6.853 Topics in Algorithmic Game Theory	November 3, 2011

Lecture 16

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NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

1 Review

We start with a review of some elements from last lecture. Let us consider a marketplace where the excess demand on goods is a well-defined vector-valued function $f(p)$ of the prices p. This happens, e.g., when traders' utility functions are strictly concave, so that they have unique utility-optimizing bundles. In this case, the excess demand function is the difference of the total demand and the total supply on each good:

$$f(p) := \sum_i x_i(p) - \sum_i e_i,$$

where $x_i(p)$ denotes the bundle that optimizes trader i's utility under price vector p. We have argued that $f(p)$ always satisfies homogeneity:

- (H): $f(\cdot)$ is *homogeneous*, i.e. $f(ap) = f(p), \forall a > 0$.

Moreover, we have argued that under very weak assumptions (namely non-satiation and quasi-concavity) it also satisfies *Walras Law* (WL):

- (WL): f satisfies *Walras's Law*, i.e. $p^T f(p) = 0, \forall p \in \mathbb{R}^k_{\geq 0}$.

Finally, we defined the property (Pos), which requires that not all goods are priced 0 at an equilibrium. I.e. if $p = 0$ then there is positive excess demand for at least one good. This is satisfied whenever there exists at least one good that somebody likes.

1.1 Gross-Substitutability

The *Gross-Substitutability condition* tries to capture a situation in which one good can be used to some degree as a substitute for another good. Put more concretely, gross-substitutability means that if the price of some goods is increased while the price of some other goods is held constant, this should increase the excess demand on the latter goods. Mathematically:

Definition 1. *The excess demand function satisfies gross-substitutability (GS), if and only if for all pairs of prices (p, p'):*

$$(GS) : \begin{pmatrix} p_i \leq p'_i, \forall i \\ p_j < p'_j, \text{for some } j \end{pmatrix} \Rightarrow f_k(p) < f_k(p'), \forall k \text{ s.t. } p_k = p'_k.$$

A stronger property is the *differential form of gross-substitutability (GS_D)*, defined as:

Definition 2. *The excess demand function satisfies the differential form of gross-substitutability if and only if $\forall r, s$, the partial derivatives $\frac{\partial f_r}{\partial p_s}$ exist, and are continuous, and the following condition holds:*

$$(GS_D) : \frac{\partial f_r}{\partial p_s} > 0, \textit{ for all } r \neq s.$$

It is immediately clear that $(GS_D) \Rightarrow (GS)$.

1.2 Properties of Price Equilibrium under Gross Substitutability

Using the properties defined up to this point, we have the following lemmas due to Arrow, Block, and Hurwicz [1]:

Lemma 1. *Suppose that the excess demand function of an exchange economy satisfies (H), (GS_D), and (Pos). Then if $\overline{p}$ is a price equilibrium,*

$$(E_+) : \overline{p}_r > 0, \forall r$$

Lemma 2. *Suppose that the excess demand function of an exchange economy satisfies (H), (GS), and (E_+). Then if $\overline{p}$ and $\overline{p}'$ are equilibrium price vectors, there exists some $\lambda > 0$ such that*

$$\overline{p}_r = \lambda \overline{p}'.$$

Put another way, Lemma 2 states that there is a unique equilibrium ray.

1.3 Weak Axiom of Revealed Preferences (WARP)

Theorem 1 ([1]). *Suppose that the excess demand function of an exchange economy satisfies (H), (WL), and (GS). If $\overline{p} > 0$ is any equilibrium price vector and $p > 0$ is any non-equilibrium vector we have*

$$\overline{p}^{\mathrm{T}} \cdot f(p) > 0.$$

2 Complexity of Exchange Markets under Gross-Substitutability

We argue that both centralized and distributed computation of price equilibrium in GS markets is easy. This follows from the following corollaries of WARP.

Corollary 1 (of WARP). *If the excess demand function satisfies (H), (WL),and (GS), it can be computed efficiently, and is c-Lipschitz, then a non-zero equilibrium price vector (if it exists) can be computed efficiently. The dependence on c and the accuracy of computation ϵ is* $\mathrm{poly}(\log(c/\epsilon))$.

