aa}(0,0) \geq \phi_{aa}(x,y) \qquad \text{for all } (x,y).$$

If $b(x,y)$ is a shifted version of $a(x,y)$,

$$b(x,y) = a(x - x_0, y - y_0),$$

then a similar maximum will occur for the appropriate value of shift. That is,

$$\phi_{ab}(x_0, y_0) \geq \phi_{ab}(x, y) \qquad \text{for all } (x, y).$$

Note that there can be other maxima, particularly if $a(x, y)$ is periodic. Nevertheless, when $b(x, y)$ is approximately equal to a shifted version of

$a(x, y)$, then the shift can be estimated by looking for maxima in $\phi_{ab}$.

The Fourier transforms of the crosscorrelations and autocorrelations are often informative. They are called *power spectra* for reasons that will become apparent, and they are denoted $\Phi_{ab}(u,v)$ and $\Phi_{aa}(u,v)$, respectively. If the Fourier transform of $a(x,y)$ is $A(u,v)$, then

$$\Phi_{aa}(u,v) = |A(u,v)|^2 = A^*(u,v)A(u,v),$$

where $A^*(u,v)$ is the complex conjugate of $A(u,v)$. Thus $\Phi_{aa}$ is always real, a property that can also be deduced from the symmetry of $\phi_{aa}$ and the fact that the transform of $a(-x,-y)$ is $A^*(u,v)$. In any case, for small $\delta u$ and $\delta v$,

$$\Phi_{aa}(u,v)\,\delta u\,\delta v$$

is the power in the rectangular region of the frequency domain lying between $u$ and $u+\delta u$ and $v$ and $v+\delta v$. This explains the origin of the term *power spectrum.*

Even when the Fourier transform of $a(x,y)$ does not converge, its power spectrum may still exist. It should also be noted that $A(u,v)$ uniquely specifies $a(x,y)$ via the inverse Fourier transform, but there is no unique function corresponding to a given $\Phi_{aa}(u,v)$. Infinitely many functions have the same autocorrelation and thus the same power spectrum. The power spectrum does not change, for example, when an image is translated, since only the phase of the Fourier transform is changed. If an object can be recognized from the power spectrum of an image, then it can be recognized independently of its position. Great hope was held out at one time, for this reason, that Fourier transform methods would be important in solving recognition problems. Unfortunately, such methods only work when the object is alone in the image and does not rotate or change size. Moreover, as we have seen, the power spectra of different objects may be the same.

Random noise provides another interesting illustration. The Fourier transform of an image in which each point has independent random noise with mean zero and standard deviation $\sigma$ is a similar random image with mean zero and standard deviation $2\pi\sigma$. The average of the power spectra of an infinite number of such random images tends to the constant $(2\pi\sigma)^2$ at all frequencies.

## 6.13 Optimal Filtering and Noise Suppression

The next section, dealing with optimal filtering, requires some patience with nontrivial mathematical manipulations. The hasty reader may choose to skip it on first reading without serious loss of continuity. It may be worthwhile returning to this section later, however, since it is the first place in this book where we introduce the tools of the calculus of variations.

Suppose that we are given the sum of the signal $b(x, y)$ and the noise $n(x, y)$. Our task is to recover, as best we can, the signal $b(x, y)$. The measure of how well we succeed will be the integral of the square of the difference between the output $o(x, y)$ and the desired signal $d(x, y)$ (figure 6-11). Usually $d(x, y)$ is just $b(x, y)$. We choose to minimize the integral of the square of the error because it leads to tractable mathematics. (This, of course, is the real reason for the popularity of least-squares methods in general.)

We have to minimize the squared error

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big(o(x,y) - d(x,y)\big)^2\, dx\, dy.$$

If we are going to use a linear system for the filtering operation, we can characterize the system by means of its point-spread function $h(x,y)$. The input to the system is

$$i(x,y) = b(x,y) + n(x,y),$$

and the output is

$$o(x,y) = i(x,y) \otimes h(x,y).$$

So

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(o^2(x,y) - 2o(x,y)d(x,y) + d^2(x,y)\right)\, dx\, dy.$$

