Rotation — outline of talk

- Properties
- Representations
- Hamilton's Quaternions
- Rotation as Unit Quaternion
- The Space of Rotations
- Photogrammetry
- Closed Form Solution of Absolute Orientation
- Division Algebras, Quaternion Analysis, Space-Time

Where do we need Rotation?

- Machine Vision
- Recognition & Orientation
- Graphics, CAD
- Virtual Reality
- Vehicle Attitude
- Robotics
- Spatial Reasoning
- Path Planning — Collision Avoidance
- Protein Folding ...

Euclidean Motion

- Translation and rotation

- Preserves distances between points
- Preserves angles between lines
- Preserves handedness

- Preserves dot-products
- Preserves triple products

- Contrast: Reflections, Skewing, Scaling

Basic Properties of Rotation

- Euler's theorem: line of fixed points — rotation axis
- Parallel Axis Theorem
- Rotation of Sphere induces Rotation of Space
- Attitude, Orientation — Rotation Relative to Reference
- Degrees Of Freedom: 3
- Note: Confusing Coincidence!

More Properties of Rotation

- Rotational velocity: easy, vector
- Poisson's formula: $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$
- Rotational velocities add
- Finite Rotations *don't* commute
- $n(n-1)/2$ DOF
- Coincidence: $n(n-1)/2 = n$ for $n = 3$
- Rotation — not "about axes" — instead "in planes"
- Confusing coincidence *also* for cross-product

Isomorphism Vectors & Skew-Symmetric Matrices

$$\mathbf{a} \times \mathbf{b} = A\,\mathbf{b}$$

$$A = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \overline{B}\,\mathbf{a}$$

$$\overline{B} = \begin{pmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{pmatrix}$$

Representations for rotation

(1) Axis and angle: $\widehat{\boldsymbol{\omega}}$ and θ — Gibbs vector: $\widehat{\boldsymbol{\omega}} \tan(\theta/2)$

(2) Euler angles (24 definitions)

(3) Orthonormal matrices — $R^T R = I$ and $\det(R) = +1$

(4) "Exponential cross-product" — $dR/d\theta = \Omega R \rightarrow R = e^{\theta\Omega}$

(5) Stereography plus bilinear complex map

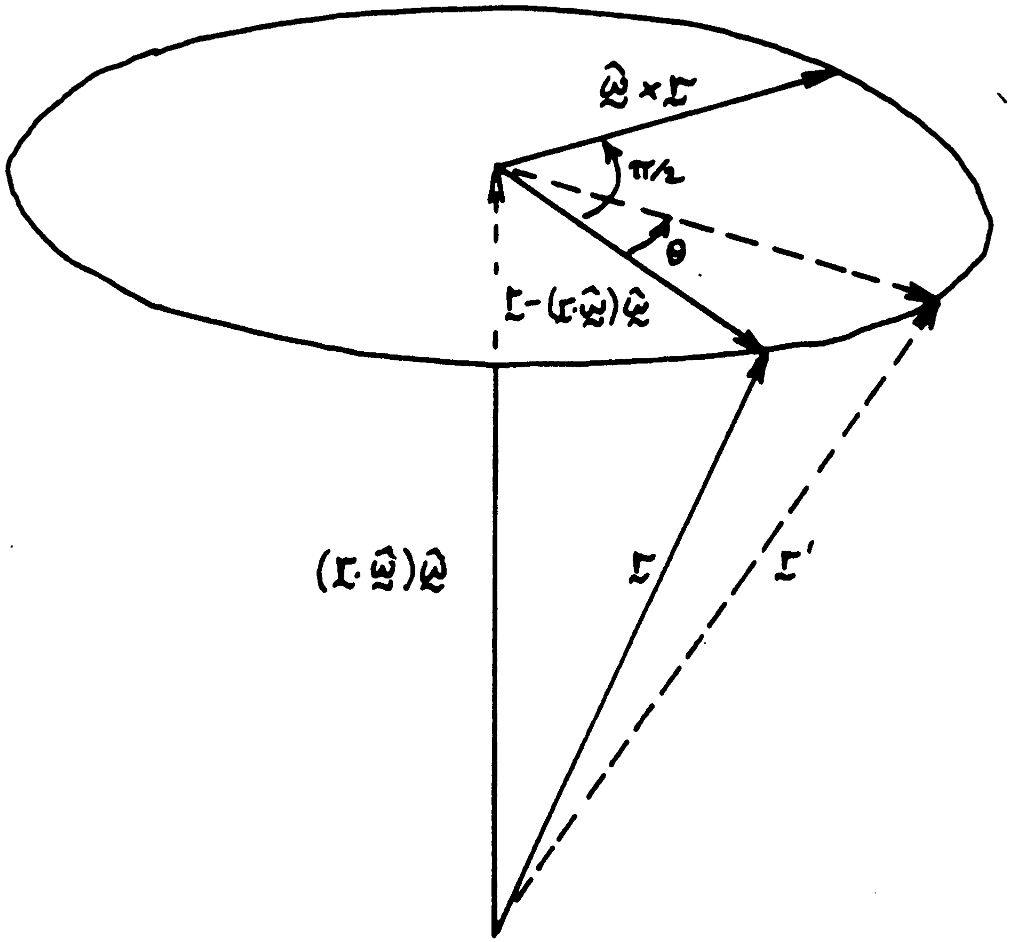
(6) Pauli spin matrices

(7) Euler *parameters*

(8) Unit quaternions

Rodrígues' Formula — Axis and Angle

Rotation about $\widehat{\boldsymbol{\omega}}$ through θ:



$$\mathbf{r}' = \cos\theta\,\mathbf{r} + (1 - \cos\theta)\,(\widehat{\boldsymbol{\omega}} \cdot \mathbf{r})\,\widehat{\boldsymbol{\omega}} + \sin\theta\,(\widehat{\boldsymbol{\omega}} \times \mathbf{r})$$

