6.896: Topics in Algorithmic Game Theory

Audiovisual Supplement to

Lecture 5

Constantinos Daskalakis

On the blackboard we defined multi-player games and Nash equilibria, and showed Nash's theorem that a Nash equilibrium exists in every game.

In our proof, we used Brouwer's fixed point theorem. In this presentation, we explain Brouwer's theorem, and give an illustration of Nash's proof.

We proceed to prove Brouwer's Theorem using a combinatorial lemma whose proof we also provide, called Sperner's Lemma.

Brouwer's Fixed Point Theorem

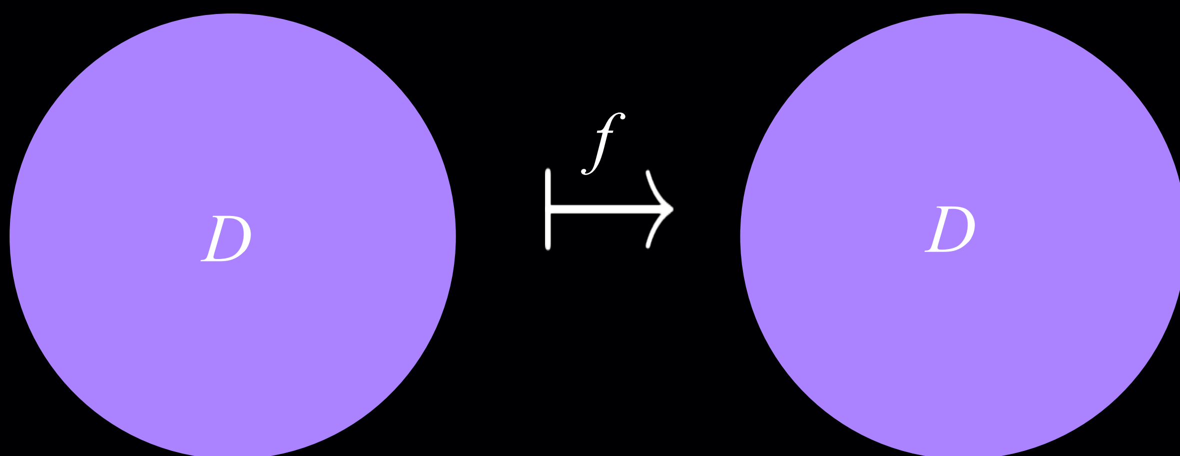
Brouwer's fixed point theorem

Theorem: Let $f : D \longrightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

Then there exists an $x \in D$ s.t. $x = f(x)$.

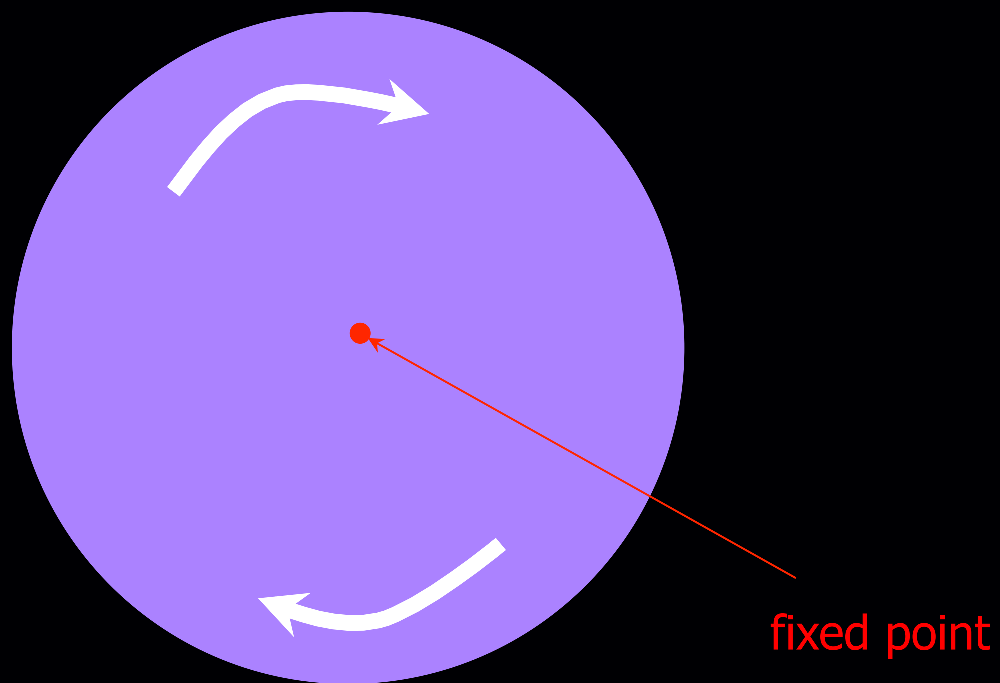
closed and bounded

Below we show a few examples, when D is the 2-dimensional disk.



N.B. All conditions in the statement of the theorem are necessary.

Brouwer's fixed point theorem



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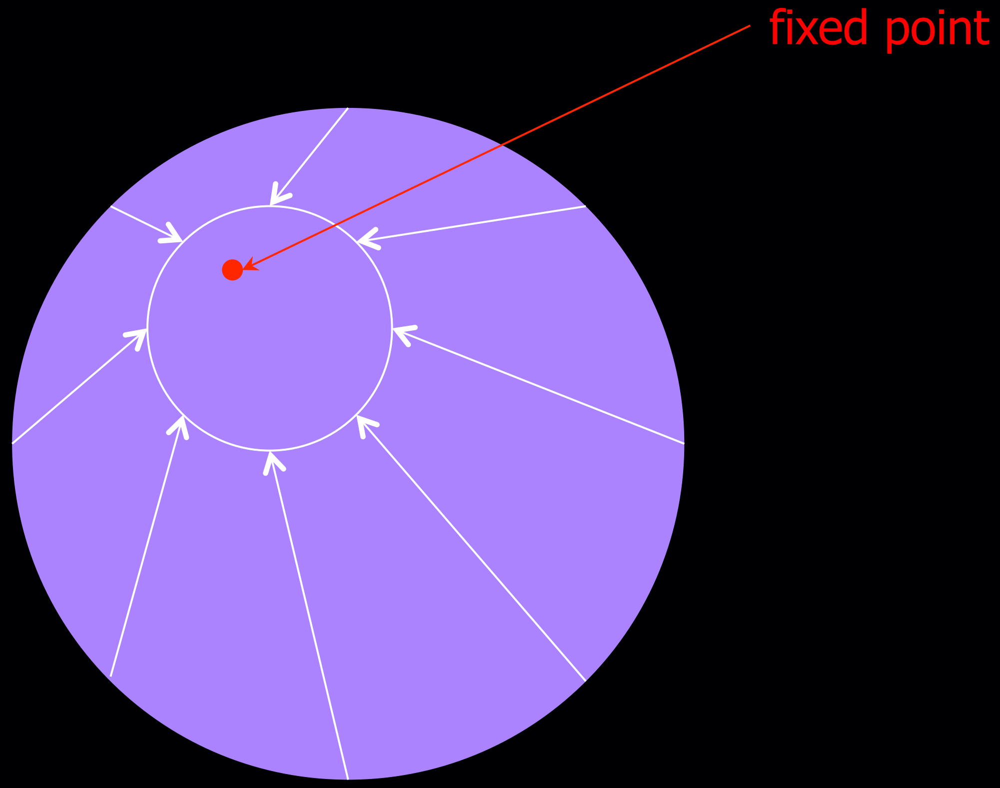
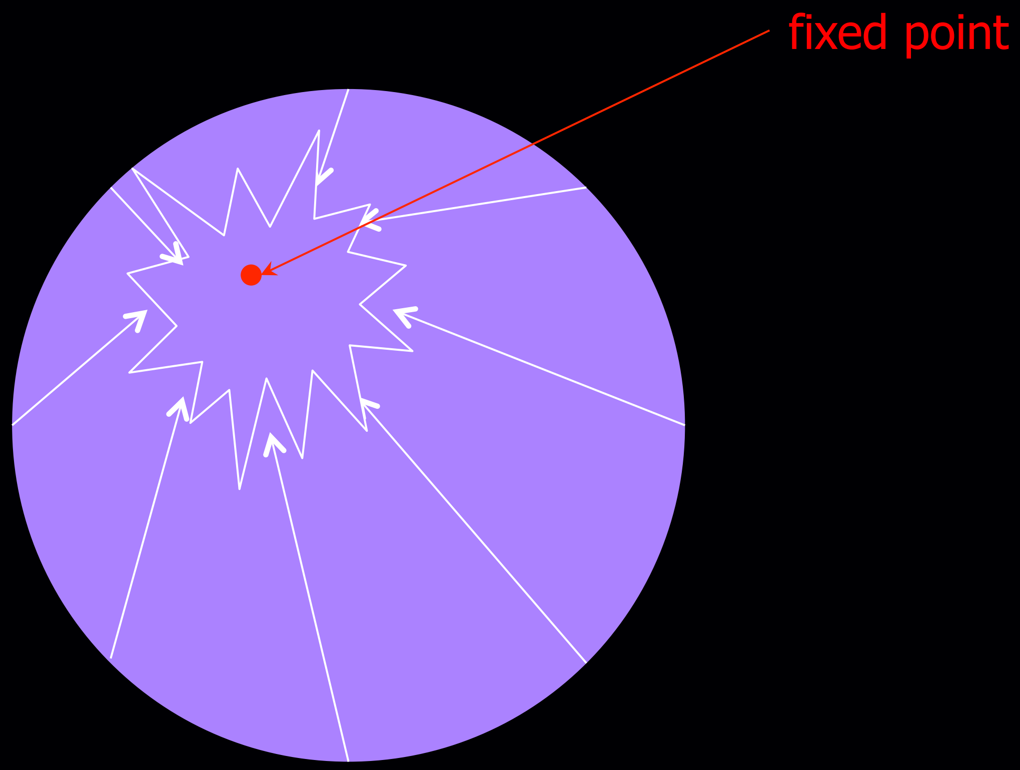


