6.896: Topics in Algorithmic Game Theory

Lecture 14

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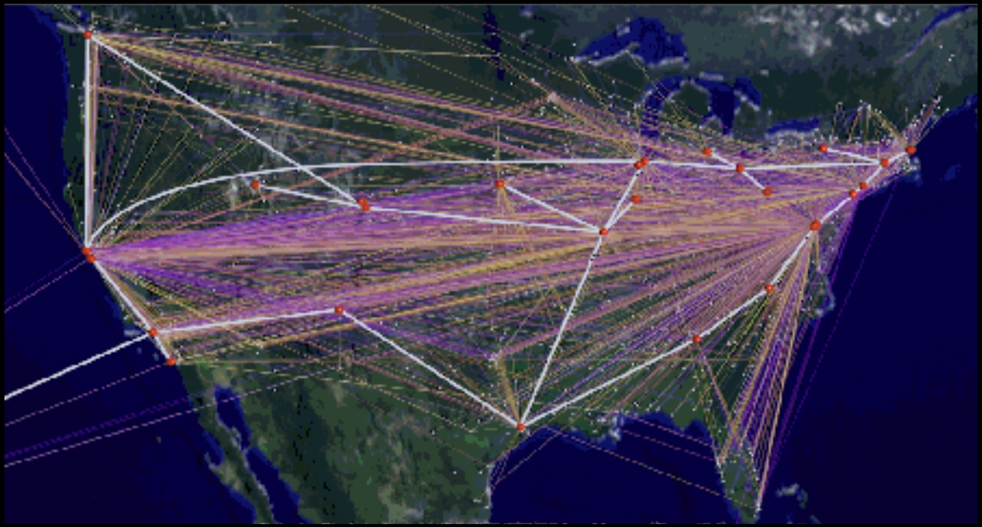
Markets



Exchange Market Model (without production)

Consider a marketplace with:

n traders (or agents)

k goods (or commodities)
assumed to be *infinitely divisible*

Utility function of trader *i*:

$$u_i \ : \ \mathcal{X}_i \subseteq \mathbb{R}^k_+ \longrightarrow \mathbb{R}_+$$

non-negative reals

consumption set for trader *i*

specifies trader *i*'s utility for bundles of goods

Endowment of trader *i*:

$$e_i \in \mathcal{X}_i$$

amount of goods trader comes to the marketplace with

Exchange Market Model (without production)

Suppose the goods in the market are priced according to some price vector $p \in \mathbb{R}^k_+$.

Under this price vector, each trader would like to sell some of her endowment and purchase an optimal bundle using her income from what s/he sold; thus she solves the following program:

$$\left.\begin{aligned} \max \quad & u_i(x) \\ \text{s.t.} \quad & p \cdot x \leq p \cdot e_i \\ & x \in \mathcal{X}_i \end{aligned}\right\} \text{Program}_i(p)$$

Note: If u_i is continuous and $\mathcal{X}_i$ is compact, then the above program has a well-defined optimum value.

Competitive (or Walrasian) Market Equilibrium

Def: A price vector $p \in \mathbb{R}^k_+$ is called a *competitive market equilibrium* iff there exists a collection of optimal solutions $x_i(p)$ to Program$_i(p)$, for all $i = 1,\ldots, n$, such that the total demand meets the total supply, i.e.

$$\underbrace{\sum_{i=1}^{n} x_i(p)}_{\text{total demand}} \leq \underbrace{\sum_{i=1}^{n} e_i}_{\text{total supply}}$$

Arrow-Debreu Theorem 1954

Theorem [Arrow-Debreu 1954]: Suppose

(i) $\mathcal{X}_i$ is closed and convex

(ii) $e_i >> 0$, for all i (all coordinates positive)

(iii a) u_i is continuous

(iii b) u_i is quasi-concave

$$u_i(x) > u_i(y) \implies u_i(\lambda x + (1-\lambda)y) > u_i(y),\ \forall \lambda \in (0,1)$$

(iii c) u_i is nonsatiated

$$\forall y \in \mathcal{X}_i,\quad \exists x \in \mathcal{X}_i \text{ s.t. } u_i(x) > u_i(y)$$

Then a competitive market equilibrium exists.

Market Clearing

Nonsatiation + quasi-concavity

➔ at equilibrium every trader spends all her budget, i.e. if $x_i(p)$ is an optimal solution to Program$_i(p)$ then

$$p \cdot x_i(p) = p \cdot e_i$$

$$\Longrightarrow \; p \cdot \left(\sum_i x_i(p) - \sum_i e_i \right) = 0$$

➔ every good with positive price is fully consumed

A market with no equilibrium

Alice has oranges and apples, but only wants apples.

Bob only has oranges, but only wants both oranges and apples.

- if oranges are priced at 0, then Bob's demand is not well-defined.

- if oranges are priced at > 0, then Alice wants more apples than there are in the market.