Exponential cross-product

Rotation about $\widehat{\boldsymbol{\omega}}$ through θ:

$$\mathbf{r} = R(\theta)\,\mathbf{r}_0$$

$$\frac{d\mathbf{r}}{d\theta} = \frac{d}{d\theta} R(\theta)\,\mathbf{r}_0$$

$$\frac{d\mathbf{r}}{d\theta} = \widehat{\boldsymbol{\omega}} \times \mathbf{r} = \Omega\,\mathbf{r} = \Omega\,R(\theta)\,\mathbf{r}_0$$

$$\frac{d}{d\theta} R(\theta)\,\mathbf{r}_0 = \Omega\,R(\theta)\,\mathbf{r}_0$$

for all $\mathbf{r}_0$:

$$\frac{d}{d\theta} R(\theta) = \Omega\,R(\theta)$$

$$R(\theta) = e^{\theta\Omega}$$

Rodrígues' Formula — Exponential Cross Product

$dR/d\theta = \Omega R \rightarrow R = e^{\theta\Omega}$

$$e^{\theta\Omega} = I + \theta\Omega + \frac{1}{2!}(\theta\Omega)^2 + \frac{1}{3!}(\theta\Omega)^3 + \frac{1}{4!}(\theta\Omega)^4 + \ldots$$

$\Omega^2 = (\widehat{\boldsymbol{\omega}}\,\widehat{\boldsymbol{\omega}}^T - I)$ and $\Omega^3 = -\Omega$

$$e^{\theta\Omega} = I + \Omega\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right) + \Omega^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \ldots\right)$$

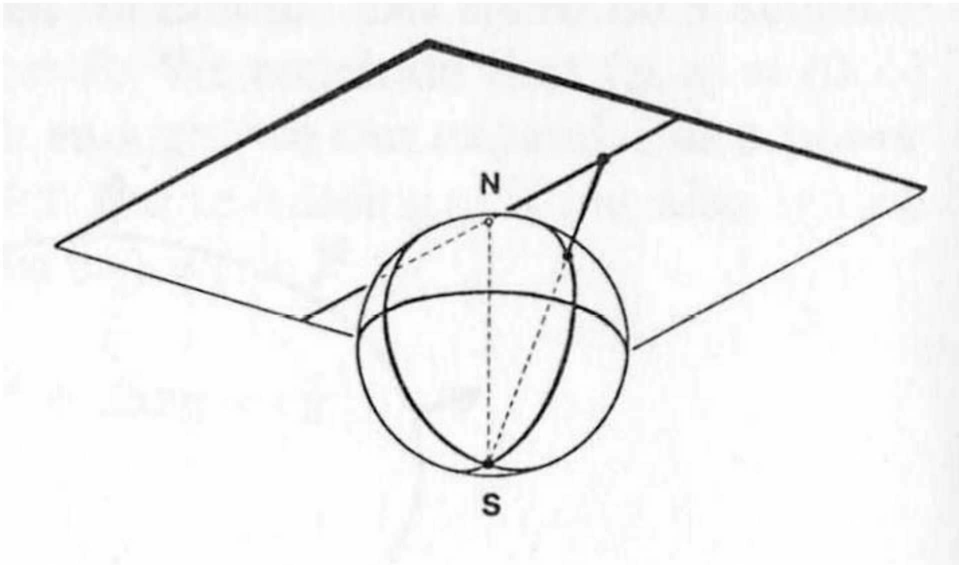
$e^{\theta\Omega} = I + (\sin\theta)\,\Omega + (1 - \cos\theta)\,\Omega^2$

$e^{\theta\Omega} = \cos\theta\, I + (\sin\theta)\,\Omega + (1 - \cos\theta)\,\widehat{\boldsymbol{\omega}}\,\widehat{\boldsymbol{\omega}}^T$

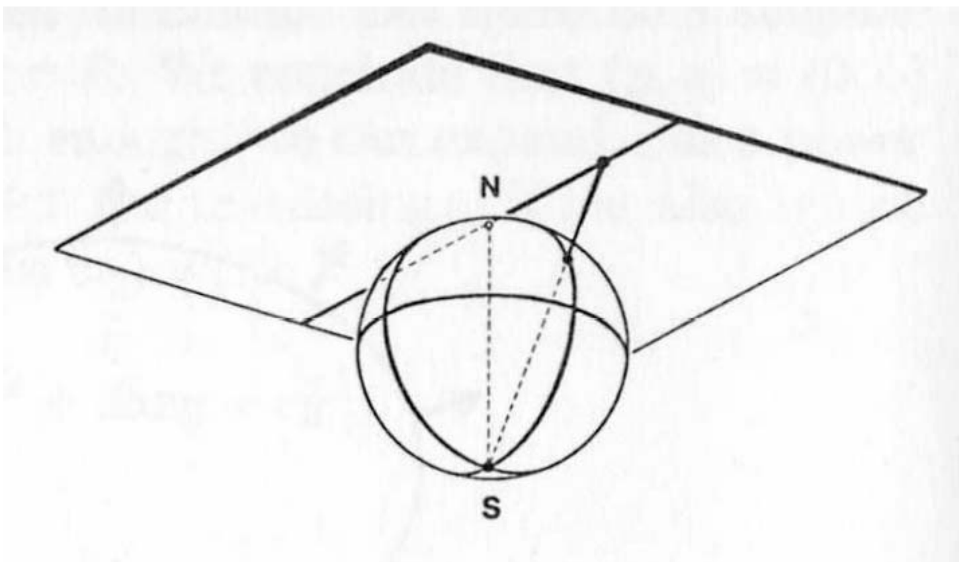
$\mathbf{r}' = e^{\theta\Omega}\,\mathbf{r}$

$\mathbf{r}' = \cos\theta\,\mathbf{r} + (1 - \cos\theta)\,(\widehat{\boldsymbol{\omega}} \cdot \mathbf{r})\,\widehat{\boldsymbol{\omega}} + \sin\theta\,(\widehat{\boldsymbol{\omega}} \times \mathbf{r})$

Stereographic Projection & Bilinear Map



$$z' = \frac{a\,z + b}{c\,z + d}$$



Desirable Properties

- Ability to rotate vectors — or coordinate system
- Ability to compose rotations
- Intuitive, non-redundant representation
- Computational efficiency

- Interpolate orientations
- Averages over range of rotations
- Derivative w.r.t rotation — optimization, LSQ
- Sampling of rotations — uniform and random

- Notion of a space of rotations

Problems with Some Representations

- Orthonormal matrices: redundant, with complex constraints
- Euler angles: inability to compose rotations, "gimbal lock"
- Gibbs vector: singularity when $\theta = \pi$
- Axis and angle: inability to compose rotations
- No notion of "space of rotations"

Hamilton and division algebras

- Algebraic couples: complex numbers as pair of reals
- Want: multiplicative inverses e.g. $z^*/\|z\|^2$
- Examples: reals, complex numbers
- Expected next: three components (vectors)
- (This was *before* Gibbs and 3-vectors)

Well, Papa, can you multiply triplets?

