6.896: Topics in Algorithmic Game Theory

Lecture 6

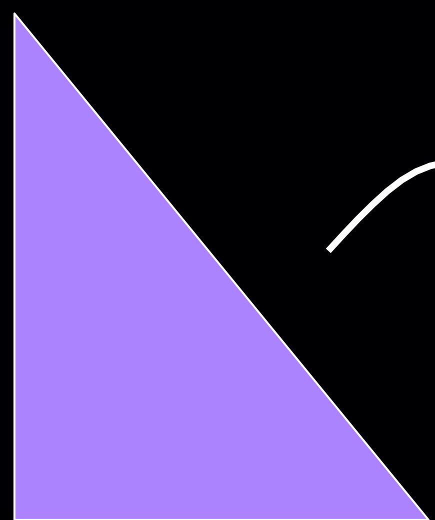
Constantinos Daskalakis

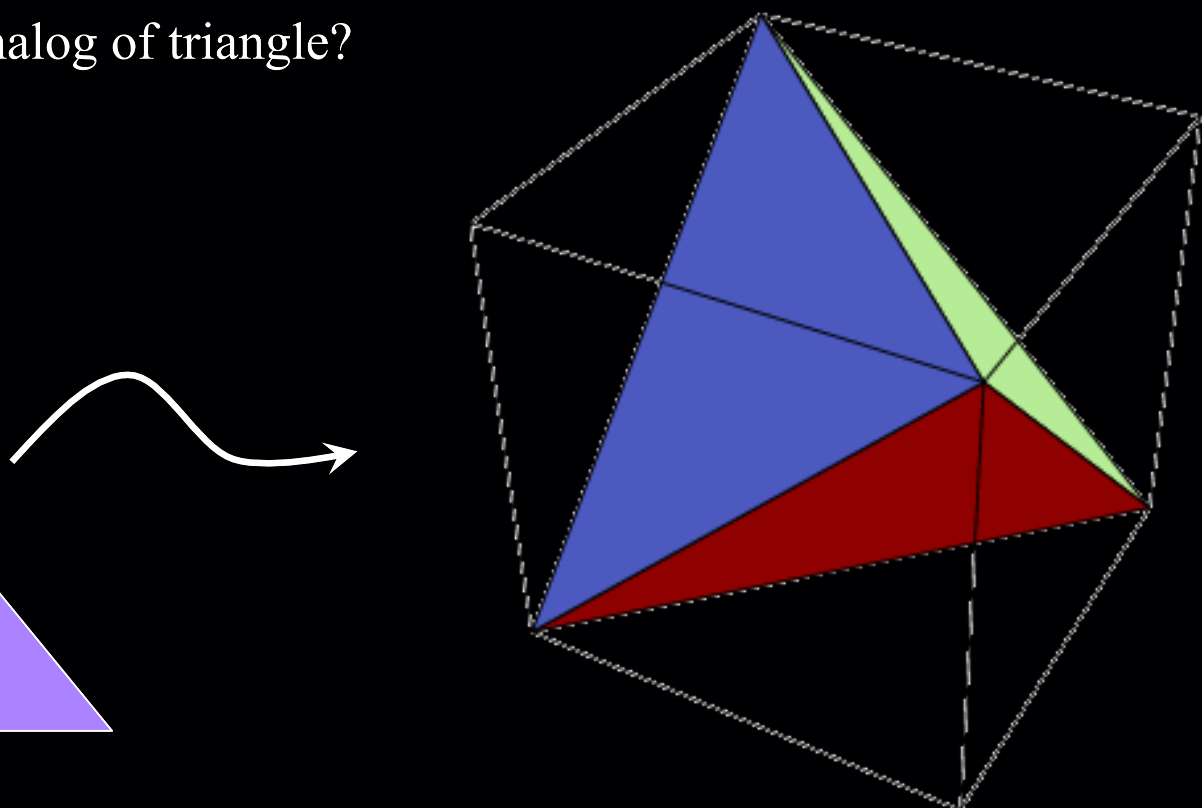
Sperner's Lemma in n dimensions

A. Canonical Triangulation of $[0,1]^n$

Triangulation

High-dimensional analog of triangle?