Brouwer’s fixed point theorem



Nash’s Proof

Visualizing Nash's Construction

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

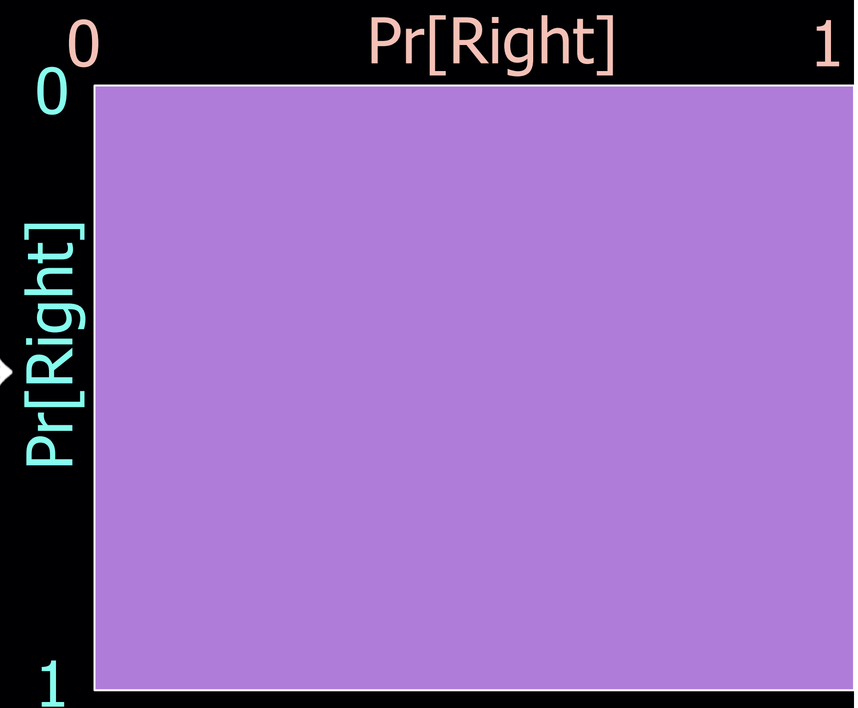
$\mapsto$

f: $[0,1]^2 \rightarrow [0,1]^2$, continuous
such that
fixed points $\equiv$ Nash eq.

Visualizing Nash's Construction

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game



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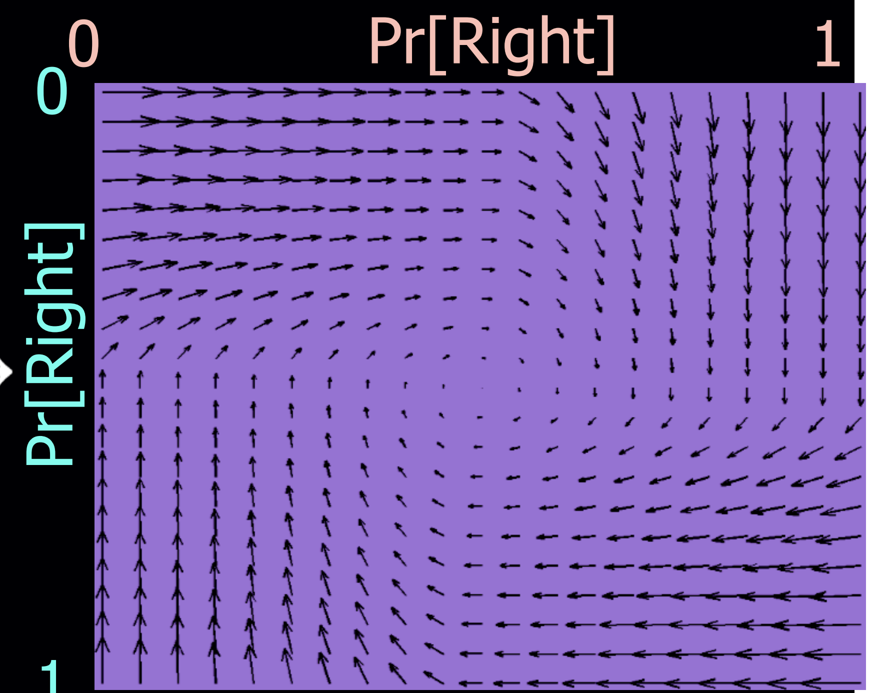