Brougham bridge — 1843 October 16th

> “And here there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triples ... An electric circuit seemed to close, and a spark flashed forth.”

Hamilton's quaternions

Insight: Can't do it with three components

Insight: Need additional square roots of -1

$i^2 = j^2 = k^2 = ijk = -1$

From which follows:

$ij = k$, $jk = i$, and $ki = j$

$ji = -k$, $kj = -i$, and $ik = -j$

Note: multiplication *not* commutative

Representations of quaternions

(1) Real and three imaginary parts $q_0 + iq_x + jq_y + kq_z$

(2) Scalar and 3-vector $(q, \mathbf{q})$

(3) 4-vector — $\mathring{\mathrm{q}}$

(4) Certain orthogonal 4×4 matrices Q

(5) Complex composite of two complex numbers

Representations of Multiplication

Real and three imaginary parts

$$(p_0 + p_x i + p_y j + p_z k)(q_0 + q_x i + q_y j + q_z k) =$$

$$\begin{aligned}
&(p_0 q_0 - p_x q_x - p_y q_y - p_z q_z) \quad + \\
&(p_0 q_x + p_x q_0 + p_y q_z - p_z q_y)\, i + \\
&(p_0 q_y - p_x q_z + p_y q_0 + p_z q_x)\, j + \\
&(p_0 q_z + p_x q_y - p_y q_x + p_z q_0)\, k
\end{aligned}$$

Scalar and 3-vector

$$(p, \mathbf{p})\,(q, \mathbf{q}) = (p\,q - \mathbf{p} \cdot \mathbf{q}, p\,\mathbf{q} + q\,\mathbf{p} + \mathbf{p} \times \mathbf{q})$$

4-vector

$$\begin{pmatrix} p_0 & -p_x & -p_y & -p_z \\ p_x & p_0 & -p_z & p_y \\ p_y & p_z & p_0 & -p_x \\ p_z & -p_y & p_x & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_x \\ q_y \\ q_z \end{pmatrix}$$

Isomorphism — Quaternions and Orthogonal Matrices

$$\mathring{p}\,\mathring{q} = P\,\mathring{q}$$

where

$$P = \begin{pmatrix} p_0 & -p_x & -p_y & -p_z \\ p_x & p_0 & -p_z & p_y \\ p_y & p_z & p_0 & -p_x \\ p_z & -p_y & p_x & p_0 \end{pmatrix}$$

P is orthogonal

P is normal if $\mathring{p}$ is a unit quaternion

P is skew-symmetrc if $\mathring{p}$ has zero scalar part

$$\mathring{p}\,\mathring{q} = \overline{Q}\,\mathring{p}$$

where

$$\overline{Q} = \begin{pmatrix} q_0 & -q_x & -q_y & -q_z \\ q_x & q_0 & q_z & -q_y \\ q_y & -q_z & q_0 & q_x \\ q_z & q_y & -q_x & q_0 \end{pmatrix}$$

Conjugate, Dot-Product, Norm, and Inverse

Not commutative: $\mathring{p}\,\mathring{q} \neq \mathring{q}\,\mathring{p}$

Associative: $(\mathring{p}\,\mathring{q})\,\mathring{r} = \mathring{p}\,(\mathring{q}\,\mathring{r})$

Conjugate: $(p, \mathbf{p})^* = (p, -\mathbf{p}) \quad \rightarrow \quad (\mathring{p}\mathring{q})^* = \mathring{q}^*\mathring{p}^*$

Dot-product: $(p, \mathbf{p}) \cdot (q, \mathbf{q}) = pq + \mathbf{p} \cdot \mathbf{q}$

Norm: $\|\mathring{q}\|^2 = \mathring{q} \cdot \mathring{q}$

$$\mathring{q}\,\mathring{q}^* = (q, \mathbf{q})\,(q, -\mathbf{q}) = (q^2 + \mathbf{q} \cdot \mathbf{q}, 0) = (\mathring{q} \cdot \mathring{q})\,\mathring{e}$$

So $\mathring{q}\,\mathring{q}^* = (\mathring{q} \cdot \mathring{q})\,\mathring{e}$ and $\mathring{q}^*\,\mathring{q} = (\mathring{q} \cdot \mathring{q})\,\mathring{e}$ where $\mathring{e} = (1, 0)$

Multiplicative inverse: $\mathring{q}^{-1} = 1/(\mathring{q} \cdot \mathring{q})\,\mathring{q}^*$

Dot-products of Products

$$(\mathring{p}\,\mathring{q}) \cdot (\mathring{p}\,\mathring{q}) = (\mathring{p} \cdot \mathring{p})(\mathring{q} \cdot \mathring{q})$$

$$(\mathring{p}\,\mathring{q}) \cdot (\mathring{p}\,\mathring{r}) = (\mathring{p} \cdot \mathring{p})(\mathring{q} \cdot \mathring{r})$$

$$(\mathring{p}\,\mathring{q}) \cdot \mathring{r} = \mathring{p} \cdot (\mathring{r}\,\mathring{q}^*)$$

