6.896: Topics in Algorithmic Game Theory

Lecture 20

Yang Cai

Recap

Games with Strict Incomplete Information

Def: A game with (independent private values and) strict incomplete information for a set of n players is given by the following ingredients:

(i) For every player i, a set of actions X_i

(ii) For every player i, a set of types T_i. A value $t_i \in T_i$ is the private information that i has

(iii) For every player i, a utility function $u_i : T_i \times X_1 \times \ldots \times X_n \to \mathcal{R}$, where $u_i(t_i, x_1, \ldots, x_n)$ is the utility achieved by player i, if his type (private information) is t_i, and the profile of actions taken by all players is $x_1, \ldots, x_n$

Strategy and Equilibrium

Def: A strategy of a player i is a function $s_i : T_i \to X_i$

Def: Equilibrium (ex-post Nash and dominant strategy)

- A profile of strategies $s_1 \ldots s_n$ is an ex-post Nash equilibrium if for all i, all $t_1 \ldots t_n$, and all x'_i we have that

$$u_i(t_i, s_i(t_i), s_{-i}(t_{-i})) \geq u_i(t_i, x'_i, s_{-i}(t_{-i}))$$

- A profile of strategies $s_1 \ldots s_n$ is a dominant strategy equilibrium if for all i, all x_{-i}, and all x'_i we have that

$$u_i(t_i, s_i(t_i), x_{-i}) \geq u_i(t_i, x'_i, x_{-i})$$

Equilibrium (cont'd)

Proposition: **Let $s_1 \ldots s_n$ be an ex-post Nash equilibrium of a game $(X_1, \ldots, X_n; T_1, \ldots, T_n; u_1, \ldots, u_n)$. Define $X_i' = \{s_i(t) | t_i \in T_i\}$, then $s_1 \ldots s_n$ is a dominant strategy equilibrium in the game $(X_1', \ldots, X_n'; T_1, \ldots, T_n; u_1, \ldots, u_n)$.**

Formal Definition of Mechanisms

General Mechanisms

Vickrey's auction and VCG are both single round and direct-revelation mechanisms.

We will give a general model of mechanisms. It can model multi-round and indirect-revelation mechanisms.

Mechanism

Def: A (general-non direct revelation) mechanism for n players is given by

(a) players' type spaces $T_1, \ldots, T_n$.

(b) players' action spaces $X_1, \ldots, X_n$.

(c) an alternative set A.

(d) players' valuation functions $v_i : T_i \times A :\rightarrow \mathcal{R}$.

(e) an outcome function $a : X_1 \times \ldots \times X_n \rightarrow A$.

(f) players' payment functions $p_i : X_1 \times \ldots \times X_n \rightarrow \mathcal{R}$.

The game with strict incomplete information induced by the mechanism has the same type spaces and action spaces, and utilities

$$u_i(t_i, x_1, \ldots, x_n) = v_i(t_i, a(x_1, \ldots, x_n)) - p_i(x_1, \ldots, x_n)$$

Implementing a social choice function

Given a social choice function $f : T_1 \times \ldots \times T_n \to A$

A mechanism implements f in dominant strategies if for ***some*** dominant strategy equilibrium $s_1, \ldots, s_n$ of the induced game, we have that for all $t_1, \ldots, t_n$, $f(t_1, \ldots, t_n) = a(s_1(t_1), \ldots, s_n(t_n))$.

Ex: Vickrey's auction implements the maximum social welfare function in dominant strategies, because $s_i(t_i) = t_i$ *is a dominant strategy equilibrium, and maximum social welfare is achieved at this equilibrium.*

Similarly we can define ex-post Nash implementation.

outcome of the social choice function

outcome of the mechanism at the equilibrium

Remark: We only requires that for ***some*** equilibrium $f(t_1, \ldots, t_n) = a(s_1(t_1), \ldots, s_n(t_n))$ and allows other equilibria to exist.

The Revelation Principle

Revelation Principle

We have defined direct revelation mechanisms in previous lectures. Clearly, the general definition of mechanisms is a superset of the direct revelation mechanisms.

But is it strictly more powerful? Can it implement some social choice functions in dominant strategy that the incentive compatible (direct revelation dominant strategy implementation) mechanism can not?

Revelation Principle

Proposition: (Revelation principle) If there exists an arbitrary mechanism that implements f in dominant strategies, then there exists an incentive compatible mechanism that implements f. The payments of the players in the incentive compatible mechanism are identical to those, obtained at equilibrium, of the original mechanism.