Proof: (Sketch). Without loss of generality, we can restrict our search space to price vectors in $[0,1]^k$, since any equilibrium can be rescaled to lie in this set (by homogeneity of the excess demand function). We can then run ellipsoid, using the separation oracle provided by the weak axiom of revealed preferences. In particular, for any non-equilibrium price vector p, we know that the price equilibrium lies in the half-space

$$S = \{x \mid x^{\mathrm{T}} \cdot f(p) > 0\}.$$

$\square$

Corollary 2. *If the excess demand function satisfies continuity, (H), (WL), (GS_D), and (Pos), then the tatonnement process (price-adjustment mechanism) described by the following differential equation converges to a market equilibrium.*

$$\frac{dp_j}{dt} = f_j(p), \forall j$$
$$p(t = 0) > 0.$$

Proof: Continuity assures that the differential equation has a solution. Moreover, because of the initial condition, it can be shown that the solution stays positive, and within the box $B = [\min p(0), \max p(0)]^k$.

By the assumptions, the price equilibria form a ray. Let $\bar{p}$ be an arbitrary equilibrium on the ray. To show convergence to a price equilibrium, we consider the following potential function:

$$V(p) = \frac{1}{2}\sum_{j=1}^{k}(p_j - \overline{p}_j)^2.$$

We have

$$\begin{aligned}
\frac{dV}{dt} &= \sum_{j=1}^{k}(p_j - \overline{p}_j) \cdot \frac{dp_j}{dt} \\
&= \sum_{j=1}^{k}(p_j - \overline{p}_j) \cdot f_j(p) \\
&= p^{\mathrm{T}} \cdot f(p) - \overline{p}^{\mathrm{T}} \cdot f(p) \\
&= -\overline{p}^{\mathrm{T}} \cdot f(p).
\end{aligned}$$

Observe that if, for some $t_0, p(t_0)$ is a price equilibrium vector, then by Lemma 1 and (WL), $f(p(t_0)) = 0$. Hence, $\overline{p}^{\mathrm{T}} f(p(t_0)) = 0$. This implies that $p(t) = p(t_0)$ for all $t \geq t_0$.

On the other hand, as long as $p(t)$ is not an equilibrium, WARP implies that

$$\frac{dV}{dt} = -\overline{p}^{T}.f(p(t)) < 0$$

implying that the L2 distance from $\overline{p}$ is monotonically decreasing. We show that in this case too $p(t)$ converges to a price equilibrium.

For a contradiction, suppose that $p(t)$ never comes closer than ϵ to the equilibrium ray. We will show that this implies that there is some δ such that for all t, $\overline{p}^{T} \cdot f(p(t)) > \delta$, yielding a contradiction, since then the absolute value of $\frac{dV}{dt}$ would remain bounded away from 0 for all t, but $V(t)$ cannot go below 0.

Let's show the above claim: Suppose $p(t)$ never comes closer than ϵ to the equilibrium ray, but the absolute value of $\frac{dV}{dt}$ comes arbitrarily close to 0. Then, for all k, there exists t_k such that $\overline{p}^{T} \cdot f(p(t_k)) < 2^{-k}$. So $\overline{p}^{T} \cdot f(p(t_k))$ converges to 0. By Property $(E)_+$ this means that $f(p(t_k))$ converges to 0. On the other hand, by compactness, $p_k := p(t_k)$ has a converging subsequence $p_{k'}$ converging to some $p^* \in B$, where p^* is at distance $\geq \epsilon$ from the equilibrium ray. Finally, since f is continuous, $f(p_{k'})$ converges to $f(p^*)$. The above imply that $f(p^*) = 0$ and p^* is in B and at distance $\geq \epsilon$ from the equilibrium ray. So p^* is a price equilibrium that is ϵ-far from the equilibrium ray, a contradiction. □

3 Exchange Model and the CES Utility Function

Recall our definition of CES utility functions. We say that trader i has a *CES utility function with exponent* ρ, if his utility for bundle $x_i \in \mathbb{R}^k$ is given by

$$u_i(x_i) = \left(\sum_j \alpha_{ij} x_{ij}^{\rho}\right)^{\frac{1}{\rho}}.$$

By convention, when $\rho < 0$, if $x_{ij} = 0$ for any j, we define $u_i(x_i)$ to equal 0. With this convention the function is continuous on $\mathbb{R}^k_{\geq 0}$. Moreover, it was an exercise in Lecture 14 to show that the function is concave whenever $-\infty < \rho \leq 1$.