Since $o^2 = (i \otimes h)^2$,

$$o^2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i(x-\xi, y-\eta)h(\xi,\eta)\, d\xi\, d\eta \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i(x-\alpha, y-\beta)h(\alpha,\beta)\, d\alpha\, d\beta,$$

and so

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} o^2(x,y)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{ii}(\xi-\alpha, \eta-\beta)h(\xi,\eta)h(\alpha,\beta)\, d\xi\, d\eta\, d\alpha\, d\beta,$$

where $\phi_{ii}(x,y)$ is the autocorrelation of $i(x,y)$. Moreover,

$$o(x,y)d(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i(x-\xi, y-\eta)h(\xi,\eta)d(x,y)\, d\xi\, d\eta,$$

and so

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} o(x,y)d(x,y)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{id}(\xi,\eta)h(\xi,\eta))\, d\xi\, d\eta,$$

where $\phi_{id}(x,y)$ is the crosscorrelation of $i(x,y)$ and $d(x,y)$. Finally we need

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2(x,y) = \phi_{dd}(0,0),$$

where $\phi_{dd}(x, y)$ is the autocorrelation of $d(x, y)$. We can now rewrite the expression for the error term to be minimized in the form

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{ii}(\xi - \alpha, \eta - \beta) h(\xi, \eta) h(\alpha, \beta) \, d\xi \, d\eta \, d\alpha \, d\beta$$

$$- 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{id}(\xi, \eta) h(\xi, \eta) \, d\xi \, d\eta + \phi_{dd}(0, 0).$$

This expression is to be minimized by finding the point-spread function $h(x, y)$. This is a problem in the *calculus of variations*. (The calculus of variations is covered in more detail in the appendix.) We shall attack the problem using the basic method of that speciality. In the typical calculus problem we look for a parameter value that results in a stationary value of a given function. In the problem here, we are looking instead for a function that leads to a stationary value of a given functional. A *functional* is an expression that depends on a function, as, for example, $E$ above depends on $h(\xi, \eta)$.

Suppose that $h(x, y)$ gives the minimum value of $E$, and let $\delta h(x, y)$ be an arbitrary function used to modify $h(x, y)$. Then $h(x, y) + \epsilon \, \delta h(x, y)$ will give a value that cannot be less than $E$, no matter what $\delta h(x, y)$ is. Let the value be $E + \delta E$. If we are truly at a minimum, then

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon}(E + \delta E) = 0 \qquad \text{for all } \delta h(x, y).$$

If this were not the case, we could reduce $E$ by adding a small multiple of $\delta h(x, y)$ to $h(x, y)$, thus contradicting the assumption that $h(x, y)$ is optimal. Now

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon}(E + \delta E)$$

$$= 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{ii}(\xi - \alpha, \eta - \beta) h(\xi, \eta) \delta h(\alpha, \beta) \, d\xi \, d\eta \, d\alpha \, d\beta$$

$$- 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{id}(\xi, \eta) \delta h(\xi, \eta) \, d\xi \, d\eta,$$

or

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon}(E + \delta E)$$

$$= -2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \phi_{id}(\xi, \eta) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{ii}(\xi - \alpha, \eta - \beta) h(\alpha, \beta) \, d\alpha \, d\beta \right]$$

$$\times\, \delta h(\xi, \eta)\, d\xi\, d\eta.$$

If this is to be zero for all $\delta h(x, y)$, then the bracketed expression must be zero, or

$$\phi_{id}(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{ii}(\xi - \alpha, \eta - \beta) h(\alpha, \beta)\, d\alpha\, d\beta,$$

that is, perhaps surprisingly,

$$\phi_{id} = \phi_{ii} \otimes h\,.$$

This simple equation for $h(x, y)$ can be solved by taking the Fourier transform,

$$\Phi_{id} = H\Phi_{ii},$$

where $\Phi_{ii}$ and $\Phi_{id}$ are the power spectra. The power spectra are thus all we need to know to design the image-restoring system under the given assumptions. The same system will be optimal for a large class of images, not just a single one. (It would, of course, not be of much interest otherwise.)