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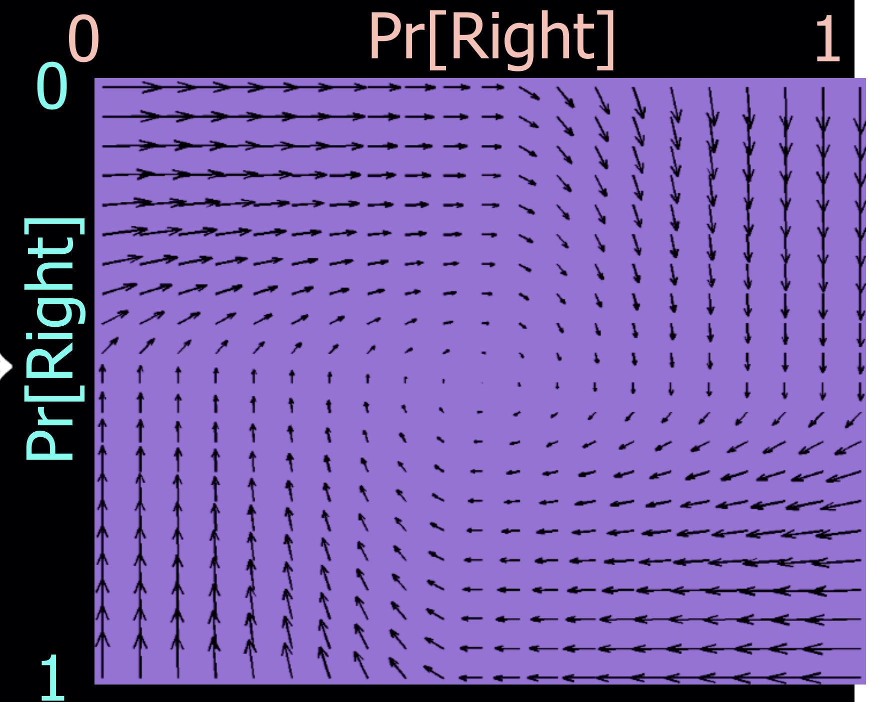
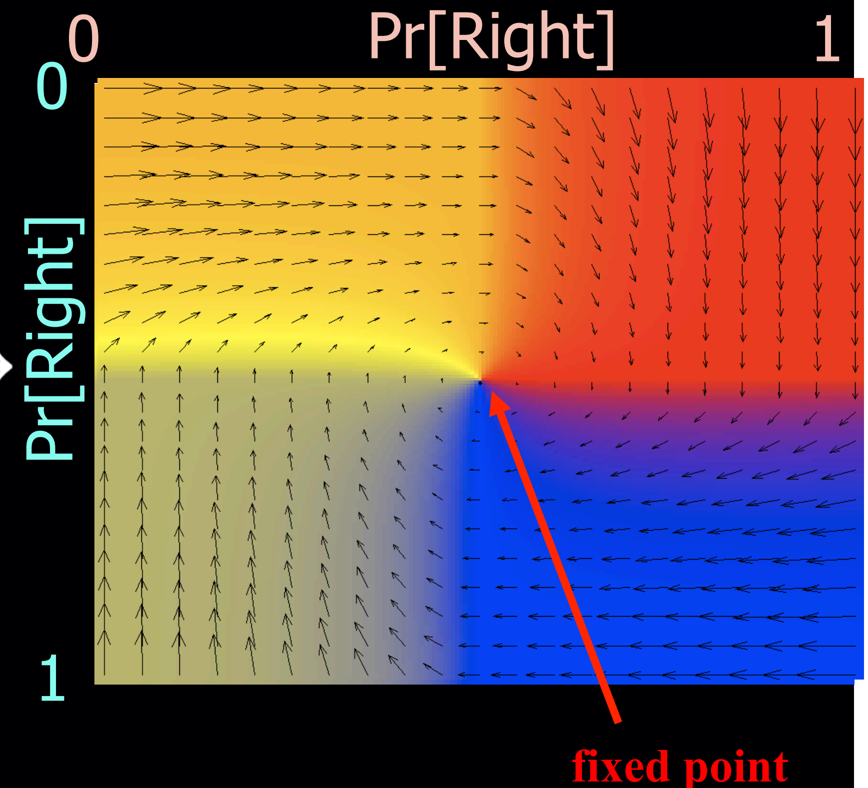


Visualizing Nash's Construction

		½	½
	Kick / Dive	Left	Right
½	Left	1 , -1	-1 , 1
½	Right	-1 , 1	1, -1

Penalty Shot Game

$\longmapsto$



Sperner's Lemma

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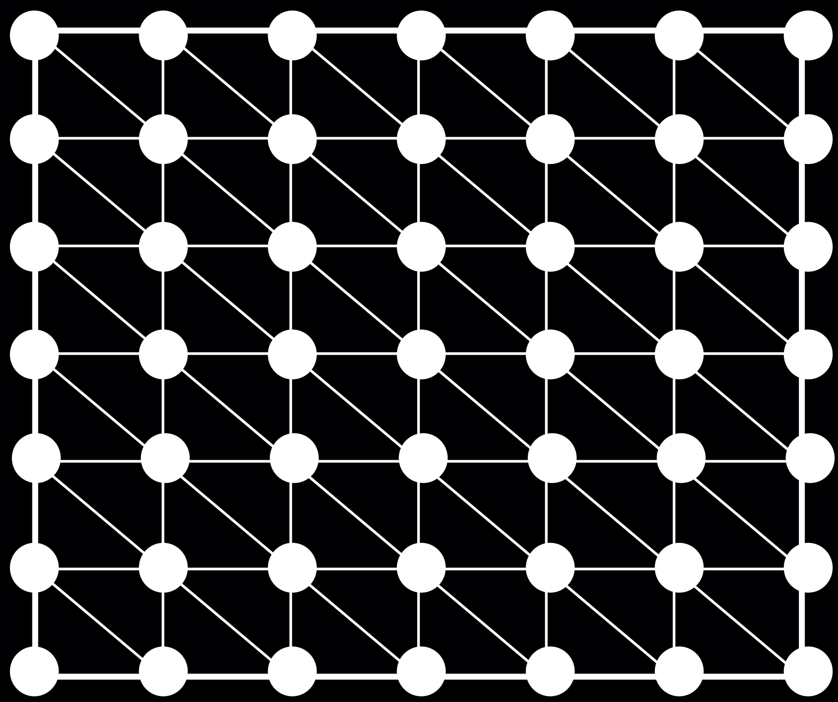