Quaternions representing Vectors

$$\mathring{r} = (0, \mathbf{r})$$

$$\mathring{r}^* = -\mathring{r}$$

$$\mathring{r} \cdot \mathring{s} = \mathbf{r} \cdot \mathbf{s}$$

$$\mathring{r}\,\mathring{s} = (-\mathbf{r} \cdot \mathbf{s}, \mathbf{r} \times \mathbf{s})$$

$$(\mathring{r}\,\mathring{s}) \cdot \mathring{t} = \mathring{r} \cdot (\mathring{s}\,\mathring{t}) = [\mathbf{r}\ \mathbf{s}\ \mathbf{t}]$$

$$\mathring{r}\,\mathring{r} = -(\mathbf{r} \cdot \mathbf{r})\,\mathring{e}$$

Representation of Rotation using Unit Quaternions

Representing Scalars: $(s, 0)$

Representing Vectors: $(0, \mathbf{v})$

Quaternion operation that maps from vectors to vectors

$$\mathring{r}' = \mathring{q}\,\mathring{r}\,\mathring{q}^*$$

$$\mathring{q}\,\mathring{r}\,\mathring{q}^* = (Q\mathring{r})\mathring{q}^* = (\overline{Q}^T Q)\mathring{r}$$

$$\begin{pmatrix} \mathring{q}\cdot\mathring{q} & 0 & 0 & 0 \\ 0 & (q_0^2 + q_x^2 - q_y^2 - q_z^2) & 2(q_x q_y - q_0 q_z) & 2(q_x q_z + q_0 q_y) \\ 0 & 2(q_y q_x + q_0 q_z) & (q_0^2 - q_x^2 + q_y^2 - q_z^2) & 2(q_y q_z - q_0 q_x) \\ 0 & 2(q_z q_x - q_0 q_y) & 2(q_z q_y + q_0 q_z) & (q_0^2 - q_x^2 - q_y^2 + q_z^2) \end{pmatrix}$$

If $\mathring{q}$ is a unit quaternion, the lower right 3×3 submatrix is orthonormal

Properties of the Mapping

$\mathring{r}' = \mathring{q}\,\mathring{r}\,\mathring{q}^*$

Scalar part: $r' = r\,(\mathring{q} \cdot \mathring{q})$

Vector part: $\mathbf{r}' = (q^2 - \mathbf{q} \cdot \mathbf{q})\,\mathbf{r} + 2(\mathbf{q} \cdot \mathbf{r})\,\mathbf{q} + 2q(\mathbf{q} \times \mathbf{r})$

Preserves dot-products: $\mathring{r}' \cdot \mathring{s}' = \mathring{r} \cdot \mathring{s} \;\rightarrow\; \mathbf{r}' \cdot \mathbf{s}' = \mathbf{r} \cdot \mathbf{s}$

Preserves triple products: $(\mathring{r}' \cdot \mathring{s}') \cdot \mathring{t}' = (\mathring{r} \cdot \mathring{s}) \cdot \mathring{t} \;\rightarrow\; [\mathbf{r}'\ \mathbf{s}'\ \mathbf{t}'] = [\mathbf{r}\ \mathbf{s}\ \mathbf{t}]$

Composition: $\mathring{p}\,(\mathring{q}\,\mathring{r}\,\mathring{q}^*)\,\mathring{p}^* = (\mathring{p}\,\mathring{q})\,\mathring{r}\,(\mathring{q}^*\,\mathring{p}^*) = (\mathring{p}\,\mathring{q})\,\mathring{r}\,(\mathring{p}\,\mathring{q})^*$

Relation with Rodrígues formula

$$\mathbf{r}' = (q^2 - \mathbf{q} \cdot \mathbf{q})\,\mathbf{r} + 2(\mathbf{q} \cdot \mathbf{r})\,\mathbf{q} + 2q(\mathbf{q} \times \mathbf{r})$$

$$\mathbf{r}' = \cos\theta\,\mathbf{r} + (1 - \cos\theta)\,(\widehat{\boldsymbol{\omega}} \cdot \mathbf{r})\,\widehat{\boldsymbol{\omega}} + \sin\theta\,(\widehat{\boldsymbol{\omega}} \times \mathbf{r})$$

$\mathbf{q}$ parallel to $\widehat{\boldsymbol{\omega}}$

$(q^2 - \mathbf{q} \cdot \mathbf{q}) = \cos\theta,\ \ 2\|\mathbf{q}\|^2 = (1 - \cos\theta),$ and $2q\|\mathbf{q}\| = \sin\theta.$

$q = \cos(\theta/2),\ \|\mathbf{q}\| = \sin(\theta/2)$

Conclusion: $\mathring{q} = \big(\cos(\theta/2),\, \widehat{\boldsymbol{\omega}}\,\sin(\theta/2)\big)$

Note that $-\mathring{q}$ represents the same mapping as $\mathring{q}$ —

since $(-\mathring{q})\,\mathring{r}\,(-\mathring{q})^* = \mathring{q}\,\mathring{r}\,\mathring{q}^*$

