6.896: Topics in Algorithmic Game Theory

Lecture 13

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multiplayer zero-sum games

Multiplayer Zero-Sum, wha?

Take an arbitrary two-player game, between Alice and Bob.

Add a third player, Eve, who does not affect Alice or Bob's payoffs, but receives payoff

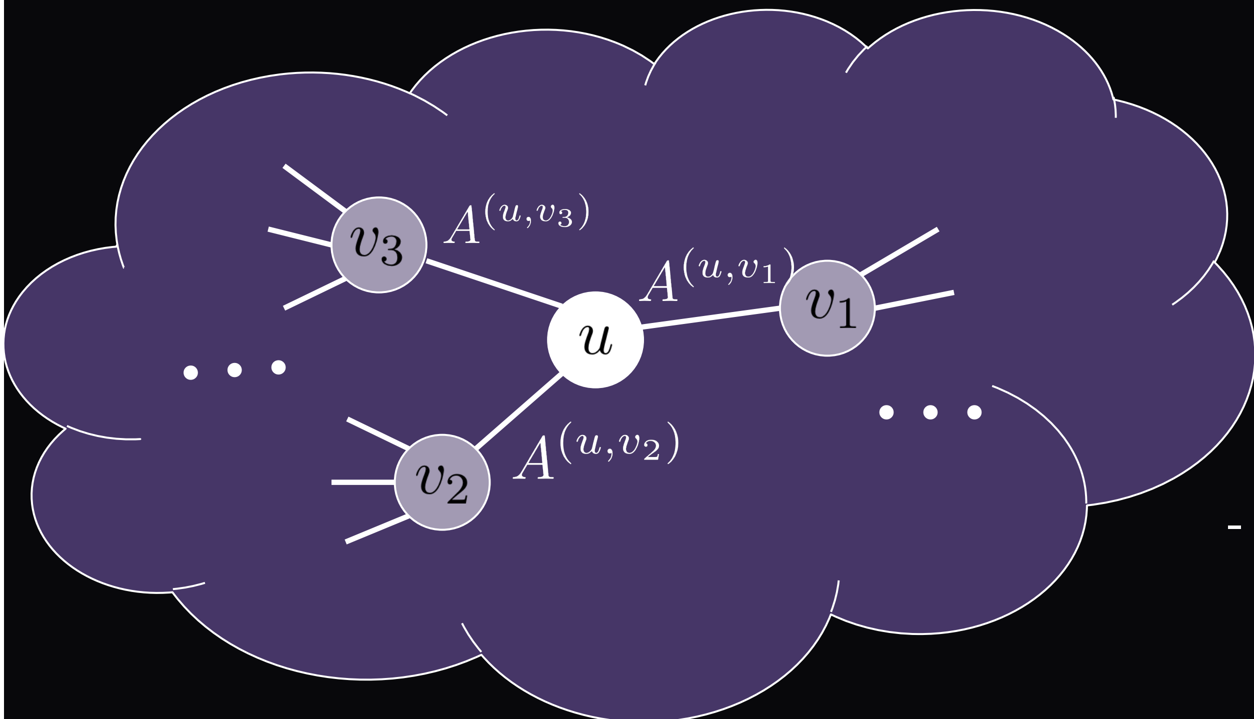
$$-(P_{\text{Alice}}(\sigma) + P_{\text{Bob}}(\sigma)), \forall \text{ outcome } \sigma$$

The game is zero-sum, but solving it is PPAD-complete.

➡ intractability even for 3 player, if *three-way* interactions are allowed.

What if only *pairwise* interactions are allowed?

Polymatrix Games



- players are nodes of a graph G
- edges are 2-player games
- player's payoff is the sum of payoffs from all adjacent edges

$$\sum_{i=1}^{3} x_u^{\mathrm{T}} A^{(u,v_i)} x_{v_i}$$

N.B. finding a Nash equilibrium is PPAD-complete for general games on the edges [D, Gold, Pap '06]

What if the total sum of players' payoffs is always 0?