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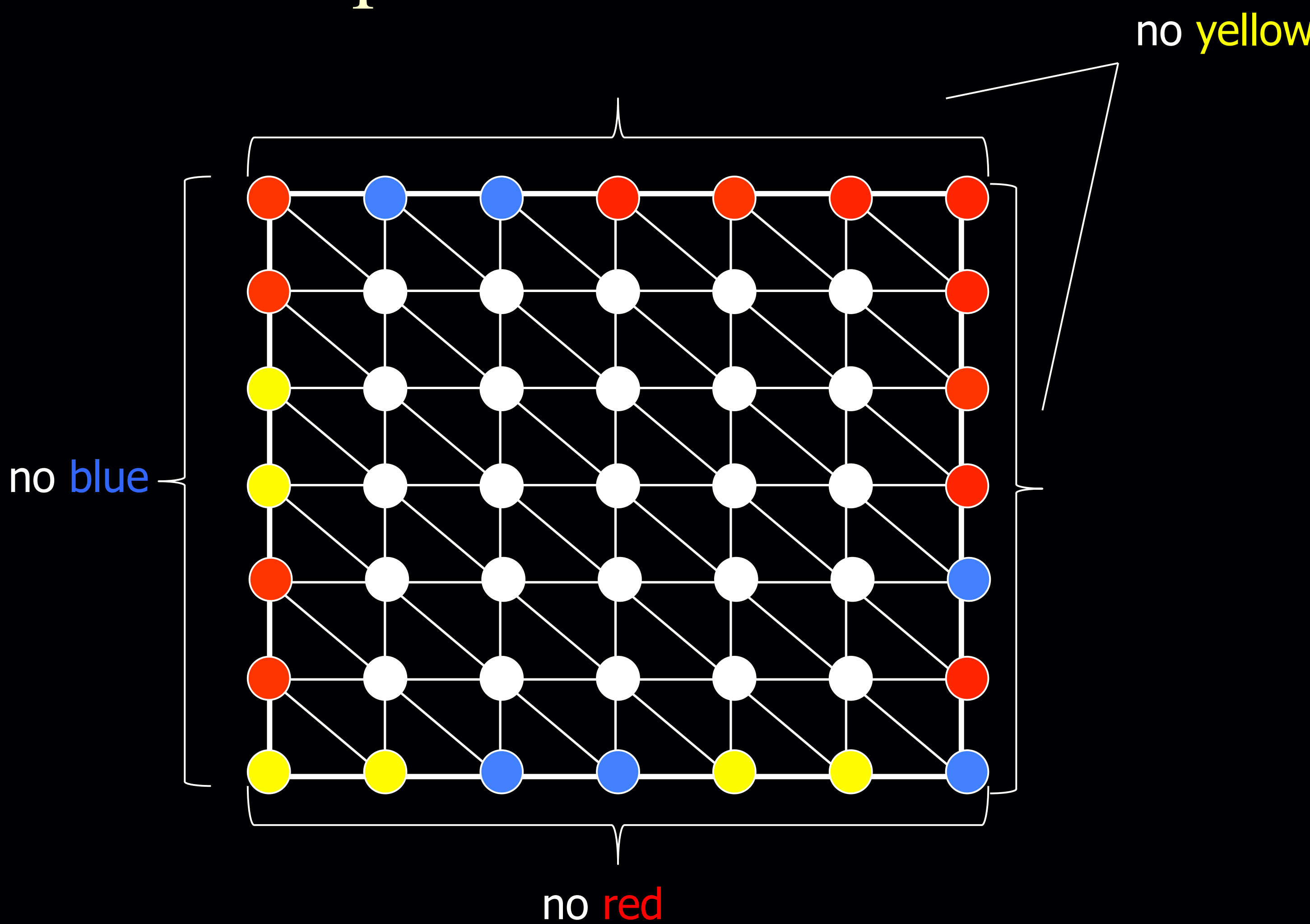
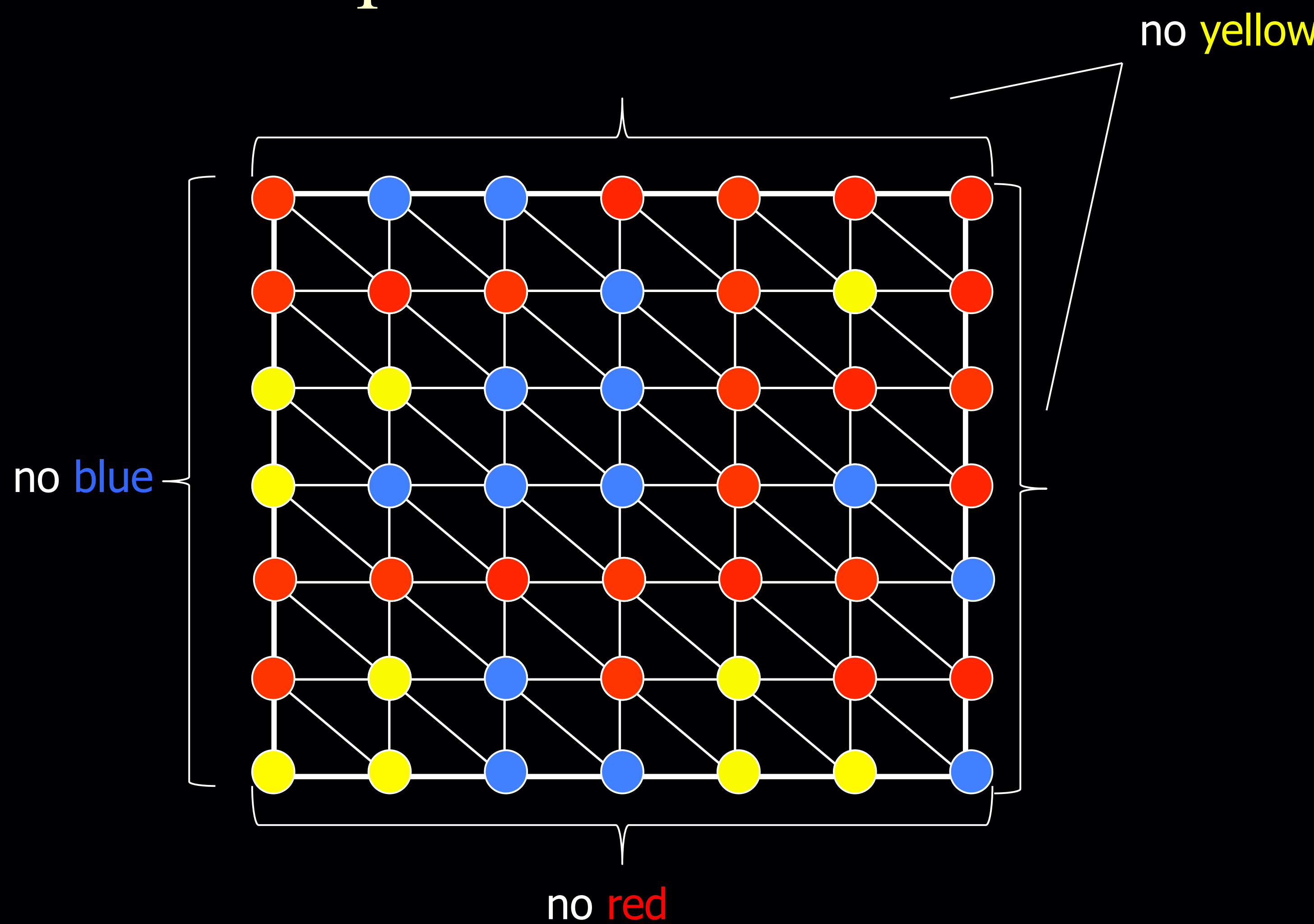


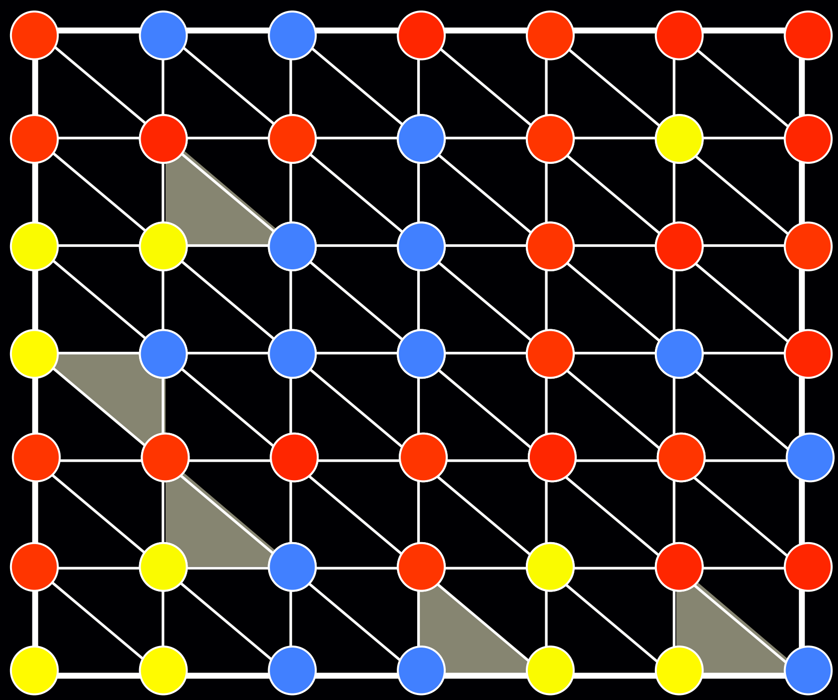
Lemma: Color the boundary using three colors in a legal way.