Proof of the Arrow-Debreu Theorem

Steps (details on the board)

simplifying assumption: u_i is strictly concave

(i) w.l.o.g. can assume that the $\mathcal{X}_i$ are compact

argument on the board; the idea is that we can replace $\mathcal{X}_i$ with

$$\mathcal{X}_i \cap \left\{ x \leq \sum_i e_i \right\}$$

without missing any equilibrium, and without introducing spurious ones

(ii) by compactness and strict concavity:

for all p, there exists a unique maximizer $x_i(p)$ of $\text{Program}_i(p)$

(iii) by the maximum theorem: $x_i(p)$ is continuous on p

(iv) *rest of the argument on the board*

Utility Functions

Linear utility function (goods are perfect substitutes)

$$u_i(x) = \sum_j a_{ij} x_j$$

Leontief (or fixed-proportion) utility function

$$u_i(x) = \min_j \{a_{ij} x_j\}$$

e.g. buying ingredients to make a cake

e.g. rate allocation on a network

Cobb-Douglas utility function

$$u_i(x) = \prod_j x_j^{a_{ij}}, \quad \text{where } \sum_j a_{ij} = 1$$

Utility Functions

CES utility functions:

$$u_i(x) = \left(\sum_j u_{ij} \cdot x_j^{\rho} \right)^{\frac{1}{\rho}}, \quad -\infty < \rho \leq 1$$

Convention: - If u_{ij}=0, then the corresponding term in the utility function is always 0.

- If $u_{ij} > 0$, x_j=0, and ρ<0, then $u_i(x)$=0 no matter what the other x_j's are.

$\rho = 1$ $\longrightarrow$ linear utility form

$\rho \to -\infty$ $\longrightarrow$ Leontief utility form

$\rho \to 0$ $\longrightarrow$ Cobb-Douglas form

elasticity of substitution: $\sigma = \dfrac{1}{1 - \rho}$

Homework

CES utility functions:

$$u_i(x) = \left(\sum_j u_{ij} \cdot x_j^{\rho} \right)^{\frac{1}{\rho}}, \quad -\infty < \rho \leq 1$$

show it is concave (2 points)

Fisher's Model

Suppose all endowment vectors are parallel…

$$e_i = m_i \cdot e, \quad m_i > 0, \quad m_i : \text{ scalar}, e : \text{ vector}$$

➔ relative incomes of the traders are independent of the prices.

Equivalently, we can imagine the following situation:

n traders, with specified money m_i

k divisible goods owned by seller; seller has q_j units of good j

Arrow-Debreu Thm ➔

(under the Arrow-Debreu conditions) there exist prices that the seller can assign on the goods so that the traders spend all their money to buy optimal bundles and supply meets demand

Fisher's Model with CES utility functions

$$u_i(x_i) = \left(\sum_j u_{ij} \cdot x_{ij}^{\rho} \right)^{\frac{1}{\rho}}, \quad -\infty < \rho \leq 1$$

- Buyers' optimization program (under price vector *p*):

$$\begin{aligned} \max \quad & u_i(x_i) \\ \text{s.t.} \quad & \sum_j x_{ij} p_j \leq m_i \end{aligned}$$

- Global Constraint:

$$\sum_i x_{ij} \leq q_j, \quad \forall j$$

$$x_{ij} \geq 0, \quad \forall j$$

Eisenberg-Gale's Convex Program

- The space of feasible allocations is:

$$\sum_i x_{ij} \le q_j, \quad \forall j$$
$$x_{ij} \ge 0, \quad \forall j$$

- But how do we aggregate the trader's optimization problems into one global optimization problem?

e.g., choosing as a global objective function the sum of the traders' utility functions won't work…

Eisenberg-Gale's Convex Program

Observation: The global optimization problem should not favor (or punish) Buyer i should he

- Doubled all her u_{ij}' s
- Split himself into two buyers with half the money

- Eisenberg and Gale's idea: Use the following objective function (take its logarithm to convert into a concave function)

$$\max \quad u_1(x_1)^{m_1} \cdot u_2(x_2)^{m_2} \cdot \ldots \cdot u_n(x_n)^{m_n}$$

Eisenberg-Gale's Convex Program

$$\max \quad u_1^{m_1} \cdot u_2^{m_2} \cdot \ldots \cdot u_n^{m_n}$$

$$\text{s.t} \quad u_i = \left(\sum_j u_{ij} x_{ij}^{\rho} \right)^{\frac{1}{\rho}}$$

$$\sum_i x_{ij} \leq q_j$$

$$x_{ij} \geq 0$$

Remarks:

- No budgets constraint!
- It is not necessary that the utility functions are CES; everything works as long as they are concave, and homogeneous

Eisenberg-Gale's Convex Program

KKT Conditions →

- interpret Langrange multipliers as prices
- primal variables + Langrange multipliers comprise a competitive eq.

1. Gives a poly-time algorithm for computing a market equilibrium in Fisher's model.
2. At the same time provides a proof that a market equilibrium exists in this model.

Homework (2 points): Show 1, 2 for linear utility functions.