6.431 Probabilistic Systems Analysis DUE: 9/14/2003

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Problem Set 3

Problem 1

First, we see that the total possible ways Billie can buy b lottery tickets out of t is $\binom{t}{b}$. Next, the total possible ways Billie can purchase g winning lottery tickets out of w winners is $\binom{w}{g}$. Similarly, the total possible ways Billie can purchase the losing lottery tickets is $\binom{t-w}{b-g}$. So, the total probability that he bought exactly g winning tickets is

$\frac{\binom{w}{g}\binom{t-w}{b-g}}{\binom{t}{b}}$.

Problem 2

We begin by noticing that we have 20 slots to fill with 5 visits to each of 4 cities. We then fix the first city that the salesman travels to. We can do this in four different ways since there are four cities. We are restricted to the next 18 slots since we cannot have the last slot contain the same city as the first slot. So we can schedule the remaining 4 visits to the chosen city in $\binom{18}{4}$ ways. For each of these assignments, we have 15 slots open for the remaining cities. We can schedule the 5 visits for a given city in these remaining slots in $\binom{15}{5}$ ways. Similarly, for the next city we can schedule the 5 visits in the remaining slots in $\binom{10}{5}$ ways. The last city's visits are then inserted into the remaining slots. So, we can schedule the visits to all of the cities in the following number of ways:

$4\binom{18}{4}\binom{15}{5}\binom{10}{5}$.

Problem 3

(a) We can think of this problem as having $n-1+k$ slots into which we will insert $n-1$ barriers and k tokens. We can do this in the following number of ways:

$\binom{n-1+k}{k}$.

(b) If we define how many tokens are in each box, we are defining the selection. Since all selections are equally probable, the probability of finding t_1 tokens in box 1, t_2 in box 2, etc. is:

$\frac{1}{\binom{n-1+k}{k}}$.

(c) We can think of this problem as fixing the number of tokens in the first box and finding all of the possible ways of distributing the remaining $k-l$ tokens in the last $n-1$ boxes. Using similar reasoning as in **(a)**, we can do this in $\binom{n-2+k-l}{k-l}$ ways. So,

$P(X_1 = l) = \frac{\binom{n-2+k-l}{k-l}}{\binom{n-1+k}{k}}$.

(d)

Problem 4

(a) We wish to compute the probability of the 5th and 7th coin flip being tails given that exactly five heads appeared in the first 11 flips. If we fix the 5th and 7th flips as tails, then the possible ways of distributing the five heads among the remaining flips is $\binom{9}{5}$. So,

$P(\{\text{5th and 7th are T}\} \mid \{\text{exactly 5 H in 1st 11 flips}\}) = \binom{9}{5}p^5(1-p)^6$.

(b) Since the coin flips are independent and identically distributed, we can think of this as a tree and multiply the probabilities along a path of the tree. So,

$P_Y(y) = p^{y-1}(1-p)$.

(c)

(i) This is taking the expectation of a Bernoulli trial. So, $E[X] = 1 \cdot (1-p) + 0 \cdot p = 1-p$.

(ii) $var(X) = E[X^2] - E[X]^2 = 1^2 \cdot (1-p) + 0^2 \cdot p - (1-p)^2 = (1-p) - (1-p)^2$.

(d)

(i) This is a binomial distribution: $p_K(k) = \binom{n}{k}p^{n-k}(1-p)^k$.

(ii) We can think of K as being the sum of n independent Bernoulli trials. So, $E[K] = E[K_1] + \cdots + E[K_n] = n(1-p)$.

(iii) $var(K) = var(K_1) + \cdots + var(K_n) = n(1-p) - n(1-p)^2$.

Problem 5

(a) We expect $E[X]$ to be larger since we are giving more weight to the larger values that X can take on. $E[Y]$ simply takes the average of the possible values of Y.

(b) $E[X] = 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} = 39.3$. $E[Y] = 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = 37$.

Problem G1

Let $l_1, \ldots, l_5$ be the dates for the lab. We can think of this like we are trying to choose five integers in a range. Let us ignore the first constraint (allowing six days after a lab date) and incorporate the last constraint (allowing 10 days after the final lab date to write the report). We can consider this problem of choosing five integers such that $1 \leq l_1 < l_2 < \ldots l_5 \leq 95$. We can do this in $\binom{95}{5}$ ways. Now we need to take into account the constraints.