Sperner's Lemma



Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Sperner’s Lemma



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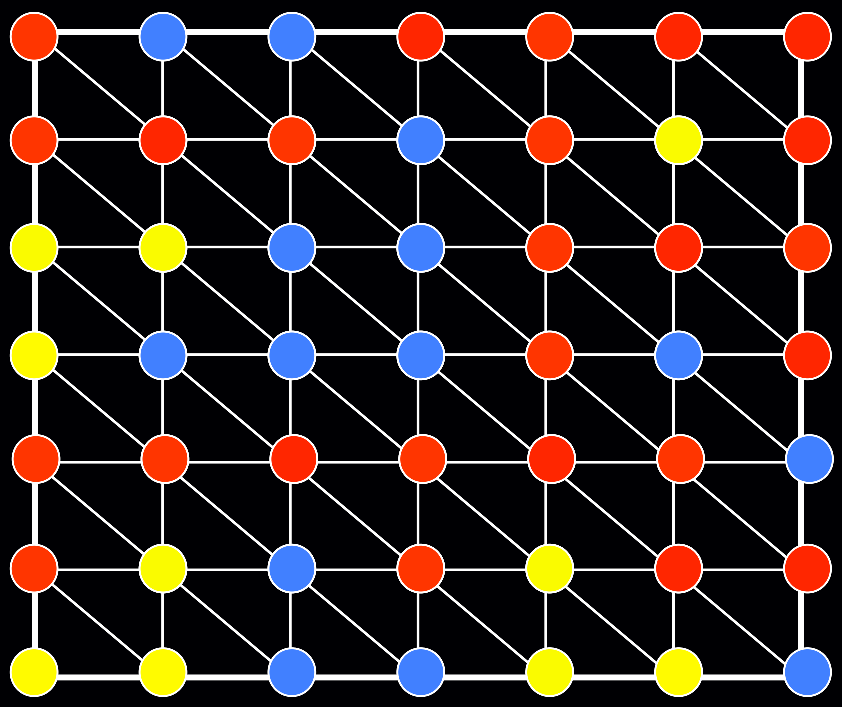
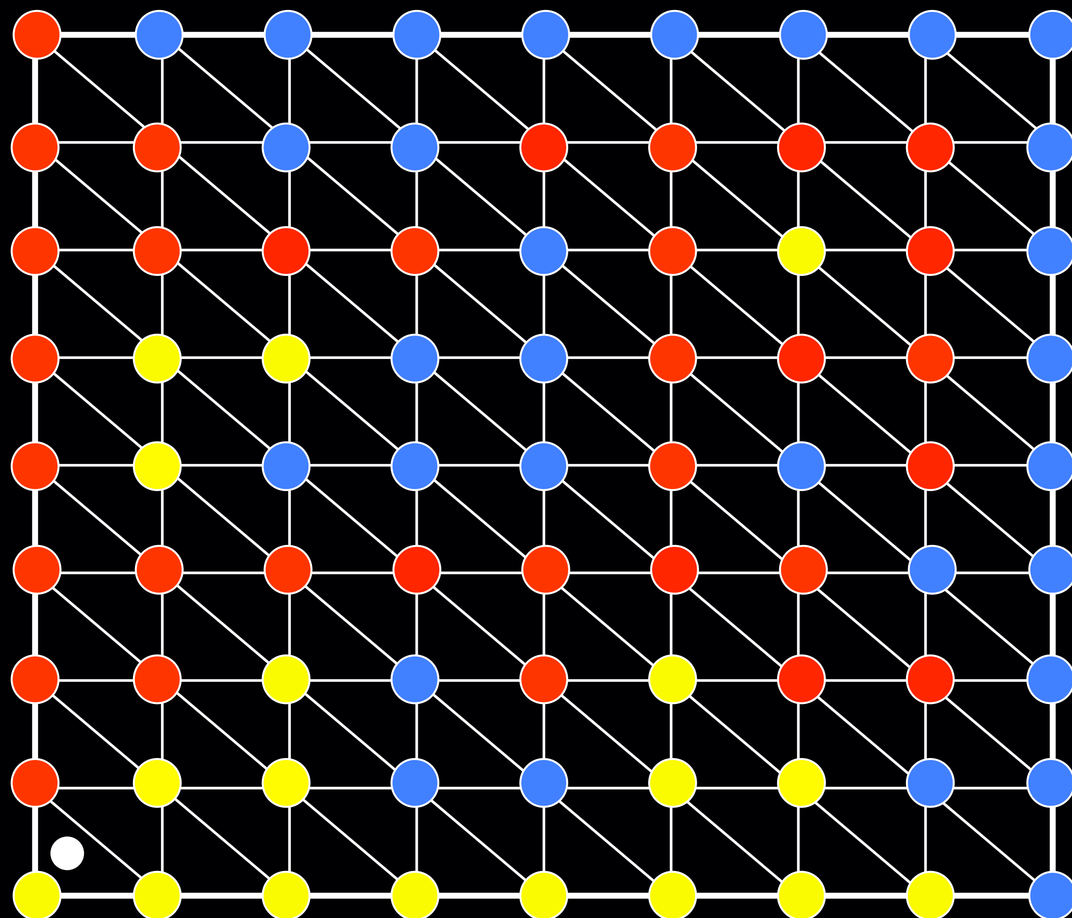
Sperner's Lemma



Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma



For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

Next we define a directed walk starting from the bottom-left triangle.

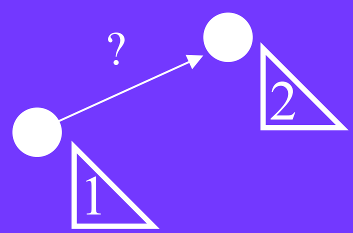
Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma

Space of Triangles

Transition Rule:

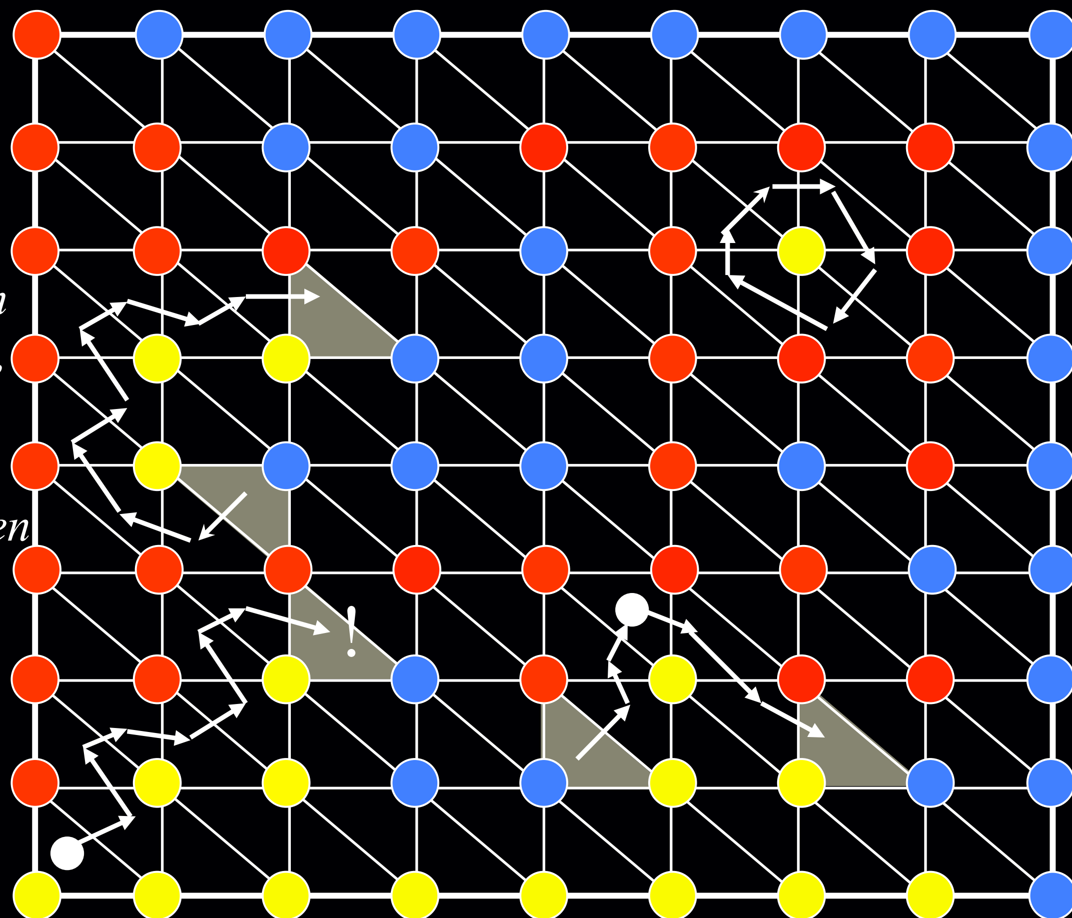
If $\exists$ ***red*** - ***yellow*** *door cross it with* ***red*** *on your left hand.*



Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma

Claim: *The walk cannot exit the square, nor can it loop around itself in a rho-shape. Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle…*



For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

Next we define a directed walk starting from the bottom-left triangle.