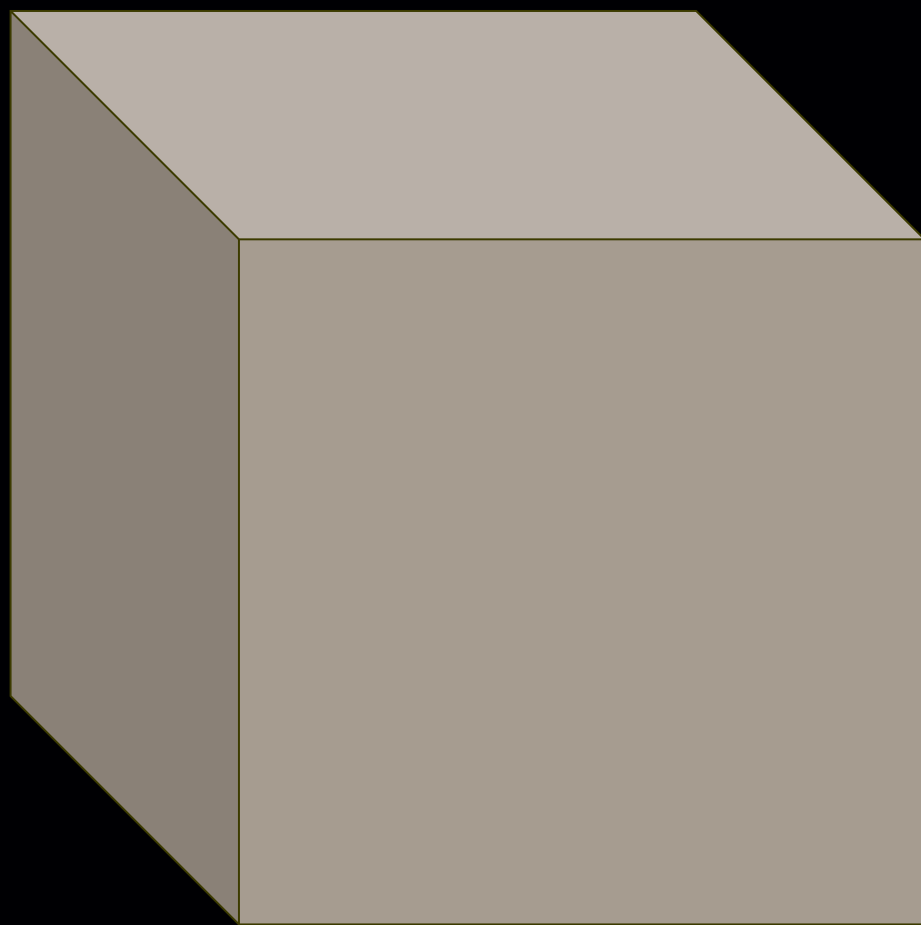


in 2 dimensions: a triangle

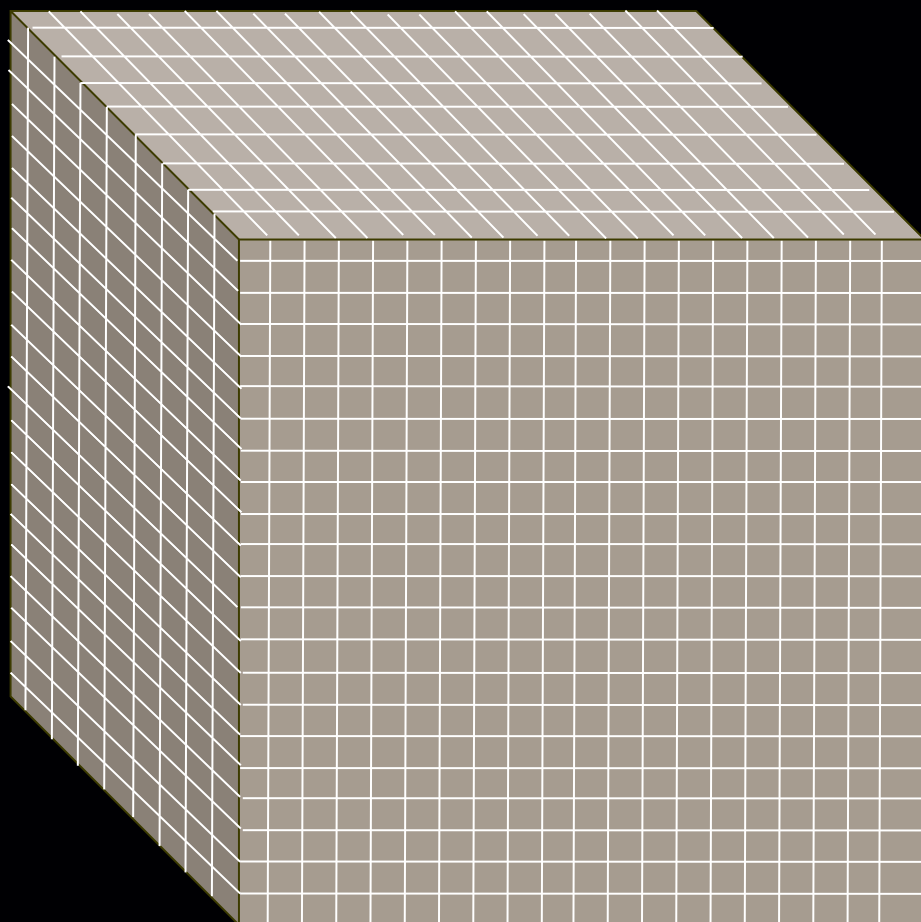
in *n* dimensions: an *n*-simplex

i.e. the convex hull of *n*+1 points in general position

