6.431 Probabilistic Systems Analysis DUE: 9/17/2003

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Problem Set 2

Problem 1

(a)

Define the events:

$B = \{\text{Billie telling the truth}\}$
$D = \{\text{Derek telling the truth}\}$.

So,

$P(D) = P(B)P(D|B) + P(B^C)P(D|B^C) = \frac{1}{4}0 + \frac{3}{4}1 = \frac{3}{4}$.

(b)

Define the event:

$T = \{\text{Trisha telling the truth}\}$.

So,

$P(T) = P(D)P(T|D) + P(D^C)P(T|D^C) = \frac{3}{4}0 + \frac{1}{4}1 = \frac{1}{4}$.

Problem 2

(i)

Since each entry in the table represents the joint probability, we need to ensure that the sum of all of the entries in the table equals 1.

(ii)

$P(S|W \cap N) = \frac{a}{a+b}$
$P(S|W \cap N^C) = \frac{c}{c+d}$
$P(S|W^C \cap N) = \frac{e}{e+f}$
$P(S|W^C \cap N^C) = \frac{g}{g+h}$
$P(S|N) = \frac{a+e}{a+b+e+f}$
$P(S|N^C) = \frac{c+g}{c+d+g+h}$

(iii)

Check to make sure that all of the entries sum to one:

$\frac{3}{30} + \frac{1}{30} + \frac{8}{30} + \frac{3}{30} + \frac{3}{30} + \frac{8}{30} + \frac{1}{30} + \frac{3}{30} = 1.$

Now, compute the probabilities in part **(ii)**:

$P(S|W \cap N) = \frac{3}{4}$
$P(S|W \cap N^C) = \frac{8}{11}$
$P(S|W^C \cap N) = \frac{3}{11}$
$P(S|W^C \cap N^C) = \frac{1}{4}$
$P(S|N) = \frac{2}{5}$
$P(S|N^C) = \frac{3}{5}$

(iv) Let us take the ratio using only what we computed above for $P(S|N)$ and $P(S|N^C)$:

$\frac{P(S|N)}{P(S|N^C)} = \frac{2}{3}$.

So, we would recommend the old drug in this case.

(v)

Let us take the ratio using only what we computed above for $P(S|W \cap N)$ and $P(S|W \cap N^C)$:

$\frac{P(S|W \cap N)}{P(S|W \cap N^C)} = \frac{33}{32}$.

So, we would recommend the new drug in this case.

(vi)

Let us take the ratio using only what we computed above for $P(S|W^C \cap N)$ and $P(S|W^C \cap N^C)$:

$\frac{P(S|W^C \cap N)}{P(S|W^C \cap N^C)} = \frac{4}{11}$.

So, we would recommend the old drug in this case.

(vii)

Problem 3

Problem 4

(a)

(i) Independent. $\frac{1}{4} = P(A \cap C) = P(A)P(C) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. **(ii)** Independent. $\frac{1}{4} = P(A \cap D) = P(A)P(D) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. **(iii)** Independent. $\frac{1}{4} = P(C \cap D) = P(C)P(D) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. **(iv)** Not independent. $0 = P(A \cap B) \neq P(A)P(B) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$.

(b)

A, C, D are NOT mutually independent. $0 = P(A \cap C \cap D) \neq P(A)P(C)P(D) = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$.

(c)

(i) Let $E = A, F = C, G = D$. From part **(a)**, we showed that A and C are independent. However, $0 = P(A \cap C|D) \neq P(A|D)P(C|D) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. See Figure 1.

(ii) Let $J = A, K = B, L = A$. So, $0 = P(A \cap B|A) = P(A|A)P(B|A) = 1 \cdot 0 = 0$. However, $0 = P(A \cap B) \neq P(A)P(B) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. See Figure 2.

(d)

The statement is true. It is simply the definition of the conditional probability where the original probability is a conditional probability. In other words, suppose we think of our new sample space as being Q instead of Ω. Then, we obtain the same equation for conditional probability, except we operate on Q. For example, let $V = B, W = C, Q = D$. Then $\frac{1}{2} = P(B \cap C|D) = P(B|C \cap D)P(C|D) = 1\frac{1}{2} = \frac{1}{2}$. See Figure 3.

Problem G1

Problem G2

(a)

We know that $P(A \cap B) = P(A)P(B)$. So,

$P((A \cap B) \cup (A^C \cap B)) = P(A \cap B) + P(A^C \cap B) = P(A)P(B) + P(A^C \cap B) = P(B)$

where we have used the fact that $A \cap B$ and $A^C \cap B$ are disjoint. The above is equivalent to:

$P(A^C \cap B) = P(B) - P(A)P(B) = P(B)(1 - P(A)) = P(B)P(A^C)$.

So A^C and B are independent. If we flip the roles of A and B in the above proof, then we show that A and B^C are independent. By using these two newly proven facts and substituting B^C in for B above, we can show that A^C and B^C are independent.

(b)

$P(A \cup B^C) = P(A) + P(B^C) - P(A \cap B^C) = P(A) + P(B^C) - P(A)P(B^C) = \frac{6}{10} + \frac{7}{10} - \frac{6}{10}\frac{7}{10} = \frac{22}{25}$

(c)

Here, we assume that A, B, C are mutually independent. So,

$P((A \cap (B \cup C)) \cup (A \cap (B \cup C)^C)) = P(A \cap (B \cup C)) + P(A \cap (B \cup C)^C) = P(A \cap (B \cup C)) + P(A \cap (B^C \cap C^C)) = P(A \cap (B \cup C)) + P(A)P(B^C)P(C^C) = P(A)$

where we have used the fact that $(A \cap (B \cup C))$ and $(A \cap (B \cup C)^C)$ are disjoint. The above is equivalent to:

$P(A \cap (B \cup C)) = P(A) - P(A)P(B^C)P(C^C) = P(A)(1 - P(B^C)P(C^C)) = P(A)P((B^C \cap C^C)^C) = P(A)P(B \cup C)$.

So A and $B \cup C$ are independent.