Starting from other triangles we do the same going forward or backward.

Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Brouwer's Fixed Point Theorem

We show that Sperner's Lemma implies Brouwer's Fixed Point Theorem. We start with the 2-dimensional Brouwer problem on the square.

2D-Brouwer on the Square

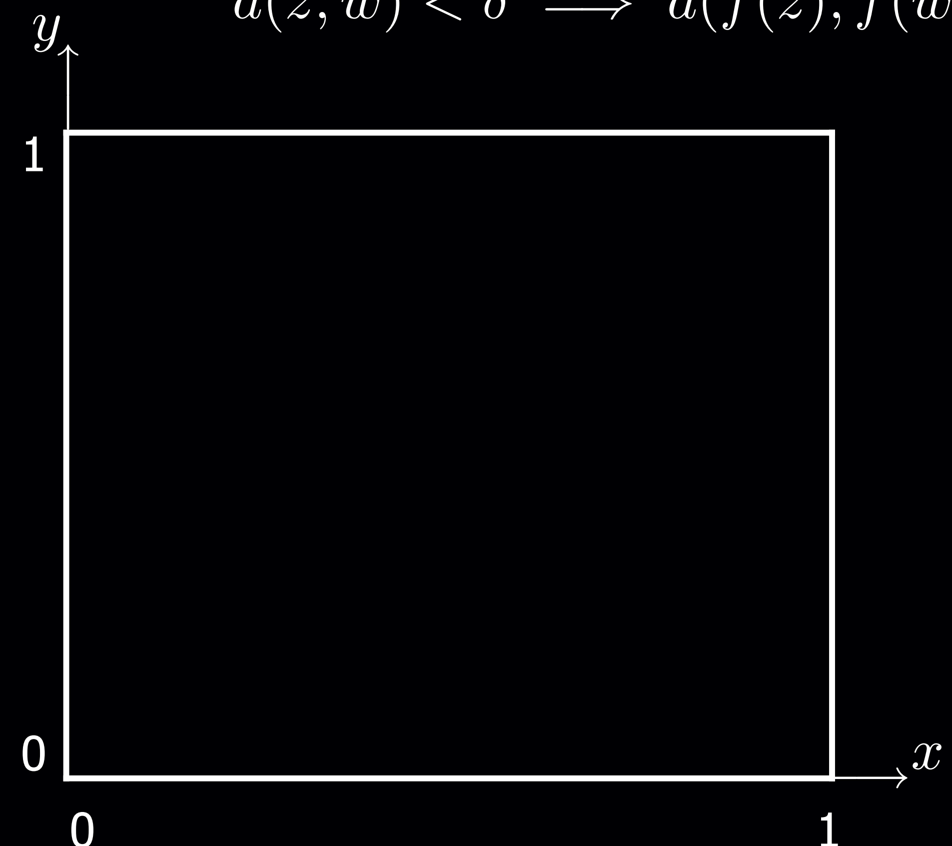
say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

→ must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0,\ \exists \delta = \delta(\epsilon) > 0,\ \text{s.t.}$$

$$d(z,w) < \delta \implies d(f(z), f(w)) < \epsilon$$



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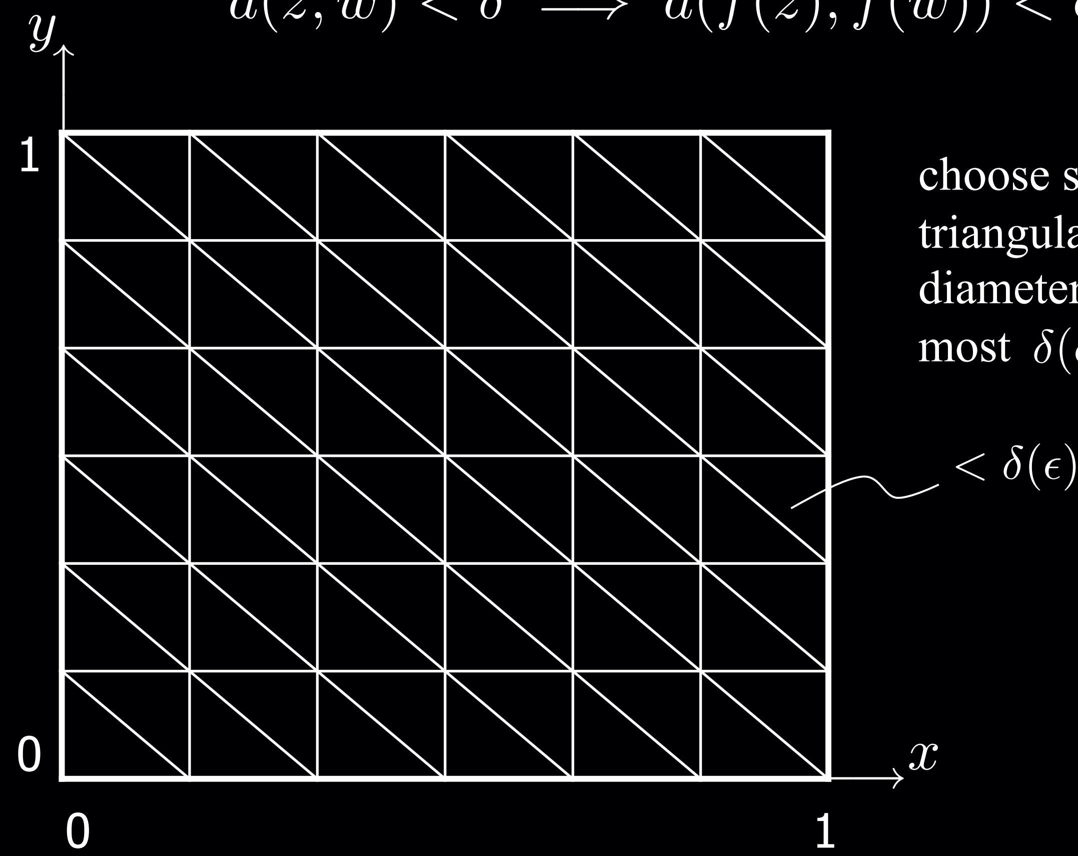

choose some ϵ and triangulate so that the diameter of cells is at most $\delta(\epsilon)$