Incentive Compatible

Def: A mechanism $(f, p_1, \ldots, p_n)$ is called *incentive compatible*, or *truthful* , or *strategy-proof* iff for all *i*, for all $t_1 \in T_1, \ldots, t_n \in T_n$ and for all $t'_i \in T_i$

$$\underbrace{v_i(t_i, a) - p_i(t_i, t_{-i})}_{\text{utility of } i \text{ if he says the truth}} \geq \underbrace{v_i(t_i, a') - p_i(t'_i, t_{-i})}_{\text{utility of } i \text{ if he lies}}$$

$a = f(t_i, t_{-i})$

$a' = f(t'_i, t_{-i})$

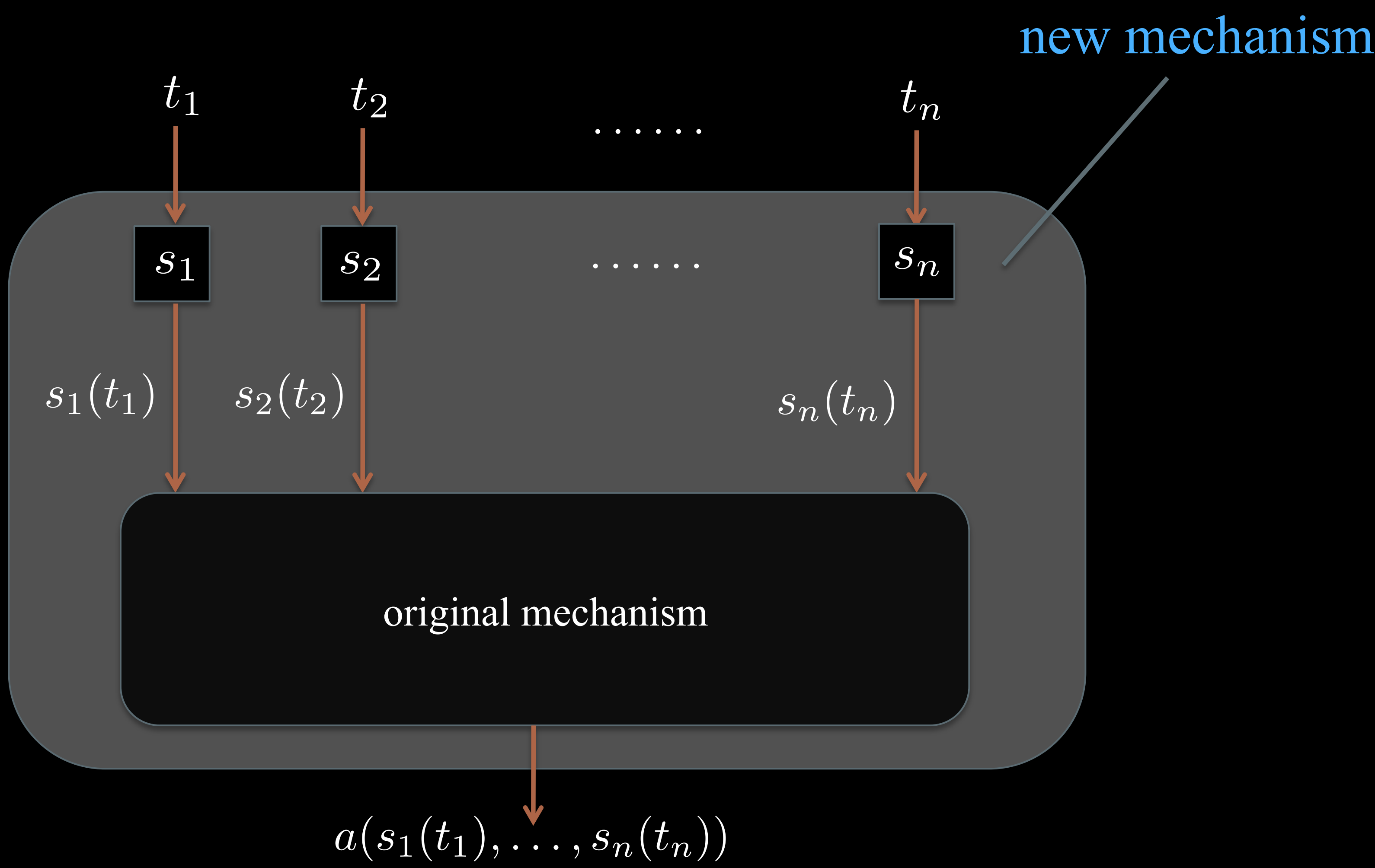
i.e. no incentive to lie!