Photogrammetry

Absolute orientation (3D to 3D) — range data

Relative orientation (2D to 2D) — binocular stereo

Exterior orientation (3D to 2D) — passive navigation

Interior orientation (2D to 3D) — camera calibration

Absolute Orientation

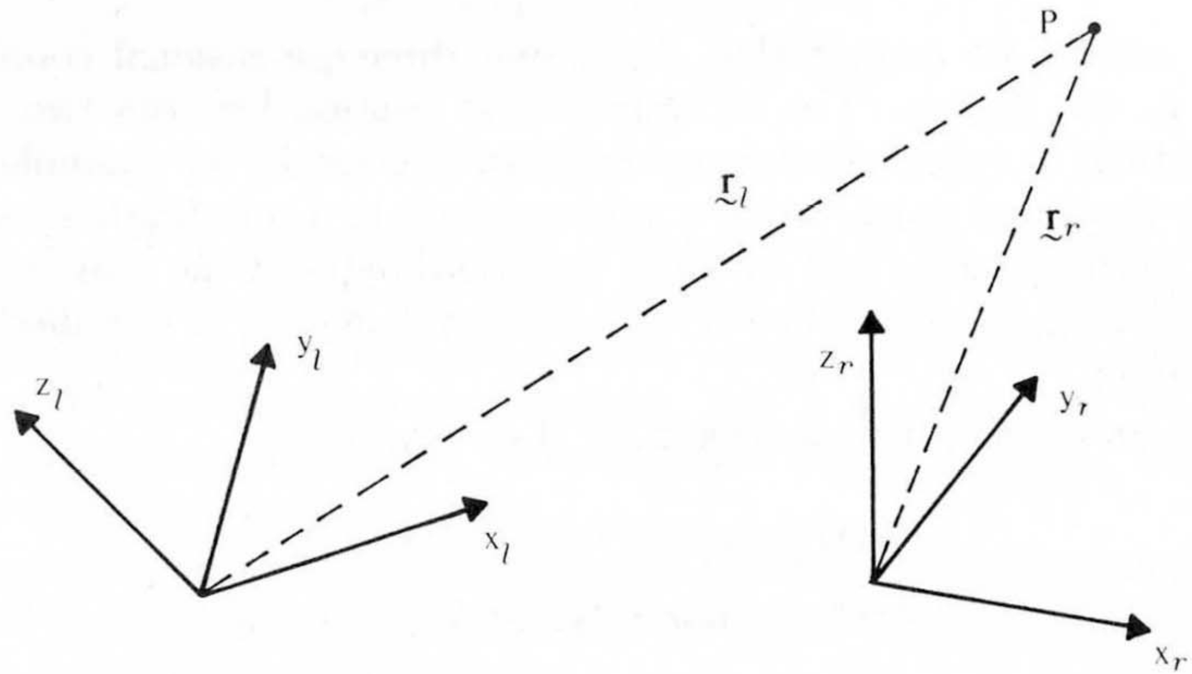


Figure 13-3. The transformation from one camera station to another can be represented by a rotation and a translation. The relation between the coordinates, $\mathbf{r}_l$ and $\mathbf{r}_r$, of a point P can be given by means of a rotation matrix and an offset vector.

Absolute Orientation

Given corresponding coordinates measured in two coordinate systems
— determine transformation between coordinate systems

OR

Given corresponding coordinates before and after motion
— determine rotation, translation.

Model $\mathbf{r}_r = R(\mathbf{r}_l) + \mathbf{r}_0$

Find (best-fit) rotation $R(\ldots)$ and translation $\mathbf{r}_0$.

given sets of corresponding $\{\mathbf{r}_{l_i}\}$ and $\{\mathbf{r}_{r_i}\}$

Least Squares Approach

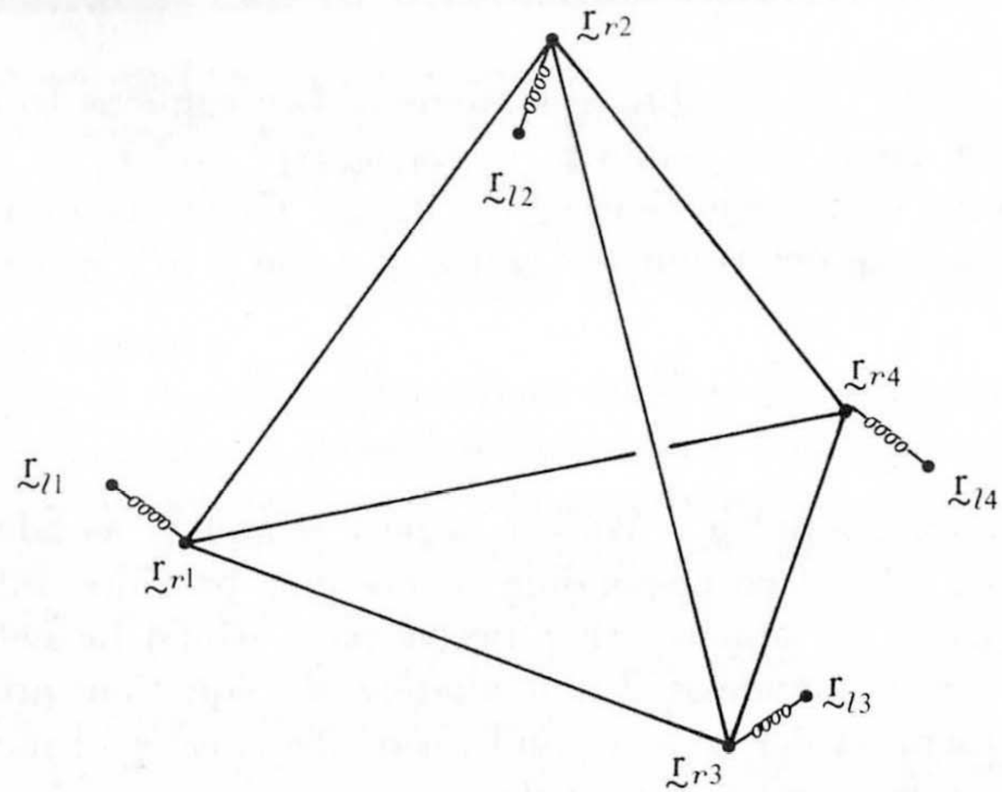


Figure 13-5. The least-squares problem can be modeled by a mechanical system in which corresponding points in the two coordinate systems are attached to each other by means of springs. The solution to the least-squares problem corresponds to the equilibrium position of the system, which minimizes the energy stored in the springs.

Finding the Translation

Minimize

$$\sum_{i=1}^{n} \left\| \mathbf{r}_{r_i} - (R(\mathbf{r}_{l_i}) + \mathbf{r}_0) \right\|^2$$

Set derivative w.r.t $\mathbf{r}_0$ equal to zero.

$$2 \sum_{i=1}^{n} \left(\mathbf{r}_{r_i} - (R(\mathbf{r}_{l_i}) + \mathbf{r}_0) \right) = 0$$

$$\sum_{i=1}^{n} \mathbf{r}_{r_i} - R \left(\sum_{i=1}^{n} \mathbf{r}_{l_i} \right) = n\, \mathbf{r}_0$$

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{r}_{r_i} - R \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{r}_{l_i} \right) = \mathbf{r}_0$$

Solution for translation: $\mathbf{r}_0 = \overline{\mathbf{r}_r} - R\,(\overline{\mathbf{r}_l})$

Finding the Rotation

Minimize

$$\sum_{i=1}^{n} ||\mathbf{r}'_{r_i} - R(\mathbf{r}'_{l_i})||^2$$

where $\mathbf{r}'_{r_i} = \mathbf{r}_{r_i} - \overline{\mathbf{r}_r}$ and $\mathbf{r}'_{l_i} = \mathbf{r}_{l_i} - \overline{\mathbf{r}_l}$

$$\sum_{i=1}^{n} ||\mathbf{r}'_{r_i}||^2 - 2\sum_{i=1}^{n} \mathbf{r}'_{r_i} \cdot R(\mathbf{r}'_{l_i}) + \sum_{i=1}^{n} ||\mathbf{r}'_{l_i}||^2$$

Maximize

$$\sum_{i=1}^{n} \mathbf{r}'_{r_i} \cdot R(\mathbf{r}'_{l_i})$$

“Differentiate w.r.t. $R(\ldots)$” ???