As an example, consider a system designed to suppress noise, that is, a system that takes the sum of the image $b(x, y)$ and the noise $n(x, y)$ and produces an output $o(x, y)$ that is as close as possible, in the least-squares sense, to the original image $b(x, y)$. Here $d(x, y) = b(x, y)$ and

$$i(x, y) = b(x, y) + n(x, y).$$

So

$$\Phi_{id} = \Phi_{bb} + \Phi_{nb},$$

and

$$\Phi_{ii} = \Phi_{bb} + \Phi_{bn} + \Phi_{nb} + \Phi_{nn},$$

as can be seen by noting the definitions of $\Phi_{ii}$ and $\Phi_{id}$. We now assume that the noise is not correlated to the signal, so that $\Phi_{bn} = \Phi_{nb} = 0$. Then

$$H = \frac{\Phi_{id}}{\Phi_{ii}} = \frac{\Phi_{bb}}{\Phi_{bb} + \Phi_{nn}} = \frac{1}{1 + \Phi_{nn}/\Phi_{bb}}.$$

It is clear what the optimal system is doing. In parts of the spectrum where the signal-to-noise ratio, $\Phi_{bb}/\Phi_{nn}$, is high, the gain is almost unity; in parts where the noise dominates, the gain is very low, approximately $\Phi_{bb}/\Phi_{nn}$, which is just the signal-to-noise ratio.

Now consider the case where the signal $b(x, y)$ is passed through a system with point-spread function $h(x, y)$ before the noise $n(x, y)$ is added. The result,

$$i = b \otimes h + n,$$

is to be passed through a system with point-spread function $h'(x, y)$. The output

$$o = i \otimes h'$$

should be as close as possible to the original image $b(x, y)$, in the least-squares sense. Here $d(x, y) = b(x, y)$, so that

$$\Phi_{id} = H\Phi_{bb} + \Phi_{nb}$$

and

$$\Phi_{ii} = H^2\Phi_{bb} + H(\Phi_{nb} + \Phi_{bn}) + \Phi_{nn}.$$

Assuming that the noise is not correlated with the signal, we have

$$H' = \frac{\Phi_{id}}{\Phi_{ii}} = \frac{H\Phi_{bb}}{H^2\Phi_{bb} + \Phi_{nn}}.$$

If the signal-to-noise ratio is high in a particular part of the spectrum, then

$$H' \approx \frac{1}{H}$$

there, while gain is limited to about $H(\Phi_{bb}/\Phi_{nn})$ in parts where $\Phi_{nn} > |H|^2\,\Phi_{bb}$. Note the similarity of this result to that derived heuristically earlier.

Finally, it may be instructive to consider the optimal filter for estimating a processed version of the image rather than the image itself. Suppose we want the least-squares estimate of

$$d(x, y) = b(x, y) \otimes p(x, y),$$

where $p(x, y)$ is the point-spread function of a processing filter. Then

$$\phi_{id} = i \star d = i \star (b \otimes p) = (i \star b) \otimes p = \phi_{ib} \otimes p,$$

so that

$$\Phi_{id} = \Phi_{ib}\,P,$$

where $P(u, v)$ is the Fourier transform of $p(x, y)$. Thus

$$H' = \frac{\Phi_{id}}{\Phi_{ii}} = \frac{\Phi_{ib}}{\Phi_{ii}}\,P.$$

The optimal filter is just the cascade of the optimal filter for recovering the image $b(x, y)$ and the processing filter $P(u, v)$. We do not need anything else.

We should note at this point that the design of the optimal filter here is much simpler than in the one-dimensional situation. This is because the impulse response in the one-dimensional case must be one-sided, since a system cannot anticipate its input. Limitations in the time domain do not translate easily into understandable limitations in the frequency domain. For example, it is hard to express the constraint that $f(t) = 0$ for $t < 0$ in terms of $F(\omega)$, the Fourier transform of $f(t)$. Fortunately, there is no such problem in the case of images, since the support of a point-spread function can extend in all directions from the origin. The *support* of a function is the region over which it is nonzero.

## 6.14 Image Models

In order to apply the optimal filtering methods, we must estimate the power spectra of the images to be processed. Looking at the spectra of a few “typical” images will quickly persuade you that most of the energy is concentrated at the lower frequencies. It is useful to know about this falloff with frequency since it helps separate the desired signal from the noise, which has a flat spectrum. The observed falloff in power with frequency is, in part, due to the fact that many objects or parts of objects are opaque and have nearly uniform brightness. The corresponding image patches are separated by discontinuities along edges where objects occlude one another.