Simplicization of $[0,1]^n$?



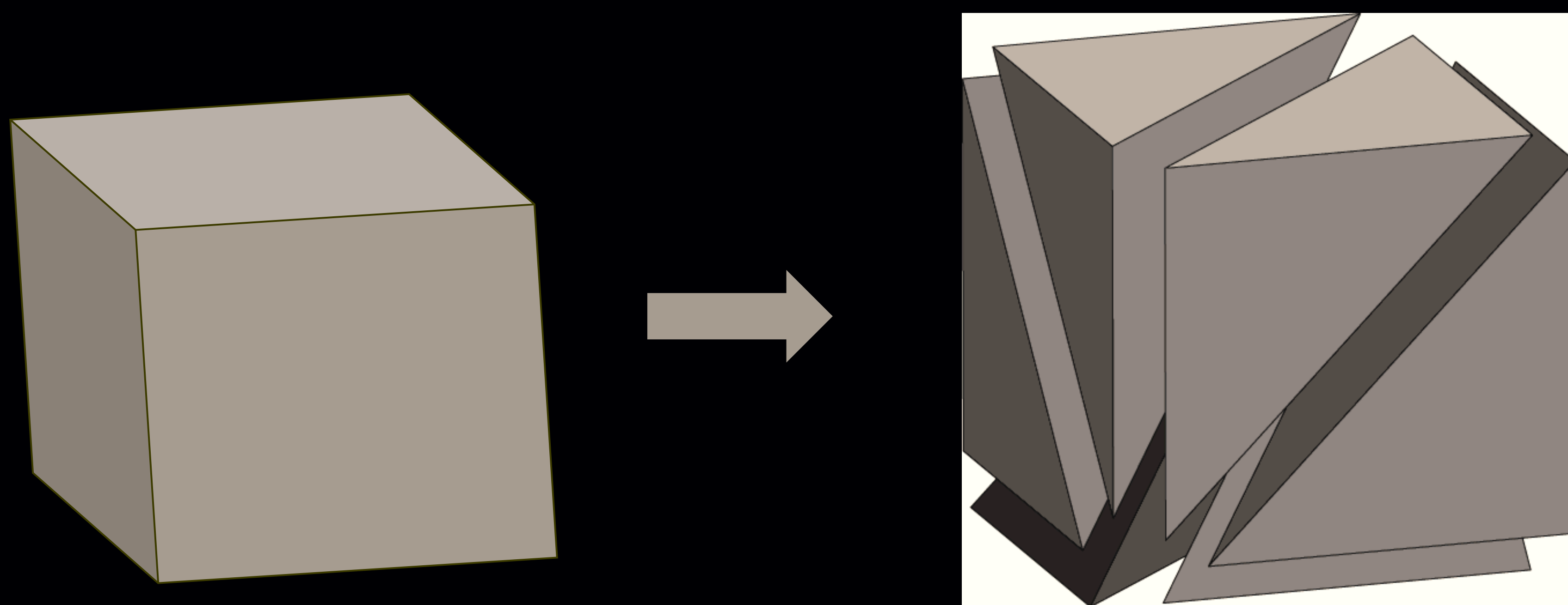
1st Step: Division into Cubelets



Divide each dimension into integer multiples of 2^{-m}, for some integer m.