Polymatrix Games

essentially the broadest class of zero-sum games we could hope to solve

Theorem [Daskalakis-Papadimitriou '09, Cai-Daskalakis'10]

If the global game is zero-sum:

- a Nash equilibrium can be found efficiently with linear-programming;
- the Nash equilibria comprise a convex set;
- if every node uses a no-regret learning algorithm, the players' behavior converges to a Nash equilibrium.

  i.e. payoffs approach equilibrium payoffs, and empirical strategies approach Nash equilibrium

strong indication that Nash eq. makes sense in this setting.

N.B. but [+ Tardos '09] the value of the nodes need not be unique.

Anonymous Games

anonymous games

Every player is (potentially) different, but only cares about how many players (of each type) play each of the available strategies.

- all players share the same set of strategies: $S = \{1, \ldots, s\}$
- payoff functions: $u_p = u_p(\sigma ; \underbrace{n_1, n_2, \ldots, n_s}_{})$

choice of p

number of the other players choosing each strategy in S

Description Size: $O(\min\{s\, n^s, n\, s^n\})$

e.g. symmetry in auctions, congestion games, social phenomena, etc.

"Congestion Games with Player- Specific Payoff Functions." Milchtaich, *Games and Economic Behavior, 1996.*

"The women of Cairo: Equilibria in Large Anonymous Games." Blonski, *Games and Economic Behavior*, 1999.

"Partially-Specified Large Games." Ehud Kalai, *WINE*, 2005.

PTAS

Theorem [Daskalakis, Papadimitriou ’07, ’08]:

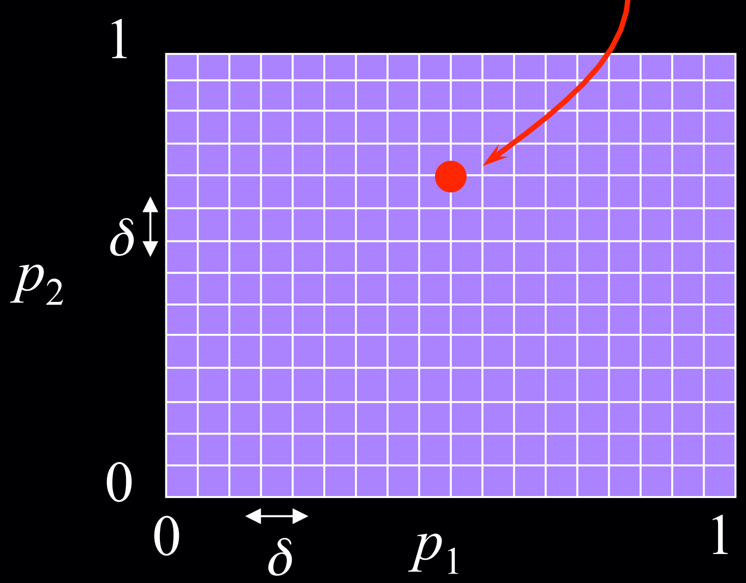
There is a PTAS for anonymous games with a constant #strategies.

Remarks: - exact computation is not known to be PPAD-complete for multi-player anonymous games with a constant number of strategies;

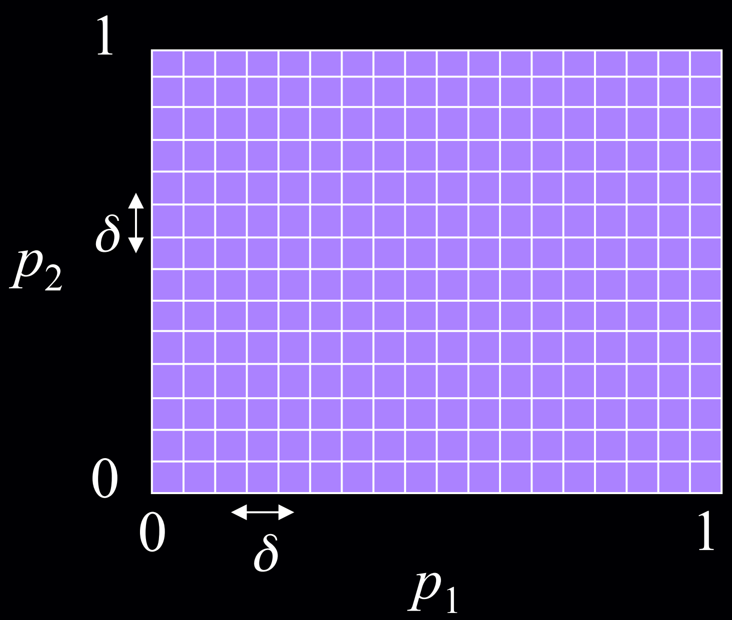
- on the flip side, if n is small and s is large (few players, many strategies) then trivially PPAD-complete, since general 2-player games can be reduced to this.