A full discussion of image models lies beyond the scope of this book, but we can get a rough idea by considering a simple rectangular patch

$$f(x, y) = \begin{cases} 1, & \text{for } |x| \le W \text{ and } |y| \le H; \\ 0, & \text{for } |x| > W \text{ or } |y| > H. \end{cases}$$

The Fourier transform is

$$F(u, v) = WH \, \frac{\sin(uW)}{uW} \, \frac{\sin(vH)}{vH}.$$

Shifting the patch just changes the phase, not the magnitude, of the transform. Ignoring the oscillations, we see that the transform falls off as $1/(uv)$. Thus, depending on the direction we choose in the frequency domain, it falls off as $1/\rho$ or $1/\rho^2$ with distance $\rho$ from the origin.

Another useful component of an image model might be the "pillbox" patch,

$$\overline{f}(r) = \begin{cases} 1, & \text{for } r \le R; \\ 0, & \text{for } r > R. \end{cases}$$

The transform in this case is

$$\overline{F}(\rho) = 2R^2 \frac{J_1(\rho R)}{(\rho R)}.$$

For large arguments, $J_1(z)$ behaves like

$$\sqrt{\frac{2}{\pi z}} \sin(z - \pi/4),$$

so that, if we ignore the oscillations, $\overline{F}(\rho)$ falls off as $1/\rho^{3/2}$ for large $\rho$.

Image models containing polygonal or circular patches tend to have power spectra falling off as some power of frequency. At higher frequencies real images fall off even more rapidly, due to the resolution limits of the optical system. In telescopes, for example, there is an absolute cutoff frequency, determined by the ratio of the aperture diameter to the wavelength of light, above which there is no transmission at all. Microscopes have a similar absolute limitation determined by the numerical aperture of the objective and the wavelength of light.

A different application of the observation that most power in images is concentrated at low frequencies can be found in image reproduction. Methods for displaying images, such as the printing of halftones, photographic reproduction, and television, have limited *dynamic range*; that is, they can only show a certain range of gray-level values. In terms of the quality of reproduction, what we are interested in is the ratio of the brightest to the darkest reproducible gray-level. One important consideration in displaying images is that small brightness differences be perceptible. Even large differences in brightness between adjacent regions may not be noticeable if the regions are themselves very bright. What is important is the relative size of the brightness difference, that is, the ratio of the difference to the smaller of the two. It is for this reason that dynamic range is measured by the ratio of the brightest to the darkest level that can be reproduced, rather than the difference.

The dynamic range of color transparencies can be over a hundred to one, while that of newsprint is often not much more than ten to one. Natural images tend to have large dynamic ranges. Usually a compromise has to be struck when they are to be reproduced. If we try to impose the variation in image brightness unchanged onto the medium, the brightest and the darkest areas will not be reproduced properly. To avoid losing detail in the highlights and shadows due to saturation, we have to compress the dynamic range.

A power function can do this compression. If the brightness of the reproduction is $b'(x,y)$ and that of the original is $b(x,y)$, then

$$b'(x,y) = \big(b(x,y)\big)^{\gamma},$$

where $0 < \gamma < 1$. Such reproductions are generally acceptable, although barely perceptible brightness differences in the original will be imperceptible in the reproduction.

Another approach is to take advantage of the fact that images usually contain large low-frequency components. A filter that attenuates low frequencies can be devised by subtracting from the image a smoothed version of the image. Such a filter will tend to reduce the dynamic range. An example is provided by a filter with a point-spread function

$$h(x,y) = \delta(x,y) - \frac{k}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}}$$

for $0 < k < 1$. The modulation-transfer function of this filter is

$$H(u,v) = 1 - k e^{-\frac{1}{2}(u^2+v^2)\sigma^2}.$$

Other smoothing functions can be used. A photographic technique for achieving a similar effect is called *unsharp masking*. Here an out-of-focus image is "subtracted," in part, from the original. Note that, in this case, sharp edges are reproduced with their full contrast. We have to be careful in applying this process, however, since the brightness values in the image are shifted around and spurious changes in the appearance of the objects may result. As we shall see later, the brightness values are used in recovering surface shape, for example.

