6.853 Topics in Algorithmic Game Theory September 15, 2011

Lecture 3

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In the previous lecture we saw that there always exists a Nash equilibrium in two-player zero-sum games. Moreover, the equilibrium enjoys several attractive properties such as polynomial-time tractability, convexity of the equilibrium set, and uniqueness of players' payoffs in all equilibria. In this lecture we explore whether we can generalize this theorem to *multi-player zero-sum games*. Before answering the question, let us formally define multi-player zero-sum games and generalize the concept of a Nash equilibrium from last lecture to these games.

1 Definitions

We start with a formal definition of multiplayer games.

Definition 1. *A (finite)* multiplayer game *is specified by:*

- *the number of players n; we denote the set of players by $[n] = \{1, 2, \dots, n\}$;*
- *for each player $p \in [n]$:*
  - *a finite set of* pure strategies *S_p available to player p;*
  - *a utility function $u_p : \prod_{p\in[n]} S_p \to \mathbb{R}$, specifying the payoff to player p for each selection of pure strategies by the players of the game.*

We often summarize this information in a tuple $\langle n, (S_p)_{p\in[n]}, (u_p)_{p\in[n]}\rangle$.

Relative to a game specification, we introduce a few useful concepts/pieces of notation:

Definition 2. *Let $\langle n, (S_p)_{p\in[n]}, (u_p)_{p\in[n]}\rangle$ be a multiplayer game. Then*

- *the set of* mixed strategies *available to player p are all distributions over S_p, denoted*

$$\Delta^{S_p} = \left\{ \underset{\sim}{x_p} \in \mathbb{R}^{S_p}_{\geq 0} \;\middle|\; \sum_{s_p\in S_p} x_p(s_p) = 1 \right\};$$

- *an element of $S := \prod_{p\in[n]} S_p$ is called a* pure strategy profile*;*
- *an element of $\Delta := \prod_{p\in[n]} \Delta^{S_p}$ is called a* mixed strategy profile*;*
- *if $s \in S$, we denote by s_p the pure strategy of player p in s; in particular, $s_p \in S_p$; we also denote by s_{-p} the vector of pure strategies of all players except p in s; in particular, $s_{-p} \in \prod_{q\neq p} S_q$;*
- *similarly, if $\underset{\sim}{x} \in \Delta$, we denote by $\underset{\sim}{x}_p$ the mixed strategy of player p in $\underset{\sim}{x}$; in particular, $\underset{\sim}{x}_p \in \Delta^{S_p}$; we also denote by $\underset{\sim}{x}_{-p}$ the vector of mixed strategies of all players except p in $\underset{\sim}{x}$; in particular, $\underset{\sim}{x}_{-p} \in \prod_{q\neq p} \Delta^{S_q}$;*
- *finally, we are quite lax in our notation and often omit $\sim$ from under vector symbols.*

When players are using randomized strategies it is assumed that they sample from their mixed strategies *independently of the other players*. Hence, given a mixed strategy profile $\underset{\sim}{x} \in \Delta$, the expected payoff of player p is given by

$$u_p(\underset{\sim}{x}) = \sum_{s\in S} u_p(s) \prod_{q\in[n]} \underset{\sim}{x}_q(s_q),$$

where, for a pure strategy profile $s \in S$, $\prod_{q \in [n]} \underset{\sim}{x}_q(s_q)$ is just the probability that s is arises, when players independently sample their mixed strategies. We use the following shorthand for the above (ugly) expression:

$$u_p(\underset{\sim}{x}) = \mathbb{E}_{s \sim \underset{\sim}{x}}\left[u_p(s)\right],$$

where it is implied in the notation "$s \sim \underset{\sim}{x}$", that $s \in S$ is drawn by having each player $q \in [n]$ independently draw a sample from his mixed strategy $\underset{\sim}{x}_q$. Having this notation in place, we define the concept of *Nash equilibrium* as follows:

Definition 3 (Nash Equilibrium). *A mixed strategy profile $\underset{\sim}{x} \in \Delta$ is a Nash equilibrium iff for all $p \in [n]$ and $\underset{\sim}{x}'_p \in \Delta^{S_p}$:*

$$u_p(\underset{\sim}{x}) \geq u_p(\underset{\sim}{x}'_p; \underset{\sim}{x}_{-p}).$$

In other words, $\underset{\sim}{x}$ is a Nash equilibrium iff no player can strictly increase his or her payoff by switching to a different mixed strategy, if the other players don't change their strategies. Notice that the expected payoff of a player is a linear function of his own mixed strategy, since

$$u_p(\underset{\sim}{x}) \equiv \sum_{s_p \in S_p} \underset{\sim}{x}_p(s_p) \cdot u_p(s_p; \underset{\sim}{x}_{-p}).$$