2nd Step: Simplicization of each Cubelet

in 3 dimensions…



note that all tetrahedra in this division use the corners 000 and 111 of the cube

Generalization to n-dimensions

For a permutation $\pi : [n] \to [n]$ of the coordinates, define:

$$\mathcal{T}_\pi := \left\{ x \in [0,1]^n \mid x_{\pi(1)} \leq x_{\pi(2)} \leq \ldots x_{\pi(n)} \right\}$$

Claim 1: The unique integral corners of $\mathcal{T}_\pi$ are the following *n*+1 points:

	$x_{\pi(1)}$	$x_{\pi(2)}$	$\ldots$	$x_{\pi(n-2)}$	$x_{\pi(n-1)}$	$x_{\pi(n)}$
$v_1^\pi =$	0	0	...	0	0	0
$v_2^\pi =$	0	0	...	0	0	1
$v_3^\pi =$	0	0	...	0	1	1
$v_4^\pi =$	0	0	...	1	1	1
$\vdots$						
$v_{n+1}^\pi =$	1	1	...	1	1	1

Simplicization

Claim 2: $\mathcal{T}_\pi$ is a simplex.

Claim 3: $\bigcup_\pi \mathcal{T}_\pi = [0,1]^n$.

Theorem: $\{\mathcal{T}_\pi\}_\pi$ is a triangulation of $[0,1]^n$.

Apply the above simplicization to each cubelet.

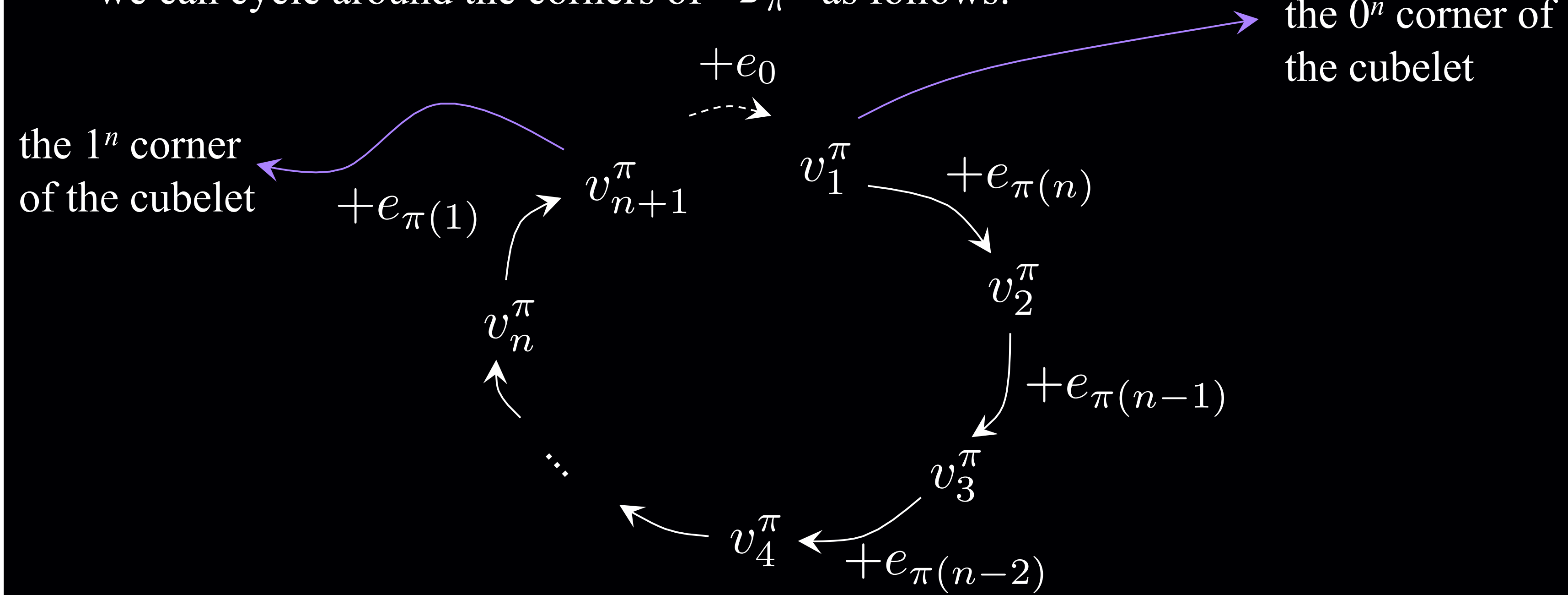
Claim 4: If two cubelets share a face, their simplicizations agree on a common simplicization of the face.