## 6.15 References

The classic reference on image processing is *Digital Image Processing* by Pratt [1978]. The first few chapters of *Digital Picture Processing* by Rosenfeld & Kak [1982] also provide an excellent introduction to the subject. Much of the two-dimensional analysis is a straightforward extension of the one-dimensional case aptly described in *Signals and Systems* by Oppenheim & Willsky [1983] and *Circuits, Signals, and Systems* by Siebert [1986]. The underlying theory of the Fourier transform is given in the standard reference *The Fourier Transform and Its Applications* by Bracewell [1965, 1978]. An enjoyable discussion of generalized functions appears in Lighthill's *Introduction to Fourier Analysis and Generalised Functions* [1978]. Even more detail can be found in *Generalized Functions: Properties and Operations* by Gel'fand & Shilov [1964]. Few texts explicitly discuss convergence factors; one that does is *Summable Series and Convergence Factors* by Moore [1966].

The basic work on optimal filtering is due to Wiener. He uses a delightfully symmetric definition of the Fourier transform in *Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications* [1966]. The optimal filter is derived using the methods of the calculus of variations, for which volume I of *Methods of Mathematical Physics* by Courant & Hilbert [1953] may be the best reference. Image models are discussed in *Pattern Models* by Ahuja & Schachter [1983].

Image processing is a relatively old field that matured more than ten years ago. A survey of early work on image processing is provided by Huang, Schreiber, & Tretiak [1971]. The classic application of image processing method has been in improving image quality, as discussed by Schreiber [1978].

Detailed structure in an image that is too fine to be resolved, yet coarse enough to produce a noticeable fluctuation in the gray-levels of neighboring picture cells, constitutes *texture*. (Note that there are other notions of what is meant by the term texture.) Texture may be periodic, nearly periodic, or random. There has been work devoted to the derivation of texture measures that allow classification. Other efforts are directed at the segmentation of images into regions of differing texture, as in the work of Bajcsy [1973] and Ehrich [1977]. Methods for the analysis of gray-level co-occurrence histograms have found application in this domain. Another approach depends on the appearance of peaks in the frequency domain. Ahuja & Rosenfeld [1981] study the relationship of mosaic image models to the notion of texture. Further references relating to image processing will be given at the end of the next chapter.

## 6.16 Exercises

**6-1** Find $k(\sigma)$ such that the family of functions

$$\delta_\sigma(x, y) = k(\sigma)\, e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}}$$

defines the unit impulse $\delta(x, y)$ as $\sigma \to 0$.

**6-2** Consider the family of functions

$$L_\delta(x, y) = \begin{cases} a, & \text{for } r \le \delta; \\ b, & \text{for } \delta < r \le 2\delta; \\ 0, & \text{for } 2\delta < r; \end{cases}$$

where $r = \sqrt{x^2 + y^2}$. For what values of $a$ and $b$ does this family define the generalized function that corresponds to the Laplacian? That is, when is the limit of the convolution of $L_\delta(x, y)$ with some given function $f(x, y)$ equal to $\nabla^2 f(x, y)$? Hint: It may help to apply the operator to the test function

$$\frac{1}{4}(x^2 + y^2),$$

whose Laplacian is known to be equal to one.

**6-3** Show that if $f(x, y)$ is separable into a product of a function of $x$ and a function of $y$, its Fourier transform $F(u, v)$ is also separable into a function of $u$ and a function of $v$.

**6-4** Show that if $f(x, y) \ge 0$ for all $x$ and $y$, then $F(0, 0) \ge |F(u, v)|$ for all $u$ and $v$. When is $F(0, 0) = F(u, v)$?

**6-5** Usually the point-spread function $h(x, y)$ of an operator used for smoothing operations is largest at the origin, $(x, y) = (0, 0)$, positive everywhere, and dies away as $x$ and $y$ tend to infinity. It can be conveniently thought of as a mass distribution. Without loss of generality we shall assume that the center of mass of this distribution lies at the origin. We need to be able to say how "spread out" such a distribution is. The *radius of gyration* of a mass distribution is the distance from its center of mass at which a point of equal mass would have to be placed in order for it to have the same inertia as the given distribution. (The inertia of a point mass is the product of the square of the distance from the origin times the mass.)

The total mass $M$ of a distribution $h(x, y)$ is just

$$M = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y)\, dx\, dy,$$

while the radius of gyration $R$ is defined by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r^2 h(x, y)\, dx\, dy = R^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y)\, dx\, dy = M\, R^2,$$

where $r^2 = x^2 + y^2$.

(a) Find the radius of gyration of a pillbox defined by

$$b_V(x,y) = \begin{cases} 1/(\pi V^2), & \text{for } r \le V; \\ 0, & \text{for } r > V. \end{cases}$$

(b) Find the radius of gyration of the Gaussian

$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}}.$$

Note that the distributions in (a) and (b) both have "unit mass" and that it may help to convert the required integrals to polar coordinates.