Hence, an equivalent definition of Nash equilibrium is the following:

Definition 4. *A mixed strategy profile $\underset{\sim}{x} \in \Delta$ is a Nash equilibrium iff for all $p \in [n]$ and $s_p, s'_p \in S_p$ such that $\underset{\sim}{x}_p(s_p) > 0$, we have*

$$u_p(s_p; \underset{\sim}{x}_{-p}) \geq u_p(s'_p; \underset{\sim}{x}_{-p}).$$

Sometimes we need to relax the Nash equilibrium conditions, allowing for a small margin of improving one's payoff. This gives rise to notions of approximate equilibrium:

Definition 5 (ϵ-approximate Nash equilibrium). *A mixed strategy profile $\underset{\sim}{x} \in \Delta$ is a ϵ-approximate Nash equilibrium iff for all $p \in [n]$ and $\underset{\sim}{x}'_p \in \Delta^{S_p}$ we have*

$$u_p(\underset{\sim}{x}) \geq u_p(\underset{\sim}{x}'_p; \underset{\sim}{x}_{-p}) - \epsilon.$$

Definition 6 (ϵ-well-supported Nash equilibrium). *A mixed strategy profile $\underset{\sim}{x} \in \Delta$ is a ϵ-well-supported Nash equilibrium iff $\forall p \in [n], s_p, s'_p \in S_p$ such that $\underset{\sim}{x}_p(s_p) > 0$, we have*

$$u_p(s_p; \underset{\sim}{x}_{-p}) \geq u_p(s'_p; \underset{\sim}{x}_{-p}) - \epsilon.$$

Notice that these two notions of approximate equilibrium are no longer equivalent. It is easy to see that an ϵ-well-supported Nash equilibrium is also an ϵ-approximate Nash equilibrium. However, the opposite is not always true.

2 Nash's Theorem

We defined multi-player games and their corresponding notions of Nash equilibrium, but never established that Nash equilibria exist in these games. Indeed, we haven't even established that Nash equilibria exist in two-player non zero-sum games. The following theorem was established in a one-page paper by John Nash in 1950 [2], i.e. twelve years after von Neumann's proof that a Nash equilibrium exists in two-player zero-sum games [4].

Theorem 1 (Nash [2]). *Every (finte) game has a Nash equilibrium.*

We will prove this theorem later in the course, using Brouwer's fixed point theorem. [1] We will also argue that (subject to complexity theoretic assumptions) this proof cannot be turned into an efficient algorithm, even for (non zero-sum) two-player games. What we explore in the next sections is whether, at least for the case of zero-sum games, the theorem can be made constructive beyond the two-player game setting.

[1] Nash's original proof [2] used Kakutani's theorem [1], but he later simplified his proof to only use Brouwer's theorem [3],

3 Nash Equilibria in Multi-player Zero-sum Games

It does not take too much thought to observe that multi-player zero-sum games are at least as hard as general two-player games. In particular,

Proposition 1. *For all $n \geq 3$, computing a Nash equilibrium in an n-player zero-sum game is at least as hard as computing a Nash equilibrium in a general $(n-1)$-player game.*

Proof: (Sketch) Here is a reduction from the computation of a Nash equilibrium in a general $(n-1)$-player game to the computation of a Nash equilibrium in an n-player zero-sum game. Introduce a new player whose payoff is minus the payoff of all other players, but who does not affect their payoff. $\square$

We have already noted above that, subject to complexity-theoretic assumptions, there is no efficient algorithm for general two-player games. Hence, the above proposition implies that there is no hope of obtaining efficient algorithms for n-player zero-sum games, when $n > 2$. Nevertheless, we show next that we can do this for a special case of multiplayer zero-sum games.

4 Nash Equilibria in Separable Multiplayer Zero-Sum Games

We define a *separable multiplayer game* as one in which each player plays a (potentially different) two-player game with a subset of the other players.

Definition 7. *A* separable n-player game *is specified by a collection of integers $\{m_p \in \mathbb{N}\}_{p\in[n]}$, representing the number of strategies available to each player, where we identify player p's strategies with the set $[m_p]$, and a collection of matrices $\{A^{(p,q)} \subseteq \mathbb{R}^{m_p \times m_q}\}_{(p,q)}$, where $(A^{(p,q)}, (A^{(q,p)})^{\mathrm{T}})$ represents the two-player game between players p and q, so that the payoff of player p under mixed strategy profile x is*

$$u_p(x) = \sum_q x_p^{\mathrm{T}} A^{(p,q)} x_q.$$

Likewise, if player p plays pure strategy $j \in [m_p]$ and the others play strategy x_{-p} then p gets:

$$u_p(j\ ;\ x_{-p}) = \sum_q e_j^{\mathrm{T}} A^{(p,q)} x_q.$$

Observe that separable multiplayer games generalize two-player games. Hence, without any restriction on these games, there is no hope of computing their equilibria efficiently. We show that, if a separable multiplayer game is zero-sum, this suffices for its equilibria to be efficiently computable.