Finding the Best-Fit Rotation

Maximize

$$\sum_{i=1}^{n} \mathbf{r}'_{r_i} \cdot R(\mathbf{r}'_{l_i})$$

$$\sum_{i=1}^{n} (\mathring{q}\mathring{r}'_{l_i}\mathring{q}^*) \cdot \mathring{r}'_{r_i}$$

$$\sum_{i=1}^{n} (\mathring{q}\mathring{r}'_{l_i}) \cdot (\mathring{r}'_{r_i}\mathring{q})$$

$$\sum_{i=1}^{n} (\overline{R_{l_i}}\mathring{q}) \cdot (R_{r_i}\mathring{q})$$

$$\mathring{q}^T \left(\sum_{i=1}^{n} \overline{R_{l_i}}^T R_{r_i} \right) \mathring{q}$$

subject to $\mathring{q} \cdot \mathring{q} = 1$

Best-Fit Rotation

Maximize $\mathring{q}^T N \mathring{q}$ subject to $\mathring{q} \cdot \mathring{q} = 1$

Where

$$N = \sum_{i=1}^{n} \overline{R_{l_i}}^T R_{r_i}$$

Use Lagrange Multiplier to incorporate constraint

Maximize $\mathring{q}^T N \mathring{q} + \lambda (1 - \mathring{q} \cdot \mathring{q})$

Differentiate w.r.t. $\mathring{q}$ (!) — Set result equal to zero:

$$2N \mathring{q} - 2\lambda \mathring{q} = 0$$

Eigenvector corresponding to largest eigenvalue of a

4×4 real symmetric matrix N constructed from elements of

$$M = \sum_{i=1}^{n} \mathbf{r}'_{l_i} \mathbf{r}'_{r_i}{}^T$$

an asymmetric 3×3 real matrix.

Characteristic equation

$$\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0$$

Simplifies, since:

$$c_3 = \text{Trace}(N) = 0$$

$$c_2 = -2\,\text{Trace}(M^T M)$$

$$c_1 = -8\,\det(M)$$

$$c_0 = \det(N)$$

Other applications

- Relative Orientation,
- Camera Calibration
- Manipulator Kinematics
- Manipulator Fingerprinting
- Spacecraft Dynamics

Desirable Properties

- Ability to rotate vectors — or coordinate system
- Ability to compose rotations
- Intuitive, non-redundant representation
- Computational efficiency

- Interpolate orientations
- Averages over range of rotations
- Derivative w.r.t rotation — optimization, LSQ
- Sampling of rotations — uniform and random