(c) Show that the mass of the convolution of two smoothing functions is the product of the masses of the two functions. Also show that, when smoothing functions are convolved, their gyration radii squared add. That is, if $f = g \otimes h$, then $R_f^2 = R_g^2 + R_h^2$, where $R_f$, $R_g$, and $R_h$ are the radii of gyration of $f$, $g$, and $h$, respectively.

(d) When a rotationally symmetric smoothing function is convolved with itself many times, it becomes indistinguishable from the Gaussian. Suppose that the pillbox is convolved with itself $n$ times. What is the value of $\sigma$ of the approximating Gaussian?

**6-6** Show that $a \star (b \otimes c) = (a \star b) \otimes c$, where $\star$ denotes correlation and $\otimes$ denotes convolution.

**6-7** The modulation-transfer function of an optical telescope is $A(u,v) = P(u,v) \otimes P(u,v)$, where $P(u,v)$ is the rotationally symmetric lowpass filter

$$P(u,v) = \begin{cases} 1, & \text{for } u^2 + v^2 \le \omega^2; \\ 0, & \text{for } u^2 + v^2 > \omega^2, \end{cases}$$

for some $\omega$, where $\omega$ is a function of the wavelength of light and the size of the collecting optics.

(a) Find $A(u,v)$. Hint: What is the overlap between two disks of equal diameter when their centers are not aligned?

(b) What is the corresponding point-spread function? Hint: What does multiplication in the frequency domain correspond to in the spatial domain?

**6-8** Consider a system that blurs images according to a Gaussian point-spread function with standard deviation $\sigma$. Suppose that the noise power spectrum is flat with power $N^2$, the signal power spectrum is also flat with power $S^2$, and that $S^2 > N^2$. (Noise is added to the image after blurring.)

(a) Sketch the modulation-transfer function of the optimal filter for deblurring the image.

(b) What is the low-frequency response?

(c) What frequency is maximally amplified?

(d) What is the maximal gain?

**6-9** It is difficult to measure the point-spread function of an optical system directly. Instead, we usually image a sharp edge between two regions with different brightnesses. In this fashion we obtain the *edge-spread function*.

(a) How would you obtain the *line-spread function* $l(x)$ from the response $e(x)$ to a unit step edge? The line-spread function is the response of the system to an impulsive line.

(b) Show that $l(x)$ is related to the point-spread function $h(r)$ by

$$l(x) = 2\int_x^\infty \frac{r}{\sqrt{r^2 - x^2}} h(r)\, dr.$$

This is the definition of the *Abel transform*. Here $l(x)$ is the Abel transform of $h(r)$. Assume that the point-spread function is rotationally symmetric.

(c) Show that the Abel transform obeys the relationships

$$\int_{-\infty}^\infty l(x)\, dx = 2\pi \int_0^\infty h(r)\, r\, dr \qquad \text{and} \qquad l(0) = 2\int_0^\infty h(r)\, dr.$$

(d) How would you recover the point-spread function from the measured line-spread function? Show that

$$h(r) = -\frac{1}{\pi}\int_r^\infty \frac{l'(x)}{\sqrt{x^2 - r^2}} dx = -\frac{1}{\pi}\int_r^\infty \sqrt{x^2 - r^2}\, \frac{d}{dx}\left(\frac{l'(x)}{x}\right) dx,$$

where $l'(x)$ is the derivative of $l(x)$ with respect to $x$. This is the inverse Abel transform.

**6-10** A rotationally symmetric function $f(x, y)$ depends only on the radius $r$ and does not depend on the polar angle $\theta$, where

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \tan^{-1}(y/x).$$

Show that the Gaussian is the only rotationally symmetric function that can be decomposed into the product of a function of $x$ and a function of $y$; that is, $f(x, y) = g(x)h(y)$. Hint: First prove that $f(x, y)$ is rotationally symmetric if and only if

$$\frac{1}{x}\frac{\partial f}{\partial x} = \frac{1}{y}\frac{\partial f}{\partial y}.$$

You can then easily prove that, for some constant $c$,

$$\frac{dg}{dx} = c\,x\,g(x) \qquad \text{and} \qquad \frac{dh}{dy} = c\,y\,h(y).$$