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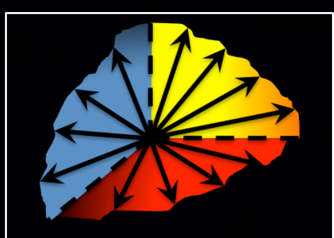
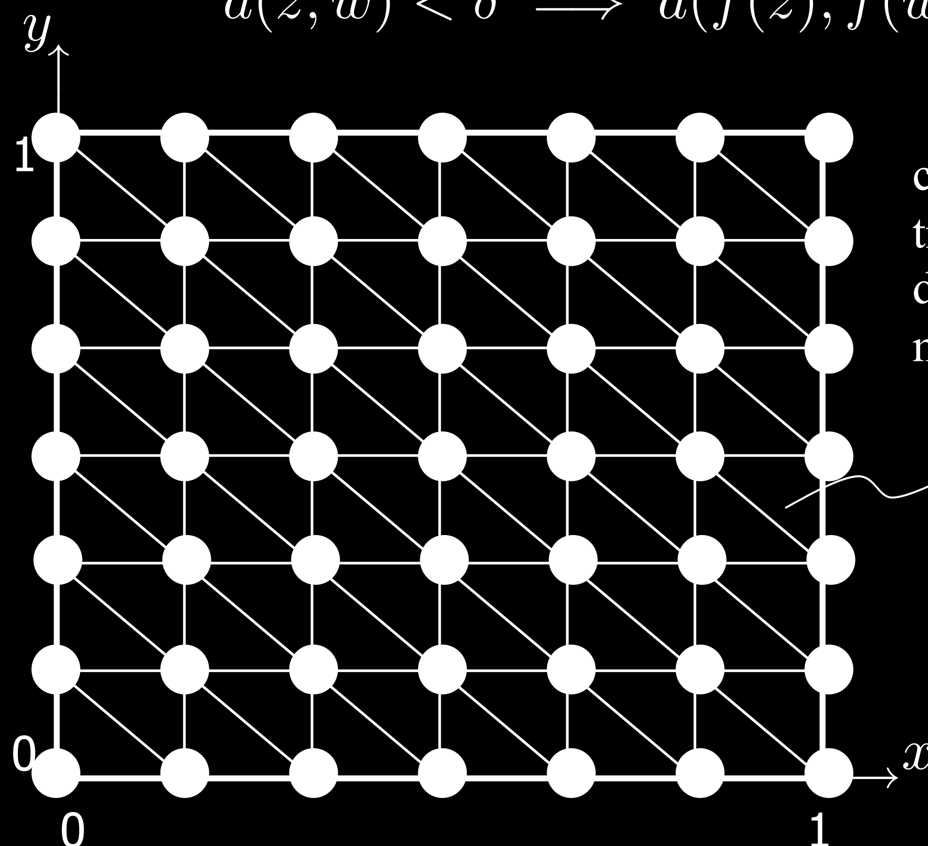
$$d(z, w) < \delta \implies d(f(z), f(w)) < \epsilon$$

color the nodes of the triangulation according to the direction of

$$f(x) - x$$





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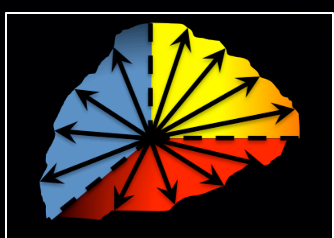
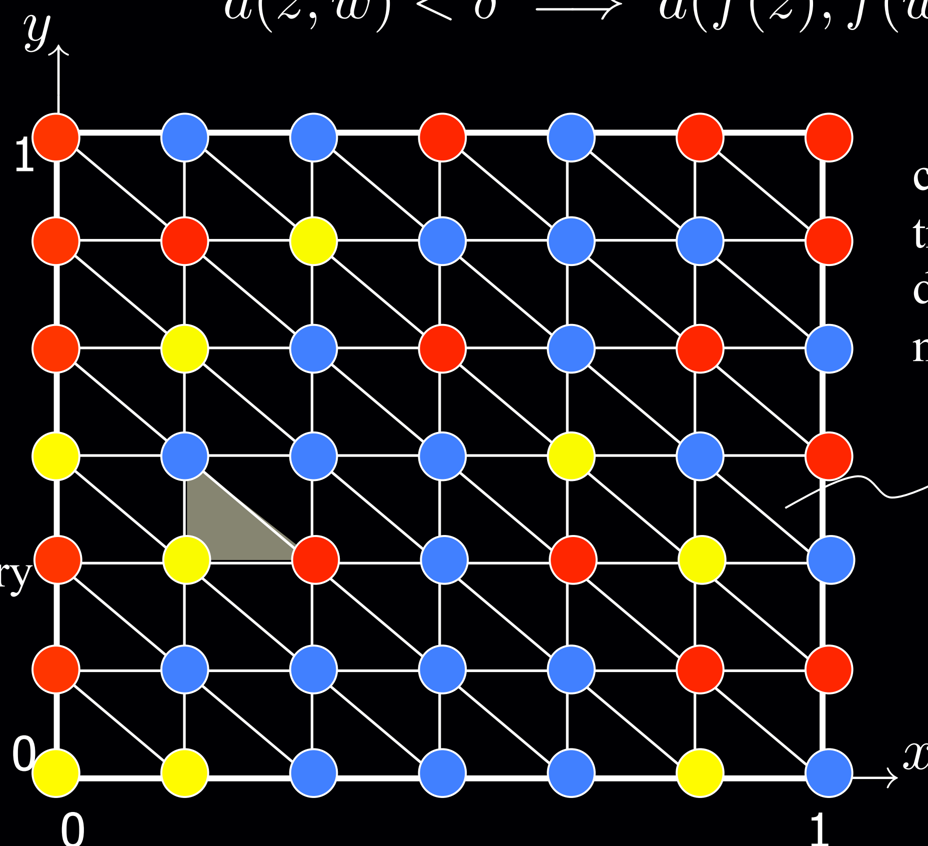


tie-break at the boundary angles, so that the resulting coloring respects the boundary conditions required by Sperner's lemma



choose some ϵ and triangulate so that the diameter of cells is at most $\delta(\epsilon)$

$< \delta(\epsilon)$

find a trichromatic triangle, guaranteed by Sperner

2D-Brouwer on the Square

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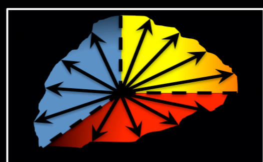
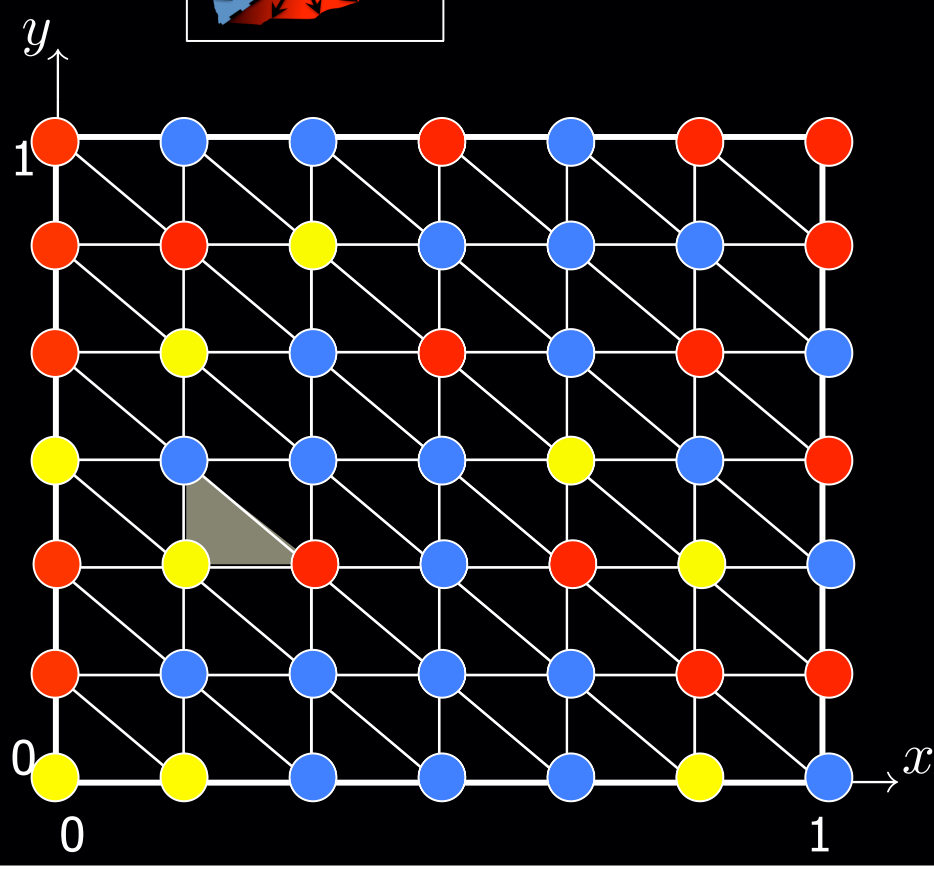




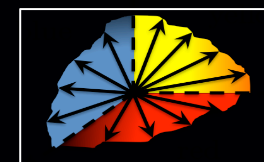
Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then