- Notion of a space of rotations

Relative Orientation — binocular stereo

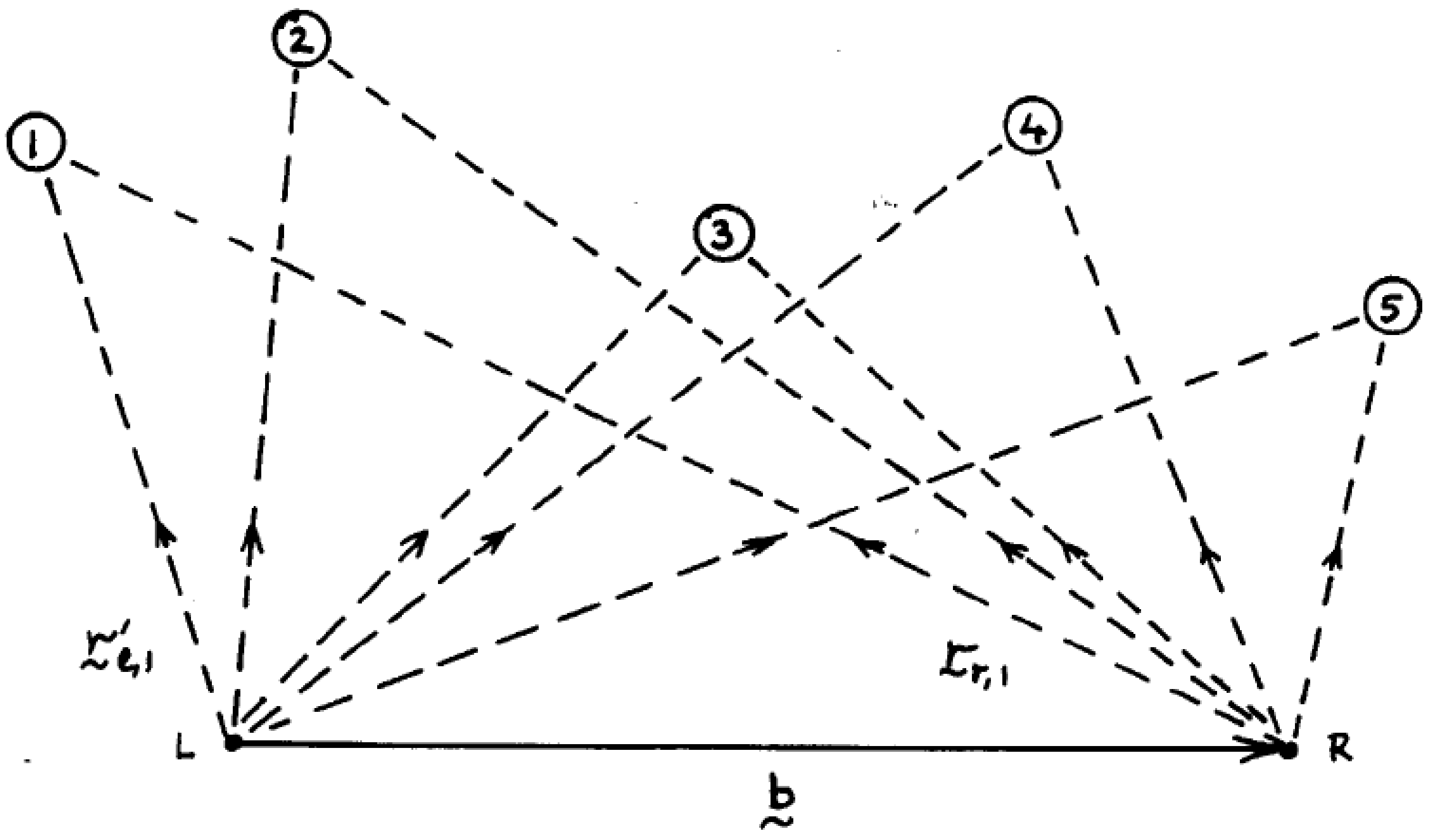
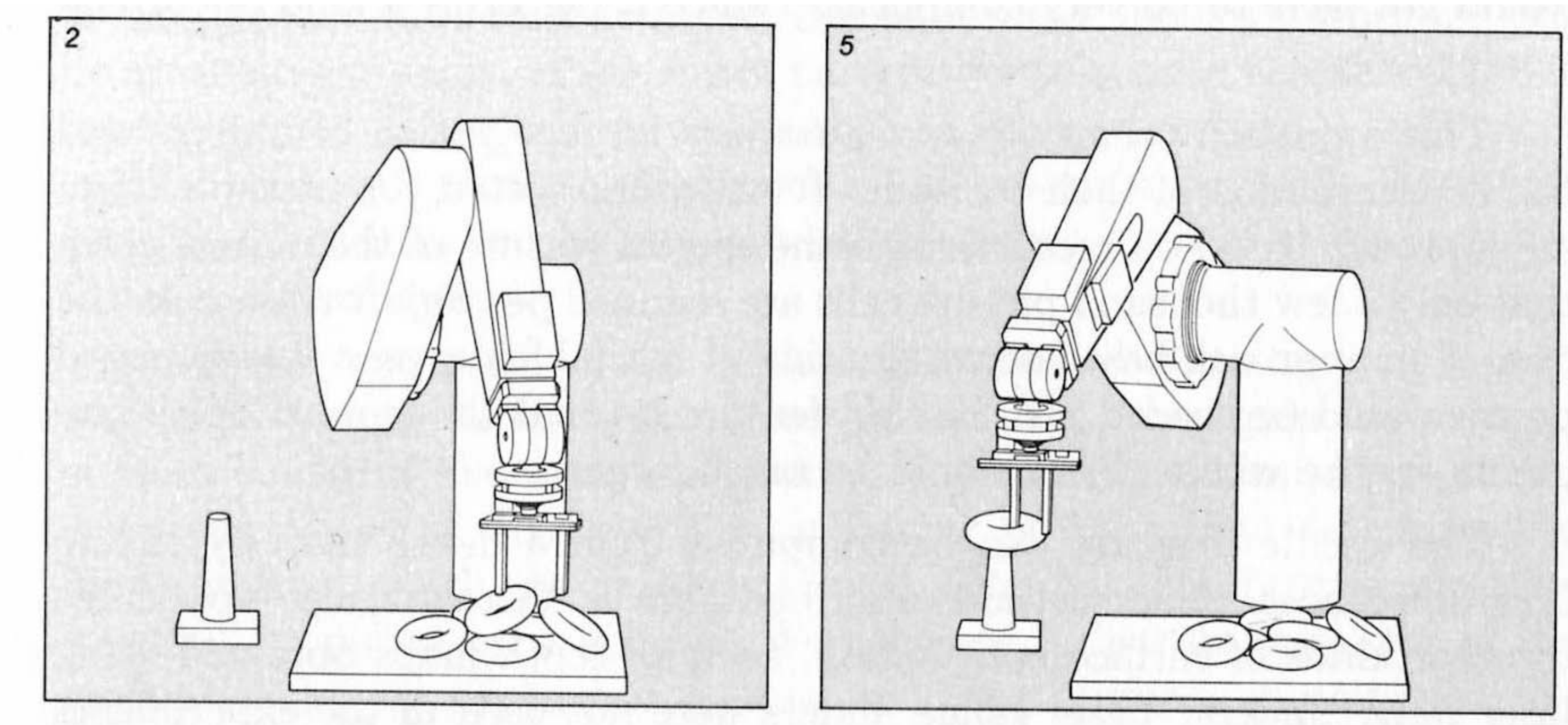


Figure 1. Points in the environment are viewed from two camera positions. The relative orientation is the direction of the baseline b, and the rotation R relating the left and right coordinate systems. The directions of rays to at least five scene points must be known in both camera coordinate systems.

Manipulator kinematic equations



Computational Issues

Composition: $\mathring{p}\,\mathring{q}$

$$(p, \mathbf{p})(q, \mathbf{q}) = (pq - \mathbf{p} \cdot \mathbf{q}, p\mathbf{q} + q\mathbf{p} + \mathbf{p} \times \mathbf{q})$$

16 $\times$s and 12 $+$s

Compare: 3×3 matrix product 27 $\times$s and 18 $+$s

Rotating vector: $\mathring{q}\,\mathring{r}\,\mathring{q}^*$

$$\mathbf{r}' = (q^2 - \mathbf{q} \cdot \mathbf{q})\,\mathbf{r} + 2\,(\mathbf{q} \cdot \mathbf{r})\,\mathbf{q} + 2\,q\,(\mathbf{q} \times \mathbf{r})$$

22 $\times$s and 16 $+$s

$$\mathbf{r}' = \mathbf{r} + 2\,q\,(\mathbf{q} \times \mathbf{r}) + 2\,\mathbf{q} \times (\mathbf{q} \times \mathbf{r})$$

15 $\times$s and 12 $+$s

Compare: 3×3 matrix-vector product 9 $\times$s and 6 $+$s

Renormalizing

Nearest unit quaternion — $\mathring{q}/\sqrt{\mathring{q}\cdot\mathring{q}}$

Nearest orthonormal matrix — $M(M^TM)^{-1/2}$

Space of rotations

Space of rotations — S^3 with antipodal points identified

Space of rotations — projective space P^3

Sampling: regular and random

Finite rotation groups — Platonic solids 12, 24, 60 elements

Getting finer sampling — sub-dividing the simplex

If $\{\mathring{q}_i\}$ is a group, so is $\{\mathring{q}'_i\}$, where $\mathring{q}'_i = \mathring{q}_0 \mathring{q}_i$