Revelation Principle

Proposition: (Revelation principle) If there exists an arbitrary mechanism that implements f in dominant strategies, then there exists an incentive compatible mechanism that implements f. The payments of the players in the incentive compatible mechanism are identical to those, obtained at equilibrium, of the original mechanism.

Proof idea: ***Simulation***

Revelation Principle (cont'd)



Proof of Revelation Principle

Proof: Let $s_1, \ldots, s_n$ be a dominant strategy equilibrium of the original mechanism such that $f(t_1, \ldots, t_n) = a(s_1(t_1), \ldots, s_n(t_n))$, we define a new direct revelation mechanism:

$$f'(t_1, \ldots, t_n) = a(s_1(t_1), \ldots, s_n(t_n))$$
$$p'_i(t_1, \ldots, t_n) = p_i(s_1(t_1), \ldots, s_n(t_n))$$

Since each s_i is a dominant strategy for player i, for every t_i, x_{-i}, x'_i, we have that

$$v_i(t_i, a(s_i(t_i), x_{-i})) - p_i(s_i(t_i), x_{-i}) \geq v_i(t_i, a(x'_i, x_{-i})) - p_i(x'_i, x_{-i})$$

Thus in particular this is true for all $x_{-i} = s_{-i}(t_{-i})$ and any $x'_i = s_i(t'_i)$ we have that

$$v_i(t_i, f'(t_i, t_{-i})) - p'_i(t_i, t_{-i}) \geq v_i(t_i, f'(t'_i, t_{-i})) - p'_i(t'_i, t_{-i})$$

which gives the definition of the incentive compatibility of the mechanism.

$\blacksquare$

Revelation Principle (cont'd)

Corollary: If there exists an arbitrary mechanism that ex-post Nash equilibrium implements f, then there exists an incentive compatible mechanism that implements f. Moreover, the payments of the players in the incentive compatible mechanism are identical to those, obtained in equilibrium, of the original mechanism.

Proof sketch: Restrict the action spaces of each player. By the previous proposition, we know in the restricted action spaces, the mechanism implements the social choice function in dominant strategies. Now we can invoke the revelation principle to get an incentive compatible mechanism.

Characterizations of Incentive Compatible Mechanisms

Characterizations

What social choice functions can be implemented?

- Only look at incentive compatible mechanisms (revelation principle)
- When is a mechanism incentive compatible?
  Characterizations of incentive compatible mechanisms.
- Maximization of social welfare can be implemented (VCG). Any others?
  Basic characterization of implementable social choice functions.

Direct Characterization

Direct Characterization

A mechanism is ***incentive compatible*** iff it satisfies the following conditions for every i and every v_{-i}:

(i) p_i does not depend on v_i, but only on the alternative $f(v_i, v_{-i})$.

i.e., for every v_{-i}, there exists a price p_a, when the chosen alternative is a, the price is p_a

(ii) The mechanism *optimizes* for each player.

i.e., for every v_i, we have alternative $f(v_i, v_{-i}) \in argmax_a(v_i(a) - p_a)$ where the quantification is over all alternatives in the range of $f(\cdot, v_{-i})$

Direct Characterization (cont'd)

Proof:

(if part) Denote $a = f(v_i, v_{-i})$, $a' = f(v_i', v_{-i})$, $p_a = p_i(v_i, v_{-i})$ and $p_a' = p_i(v_i', v_{-i})$. Since the mechanism optimizes for i, the utility of i when telling the truth is not less than the utility when lying.

Direct Characterization (cont'd)

Proof (cont):

(only if part; (i)) If for some v_i, v_i', $f(v_i, v_{-i}) = f(v_i', v_{-i})$ but $p_i(v_i, v_{-i}) \neq p_i(v_i', v_{-i})$. WLOG, we assume $p_i(v_i, v_{-i}) \geq p_i(v_i', v_{-i})$. Then a player with type v_i will increase his utility by declaring v_i'.

(only if part; (ii)) If $f(v_i, v_{-i}) \notin argmax_a(v_i(a) - p_a)$, we fix $a' \in argmax_a(v_i(a) - p_a)$ and $f(v_i', v_{-i}) = a'$ Now a player with type v_i will increase his utility by declaring v_i'.