sketch of algorithm for 2 strategies

- since 2 strategies per player, Nash equilibrium lies in $[0,1]^n$
- discretize $[0,1]^n$ into multiples of δ, and restrict search to the discrete space
- pick best point in discrete space



sketch for 2 strategies (cont.)



*F*irst trouble:

size of search space $\left(\frac{1}{\delta}\right)^n$

but will deal with this later

*Basic Q*uestion:

what grid size δ is required for ε - approximation?

if function of ε only $\Rightarrow$ PTAS

if function also of n $\Rightarrow$ nothing

sketch for 2 strategies (cont.)

Theorem [Daskalakis, Papadimitriou '07]:

Given

- n ind. Bernoulli's X_i with expectations p_i , i =1,…, n
- a constant δ independent of n

there exists another set of Bernoulli's Y_i with expectations q_i such that

q_i's are integer multiples of δ

$$\left\| \sum_i X_i - \sum_i Y_i \right\|_{\mathrm{TV}} \leq O(\sqrt{\delta})$$

N.B. argument from last lecture gives $n \cdot \delta$

in fact: $\forall j$: $$\left\| \sum_{i \neq j} X_i - \sum_{i \neq j} Y_i \right\|_{\mathrm{TV}} \leq O(\sqrt{\delta})$$

The TV Bound

How much does player p's payoff from pure `strategy σ change if we replace $X = (X_1, X_2, \ldots, X_n)$ with $Y = (Y_1, Y_2, \ldots, Y_n)$?

$$|u_p(\sigma\ ;\ X_{-p}) - u_p(\sigma\ ;\ Y_{-p})| \leq \ldots \leq u_{\max} \left\| \sum_{q \neq p} X_q - \sum_{q \neq p} Y_q \right\|$$

Given previous theorem, can guarantee that there exists a discretized point making the above difference at most $\epsilon/2$ by selecting $\delta = (\epsilon/2)^2$.

Completing the algorithm

discretization + dynamic programming + TV bound

assume that the players only use mixed strategies in probabilities that are multiples of $\delta = (\epsilon/2)^2$.

complete this step (2 points)

enough to guarantee a discretized ϵ - Nash equilibrium

Resulting running time $n^{O(1/\epsilon^2)}$ for 2 strategies.

The first probabilistic approximation theorem

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there exists another set of Bernoulli's Y_i with expectations q_i such that

q_i's are integer multiples of δ

$$\left\| \sum_i X_i - \sum_i Y_i \right\|_{\mathrm{TV}} \leq O(\sqrt{\delta})$$

argument from last time gives $n \cdot \delta$

in fact: $\forall j$: $$\left\| \sum_{i \neq j} X_i - \sum_{i \neq j} Y_i \right\|_{\mathrm{TV}} \leq O(\sqrt{\delta})$$

proof of approximation result

- rounding p_i's to the closest multiple of δ *gives total variation* $n\delta$
- probabilistic rounding up or down quickly runs into problems
- what works:

Law of Rare Events

$+$

CLT

Poisson Approximations

(Stein's Method)

Berry-Esséen

proof of approximation result

Intuition:

If p_i's were small $\Rightarrow$ $\sum_i X_i$ would be close to a Poisson with mean $\sum_i p_i$

$\Rightarrow$ define the q_i's so that $\sum_i q_i \approx \sum_i p_i$

$$
\begin{tikzcd}
\sum_i X_i \arrow[dr, leftrightarrow] & & & \sum_i Y_i \\
& Poisson\left(\sum_i p_i\right) \arrow[r, leftrightarrow] & Poisson\left(\sum_i q_i\right) \arrow[ur, leftrightarrow] &
\end{tikzcd}
$$

proof of approximation result

Poisson approximation is only good for small values of p_i's. (LRE)