Lemma 3. *Suppose $\rho < 1$. Under price vector p, trader i with a CES utility function as above and endowment vector e_i will have a unique feasible, utility optimizing bundle, defined as follows:*

$$x_{ij}(p) = \frac{\alpha_{ij}^{\frac{1}{1-\rho}}}{p_j^{\frac{1}{1-\rho}}} \cdot \frac{\sum_t p_t e_{it}}{\sum_t \alpha_{it}^{\frac{1}{1-\rho}} p_t^{-\frac{\rho}{1-\rho}}}. \tag{1}$$

Proof: Note that the utility maximizing, feasible bundle x_i under price vector p is this a solution to the following program:

$$\max u_i(x_i) \text{ subject to } \sum_j p_j x_{ij} \leq \sum_j p_j e_{ij}.$$

When $\rho < 1$, u_i is strictly concave. Hence the maximizer is unique. By writing the KKT conditions of the program we obtain Eq. (1). $\square$

3.1 Convex Program for the Case $\rho \in [-1, 0)$

Next, we provide a convex program for computing a price equilibrium, when $-1 \leq \rho < 0$. The price equilibrium is "just" a solution to the program

$$\text{find } p \text{ such that: } \sum_i x_{ij}(p) \leq \sum_i e_{ij}, \forall j, \tag{2}$$

where $x_{ij}(p)$ is as defined in (1).

We transform this program to a convex one. First, let us set $f_{ij}(p) = p_j^{\frac{1}{1-\rho}} x_{ij}(p)$, and make the change of variables $\sigma_j = p_j^{\frac{1}{1-\rho}}$. We can then rewrite (2) as

$$\text{find } p \text{ such that: } \sum_i p_j^{\frac{1}{1-\rho}} x_{ij}(p) \leq p_j^{\frac{1}{1-\rho}} \sum_i e_{ij}, \forall j;$$

or, equivalently:

$$\text{find } \sigma \text{ such that: } \sum_i f_{ij}(\sigma) \leq \sigma_j \sum_i e_{ij}, \forall j, \tag{3}$$

where $f_{ij}(\sigma) = \alpha_{ij}^{\frac{1}{1-\rho}} \cdot \dfrac{\sum_t \sigma_t^{1-\rho} e_{it}}{\sum_t \alpha_{it}^{\frac{1}{1-\rho}} \sigma_t^{-\rho}}$.

We claim that (3) is a convex program. Indeed, the function on the RHS of every constraint is linear. We claim that the function on the LHS of every constraint is a convex function as a sum of convex functions. This follows from:

Lemma 4. *For all i, j: $f_{ij}(\sigma) = \alpha_{ij}^{\frac{1}{1-\rho}} \cdot \dfrac{\sum_t \sigma_t^{1-\rho} e_{it}}{\sum_t \alpha_{it}^{\frac{1}{1-\rho}} \sigma_t^{-\rho}}$ is a convex function on $\mathbb{R}^k_{\geq 0}$.*

Proof: For an incorrect proof of this, see [2]. A correct proof is left as an exercise. $\square$

3.2 Convex Program for the Case $\rho = 1$

$\rho = 1$ corresponds to linear utility functions. Consider the following system of equations on x's and ψ's:

$$\forall i, j : \sum_t \alpha_{it} x_{it} \geq \alpha_{ij} \sum_t e_{it} \cdot \exp(\psi_t - \psi_j)$$

$$\forall j : \sum_i x_{ij} = \sum_i e_{ij}.$$

The above program is clearly convex as the function on the RHS of the above constraints is convex and the function on the LHS is linear. We claim that any feasible solution gives an equilibrium by setting $p_j = \exp(\psi_j)$, for all j. This is left as an exercise.

References

[1] K. Arrow, J. Block, and L. Hurwicz. On the Stability of Competitive Equilibrium, II. *Econometrica*, 27:82–109, 1959.

[2] B. Codenotti, B. McCune, S. Penumatcha and K. Varadarajan. Market equilibrium for CES exchange economies: Existence, multiplicity, and computation. *Proceedings of the Foundations of Software Technology and Theoretical Computer Science*, 2005.