$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

Proof of Claim

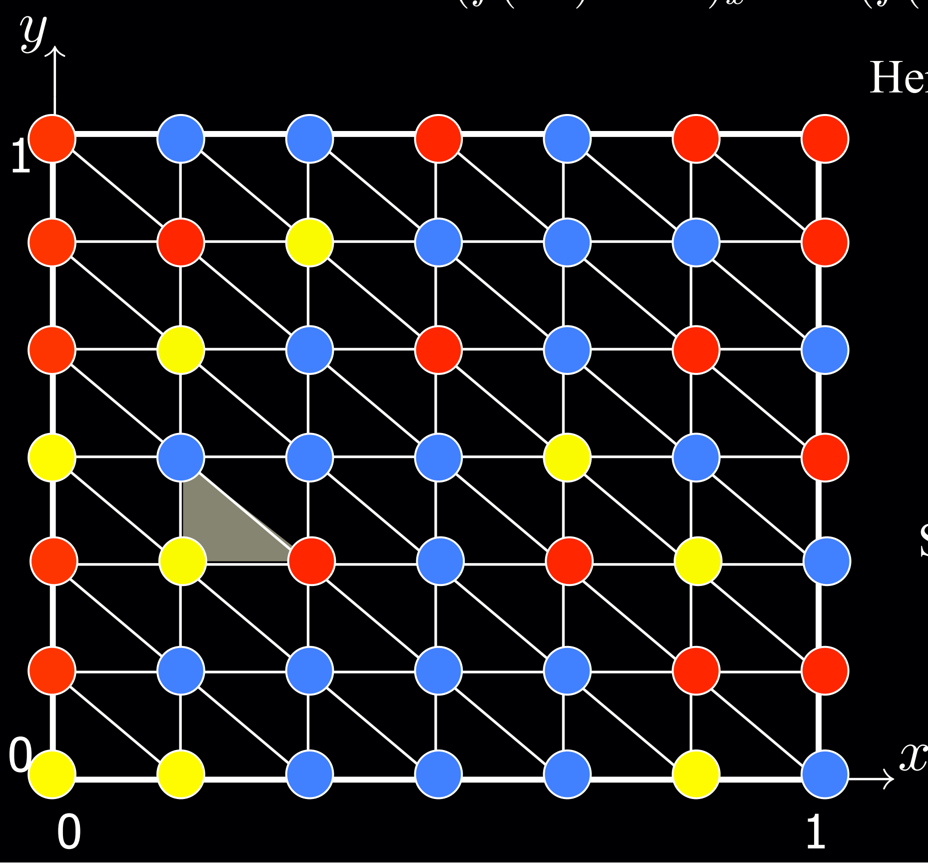
Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then $|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta$.

Proof: Let $z^{\mathrm{Y}}, z^{\mathrm{R}}, z^{\mathrm{B}}$ be the yellow/red/blue corners of a trichromatic triangle.



By the definition of the coloring, observe that the product of

$$(f(z^Y) - z^Y)_x \text{ and } (f(z^B) - z^B)_x \text{ is } \leq 0.$$



Hence:

$$\begin{aligned}
&|(f(z^Y) - z^Y)_x| \\
&\quad \leq |(f(z^Y) - z^Y)_x - (f(z^B) - z^B)_x| \\
&\quad \leq |(f(z^Y) - f(z^B))_x| + |(z^Y - z^B)_x| \\
&\quad \leq d(f(z^Y), f(z^B)) + d(z^Y, z^B) \\
&\quad \leq \epsilon + \delta.
\end{aligned}$$

Similarly, we can show:

$$|(f(z^Y) - z^R)_y| \leq \epsilon + \delta.$$

2D-Brouwer on the Square

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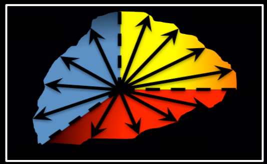

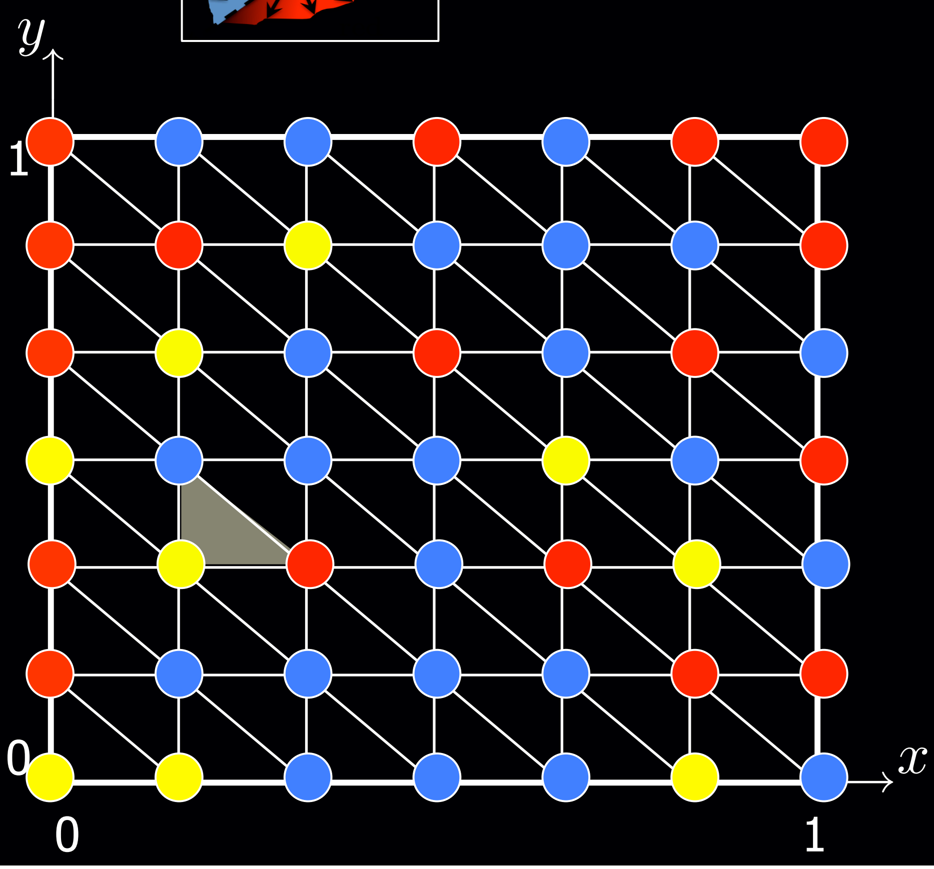

Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then

$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

choosing $\delta = \min(\delta(\epsilon), \epsilon)$

$$|f(z^Y) - z^Y|_\infty < 2\epsilon.$$

2D-Brouwer on the Square

Finishing the proof of Brouwer's Theorem:

- pick a sequence of epsilons: $\epsilon_i = 2^{-i}, i = 1, 2, \ldots$
- define a sequence of triangulations of diameter: $\delta_i = \min(\delta(\epsilon_i), \epsilon_i), i = 1, 2, \ldots$
- pick a trichromatic triangle in each triangulation, and call its yellow corner $z_i^{\mathrm{Y}}, i = 1, 2, \ldots$
- by compactness, this sequence has a converging subsequence $w_i, \quad i = 1, 2, \ldots$ with limit point w^*

Claim: $f(w^*) = w^*$.

Proof: Define the function $g(x) = d(f(x), x)$. Clearly, g is continuous since $d(\cdot, \cdot)$ is continuous and so is f. It follows from continuity that

$$g(w_i) \longrightarrow g(w^*), \text{ as } i \rightarrow +\infty.$$

But $0 \leq g(w_i) \leq 2^{-i+1}$. Hence, $g(w_i) \longrightarrow 0$. It follows that $g(w^*) = 0$.

Therefore, $d(f(w^*), w^*) = 0 \implies f(w^*) = w^*$. ■