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Weak Monotonicity

Weak Monotonicity

- The direct characterization involves both the social choice function and the payment functions.

- ***Weak Monotonicity*** provides a partial characterization that only involves the social choice function.

Weak Monotonicity (WMON)

Def: A social choice function f satisfies Weak Monotonicity (WMON) if for all i, all v_{-i} we have that

$$f(v_i, v_{-i}) = a \neq b = f(v'_i, v_{-i}) \quad \Longrightarrow \quad v_i(a) - v_i(b) \geq v'_i(a) - v'_i(b)$$

i.e. WMON means that if the social choice changes when a single player changes his valuation, then it must be because the player increased his value of the new choice relative to his value of the old choice.

Weak Monotonicity

Theorem: If a mechanism $(f, p_1, \ldots, p_n)$ is incentive compatible then f satisfies ***WMON***. If all domains of preferences V_i are convex sets (as subsets of an Euclidean space) then for every social choice function that satisfies ***WMON*** there exists payment function $p_1, \ldots, p_n$ such that $(f, p_1, \ldots, p_n)$ is incentive compatible.

Remarks: (i) We will prove the first part of the theorem. The second part is quite involved, and will not be given here.

(ii) It is known that WMON is not a sufficient condition for incentive compatibility in general non-convex domains.

Weak Monotonicity (cont'd)

Proof: (First part) Assume first that $(f, p_1, \ldots, p_n)$ is incentive compatible, and fix i and v_{-i} in an arbitrary manner. The direct characterization implies the existence of fixed prices p_a for all $a \in A$ (that do not depend on v_i) such that whenever the outcome is a then i pays exactly p_a.

Assume $f(v_i, v_{-i}) = a \neq b = f(v'_i, v_{-i})$. Since the mechanism is incentive compatible, we have

$$v_i(a) - p_a \geq v_i(b) - p_b$$
$$v'_i(a) - p_a \leq v'_i(b) - p_b$$

Thus, we have

$$v_i(a) - v_i(b) \geq p_a - p_b \geq v'_i(a) - v'_i(b)$$

Minimization of Social Welfare

We know ***maximization*** of social welfare function can be implemented.

How about ***minimization*** of social welfare function?

No! Because of WMON.

Minimization of Social Welfare

Assume there is a single good. WLOG, let $v_1 < v_2 < \cdots < v_n$. In this case, player 1 wins the good.

If we change v_1 to v_1', such that $v_1' > v_n$. Then player 2 wins the good. Now we can apply the WMON.

The outcome changes when we change player 1's value. But according to WMON, it should be the case that $v_1 - 0 \geq v_1' - 0$. But $v_1' > v_1$. Contradiction.

Weak Monotonicity

WMON is a good characterization of implementable social choice functions, but is a ***local*** one.

Is there a ***global*** characterization?

Weighted VCG

Affine Maximizer

Def: A social choice function f is called an ***affine maximizer*** if for some subrange $A' \subseteq A$, for some weights $w_1, \ldots, w_n \in \mathcal{R}^+$ and for some outcome weights $c_a \in \mathcal{R}$, for every $a \in A'$, we have that

$$f(v_1, \ldots, v_n) \in argmax_{a \in A'}(c_a + \sum_i w_i v_i(a))$$

Payments for Affine Maximizer

Proposition: Let f be an ***affine maximizer***. Define for every i,

$$p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i}(w_j/w_i)v_j(a) - c_a/w_i$$

where h_i is an arbitrary function that does not depend on v_i. Then, $(f, p_1, \ldots, p_n)$ is incentive compatible.

Payments for Affine Maximizer

Proof: First, we can assume wlog $h_i = 0$. The utility of player i if alternative a is chosen is $v_i(a) + \sum_{j \neq i}(w_j/w_i)v_j(a) + c_a/w_i$. By multiplying by $w_i > 0$ this expression is maximized when $c_a + \sum_j w_j v_j(a)$ is maximized which is what happens when i reports truthfully.

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Roberts Theorem

Theorem [Roberts 79]: If $|A| \geq 3$, f is onto A, $V_i = \mathcal{R}^A$ for every i, and $(f, p_1, \ldots, p_n)$ is incentive compatible then f is an affine maximizer.

Remark: The restriction $|A| \geq 3$ is crucial (as in Arrow's theorem), for the case $|A| = 2$, there do exists incentive compatible mechanisms beyond VCG.