For intermediate values of p_i's, Normals are better. (CLT)

$$\sum_i X_i \overset{\textit{Berry-Esséen}}{\longleftrightarrow} \mathcal{N}\left(\sum_i p_i, \sum_i p_i(1-p_i)\right) \longleftrightarrow \mathcal{N}\left(\sum_i q_i, \sum_i q_i(1-q_i)\right) \overset{\textit{Berry-Esséen}}{\longleftrightarrow} \sum_i Y_i$$

Anonymous Games Summary

2-strategies per player: $n^{O(1/\epsilon^2)}$ [DP '07]

constant #strategies per player: $n^{f(s) 1/\epsilon^6}$ [DP '08]

bad function of s

is there a faster PTAS?

Theorem [Daskalakis '08]:

There is an oblivious PTAS with running time $poly(n) \cdot (1/\epsilon)^{O(1/\epsilon^2)}$

the underlying structural result…

Theorem [D'08]: In every anonymous game there exists an ε-approximate Nash equilibrium in which

- either all players who mix play the same mixed strategy
- or, at most $1/\epsilon^3$ mix, and they choose mixed strategies which are integer multiples of ϵ^2

the corresponding symmetry…

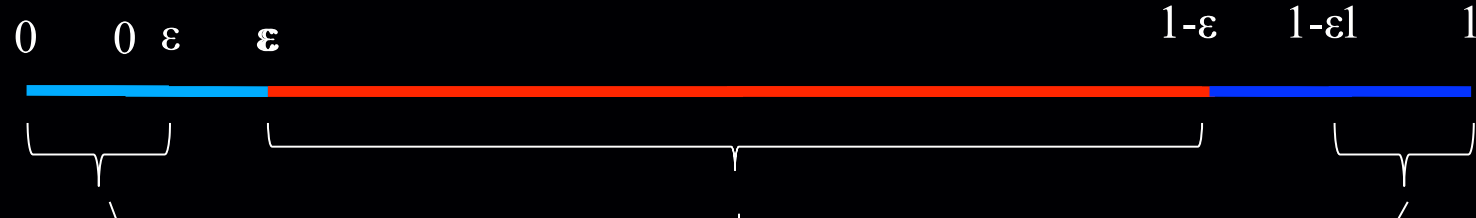
Lemma:

- The sum of $m \geq k^3$ indicators X_i with expectations in $[1/k, 1-1/k]$ is $O(1/k)$-close in total variation distance to a Binomial distribution with the same mean and variance

… i.e. close to a sum of indicators with the same expectation

[tightness of parameters by Berry-Esséen]

proof of structural result



round some of the X_i's falling here to 0 and some of them to ε so that the total mean is preserved to within ε

similarly

- if more than $1/\varepsilon^3$ X_i's are left here, appeal to previous slide (Binomial appx)

- o.w. use Dask. Pap. '07 (exists rounding into multiples of ε^2)

Anonymous Games Summary

2-strategies per player: $n^{O(1/\epsilon^2)}$ [DP '07]

$\text{poly}(n) \cdot (1/\epsilon)^{O(1/\epsilon^2)}$ [D '08]

constant #strategies per player: $n^{f(s) 1/\epsilon^6}$ [DP '08]

bad function of *s*

Is there an even faster PTAS?

Theorem [Daskalakis, Papadimitriou ’08]:

There is a non-oblivious PTAS with running time

$$poly(n) \cdot (1/\epsilon)^{O(\log^2(1/\epsilon))}$$

the underlying probabilistic result [DP ’08]:

If two sums of indicators have equal moments up to moment *k* then their total variation distance is $O(2^{-k})$.

Anonymous Games Summary

2-strategies per player: $n^{O(1/\epsilon^2)}$ [DP '07]

$\mathrm{poly}(n) \cdot (1/\epsilon)^{O(1/\epsilon^2)}$ [D '08]

$\mathrm{poly}(n) \cdot (1/\epsilon)^{O(\log^2(1/\epsilon))}$ [DP '09]

is there an FPTAS?

constant #strategies per player: $n^{f(s) 1/\epsilon^6}$

bad function of *s*