Finite Rotation Groups

$a = (\sqrt{5} - 1)/4$, $b = 1/2$, $c = 1/\sqrt{2}$, $d = (\sqrt{5} + 1)/4$

Rotation group of the tetrahedron

$$
\begin{array}{llll}
(1, 0, 0, 0) & (0, 1, 0, 0) & (0, 0, 1, 0) & (0, 0, 0, 1) \\
(b, b, b, b) & (b, b, b, -b) & (b, b, -b, b) & (b, b, -b, -b) \\
(b, -b, b, b) & (b, -b, b, -b) & (b, -b, -b, b) & (b, -b, -b, -b)
\end{array}
$$

Rotation group of the hexahedron / octahedron

$$
\begin{array}{llll}
(1, 0, 0, 0) & (0, 1, 0, 0) & (0, 0, 1, 0) & (0, 0, 0, 1) \\
(0, 0, c, c) & (0, 0, c, -c) & (0, c, 0, c) & (0, c, 0, -c) \\
(0, c, c, 0) & (0, c, -c, 0) & (c, 0, 0, c) & (c, 0, 0, -c) \\
(c, 0, c, 0) & (c, 0, -c, 0) & (c, c, 0, 0) & (c, -c, 0, 0) \\
(b, b, b, b) & (b, b, b, -b) & (b, b, -b, b) & (b, b, -b, -b) \\
(b, -b, b, b) & (b, -b, b, -b) & (b, -b, -b, b) & (b, -b, -b, -b)
\end{array}
$$

Finite Rotation Groups

$a = (\sqrt{5} - 1)/4$, $b = 1/2$, $c = 1/\sqrt{2}$, $d = (\sqrt{5} + 1)/4$

Rotation group of the dodecahderon / icosahedron

(1,	0,	0,	0)	(0,	1,	0,	0)	(0,	0,	1,	0)	(0,	0,	0,	1)
(0,	a,	b,	d)	(0,	a,	b,	$-d$)	(0,	a,	$-b$,	d)	(0,	a,	$-b$,	$-d$)
(0,	b,	d,	a)	(0,	b,	d,	$-a$)	(0,	b,	$-d$,	a)	(0,	b,	$-d$,	$-a$)
(0,	d,	b,	a)	(0,	d,	a,	$-b$)	(0,	d,	$-a$,	b)	(0,	d,	$-a$,	$-b$)
(a,	0,	d,	b)	(a,	0,	d,	$-b$)	(a,	0,	$-d$,	b)	(a,	0,	$-d$,	$-b$)
(b,	0,	a,	d)	(b,	0,	a,	$-d$)	(b,	0,	$-a$,	d)	(b,	0,	$-a$,	$-d$)
(d,	0,	b,	a)	(d,	0,	b,	$-a$)	(d,	0,	$-b$,	a)	(d,	0,	$-b$,	$-a$)
(a,	b,	0,	d)	(a,	b,	0,	$-d$)	(a,	$-b$,	0,	d)	(a,	$-b$,	0,	$-d$)
(b,	d,	0,	a)	(b,	d,	0,	$-a$)	(b,	$-d$,	0,	a)	(b,	$-d$,	0,	$-a$)
(d,	a,	0,	b)	(d,	a,	0,	$-b$)	(d,	$-a$,	0,	b)	(d,	$-a$,	0,	$-b$)
(a,	d,	b,	0)	(a,	d,	$-b$,	0)	(a,	$-d$,	b,	0)	(a,	$-d$,	$-b$,	0)
(b,	a,	d,	0)	(b,	a,	$-d$,	0)	(b,	$-a$,	d,	0)	(b,	$-a$,	$-d$,	0)
(d,	b,	a,	0)	(d,	b,	$-a$,	0)	(d,	$-b$,	a,	0)	(d,	$-b$,	$-a$,	0)
(b,	b,	b,	b)	(b,	b,	b,	$-b$)	(b,	b,	$-b$,	b)	(b,	b,	$-b$,	$-b$)
(b,	$-b$,	b,	b)	(b,	$-b$,	b,	$-b$)	(b,	$-b$,	$-b$,	b)	(b,	$-b$,	$-b$,	$-b$)

Division algebras

Recursive definition of multiplication and conjugation

Conjugation: $(a, b)^* = (a^*, -b)$

Multiplication: $(a, b)(c, d) = (a\,c - d\,b^*, a^*\,d + c\,b)$

Real, Complex, Quaternion, Octonion, Sedenions, ...

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ ...

Cayley numbers order n have 2^n components

Loss of properties:

Ordering, Commutativity, Associativity, Lack of zero divisors...

Hermaphrodite Monster

Tait's view of:

Gibbs' vector analysis — dot-product and cross-product

Hamilton's quaternions and Grassman algebra of extensions

Esoterica

Existence of Division Algebras

There exists (n) pointwise linearly independent smooth vector fields for the n-sphere only when $(n+1) = 1, 2, 4, 8$ — this can be shown to imply that division algebras over the reals can only occur for these dimensions.

Quaternion *analysis* (alá complex analysis)

Application to space-time (Relativity)

“Time is said to have only one dimension, and space to have three dimensions. ... The mathematical quaternion partakes of both these elements; in technical language it may be said to be ‘time plus space’...”

Clifford algebras

Bott periodicity

Evangelists

www.quaternion.com

Space-time: $(i\,t, \mathbf{r}) = (i\,t, x, y, z)$

String Theory

Octonions explain some curious features of string theory:

Lagrangian for classical superstring involves relationships between vectors and spinors in Minkowski spacetime which holds only in 3, 4, 6, and 10 dimensions (i.e. 2 more than the dimensions of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ ...)

Can treat spinor as pair of elements of corresponding division algebra

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Q: How many dimensions does superstring theory postulate?

A: More than the villainous inconstancy of man's disposition is able to bear (Shakespeare).

Parting thoughts

> The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on.
>
> The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete.
>
> The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings.
>
> But the octonions are the crazy old uncle nobody lets out of the attic: they are nonassociative.

From: “The Octonions” by John C. Baez
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