Theorem 2. *A Nash equilibrium of a separable multi-player zero-sum game can be found efficiently with linear programming.*

Proof: Suppose we are given a separable multiplayer zero-sum game (as in Definition 7). In terms of the given game, we define the following linear program:

$$\begin{aligned} & \min \sum_p w_p \\ \text{s.t.} \quad & w_p \geq u_p(j\ ;\ x_{-p}),\ \forall p,\ \forall j \in [m_p]; \\ \forall p: \quad & \sum_{j\in[m_p]} x_p(j) = 1 \ \text{ and } \ x_p(j) \geq 0, \forall j \in [m_p]. \end{aligned}$$

Notice that this is truly a linear program as, for all p, $u_p(j\ ;\ x_{-p})$ is a linear function. We argue that any optimal solution of the linear program gives a Nash equilibrium of the game via the following sequence of claims.

Claim 1. *Let $x = (x_1, \ldots, x_n)$ be a Nash equilibrium of the given game (guaranteed to exist by Nash's theorem). For all p, let also $w_p = u_p(x)$. Then $(x_1, \ldots, x_n\ ;\ w_1, \ldots, w_n)$ is a feasible solution of the linear program of value 0.*

Proof: Since x is a Nash equilibrium, clearly $w_p \equiv u_p(x) \geq u_p(j\ ;\ x_{-p}),\ \forall p,\ \forall j \in [m_p]$. So $(x_1, \ldots, x_n\ ;\ w_1, \ldots, w_n)$ is a feasible solution of the LP. Moreover, $\sum_p w_p = \sum_p u_p(x) = 0$, since the game is zero-sum. $\square$

Claim 2. *The above LP has value* 0.

Proof: By Claim 1, it follows that the value of the LP is at most 0. Let now $(x_1, \ldots, x_n\ ;\ w_1, \ldots, w_n)$ be any feasible solution. We will argue that the value of the solution is ≥ 0. Indeed, for all p, it follows from

$$w_p \geq u_p(j\ ;\ x_{-p}),\ \forall j \in [m_p],$$

that

$$w_p \geq u_p(x).$$

Hence, $\sum_p w_p \geq \sum_p u_p(x) = 0$, where the last equality follows from the fact that the game is zero-sum. $\square$

Claim 3. *Let* $(x_1, \ldots, x_n\ ;\ w_1, \ldots, w_n)$ *be an optimal solution to the above LP. Then* $(x_1, \ldots, x_n)$ *is a Nash equilibrium.*

Proof: We argued in the proof of Claim 2 that any feasible solution of the LP satisfies:

$$w_p \geq u_p(x), \forall p; \tag{1}$$

$$\sum_p w_p \geq \sum_p u_p(x) = 0. \tag{2}$$

Combining the second inequality above with the fact that the LP has value 0 (which also follows from Claim 2), it follows that:

$$\sum_p w_p = \sum_p u_p(x).$$

Using the latter with Eq (1) we obtain that

$$w_p = u_p(x), \forall p.$$

Combining the latter with the LP feasibility constraints we obtain that:

$$u_p(x) = w_p \geq u_p(j\ ;\ x_{-p}),\ \forall p,\ \forall j \in [m_p],$$

which implies that x is a Nash equilibrium. $\square$ $\square$

Remark 1 (Exercise)**.** *Observe that in the proof of Theorem 2 we used Nash's theorem (Theorem 1). Hence, while we argued successfully that Nash equilibria can be computed efficiently in separable multi-player zero-sum games, we haven't provided a proof that Nash equilibria exist in these games. Prove the existence of equilibria in these games using linear programming duality, i.e. without resorting to Nash's theorem.*

References

[1] S. Kakutani. A generalisation of Brouwer's fixed point theorem. *Duke Math. J.*, 7:457–459, 1941.

[2] J. Nash. Equilibrium Points in n-Person Games. *Proceedings of the National Academy of Sciences*, 36(1):48–49, 1950.

[3] J. Nash. Noncooperative Games. *Annals of Mathematics*, 54:289–295, 1951.

[4] John von Neumann. Zur Theorie der Gesellshaftsspiele. *Mathematische Annalen,* 100:295320, 1928.