Cycle of a Simplex

Letting $e_i, i = 1, \ldots, n,$ denote the unit vector along dimension i, and

$$e_0 = (-1, -1, \ldots, -1)$$

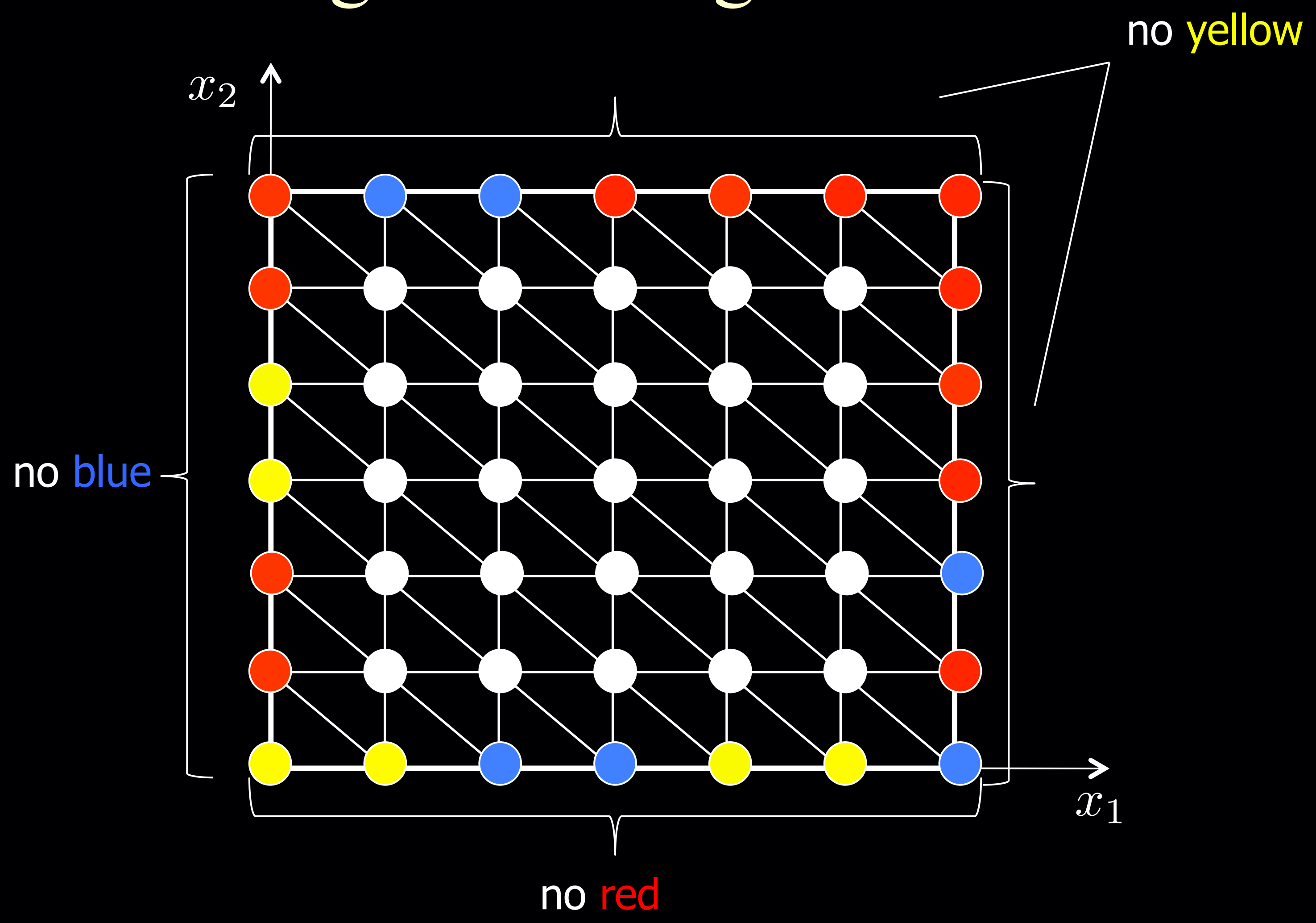
we can cycle around the corners of $\mathcal{T}_\pi$ as follows:



Claim: Hamming weight is increasing from v_1^π to v_{n+1}^π.

B. Legal Coloring

Legal Coloring in 2-d



Legal Coloring

2-dimensional Sperner

↳ 3 colors: blue (1), red (2), yellow (0)

↳ (P_2): None of the vertices on the left (x_1=0) side of the square uses blue, no vertex on the bottom side (x_2=0) uses red, and no vertex on the other two sides uses yellow.

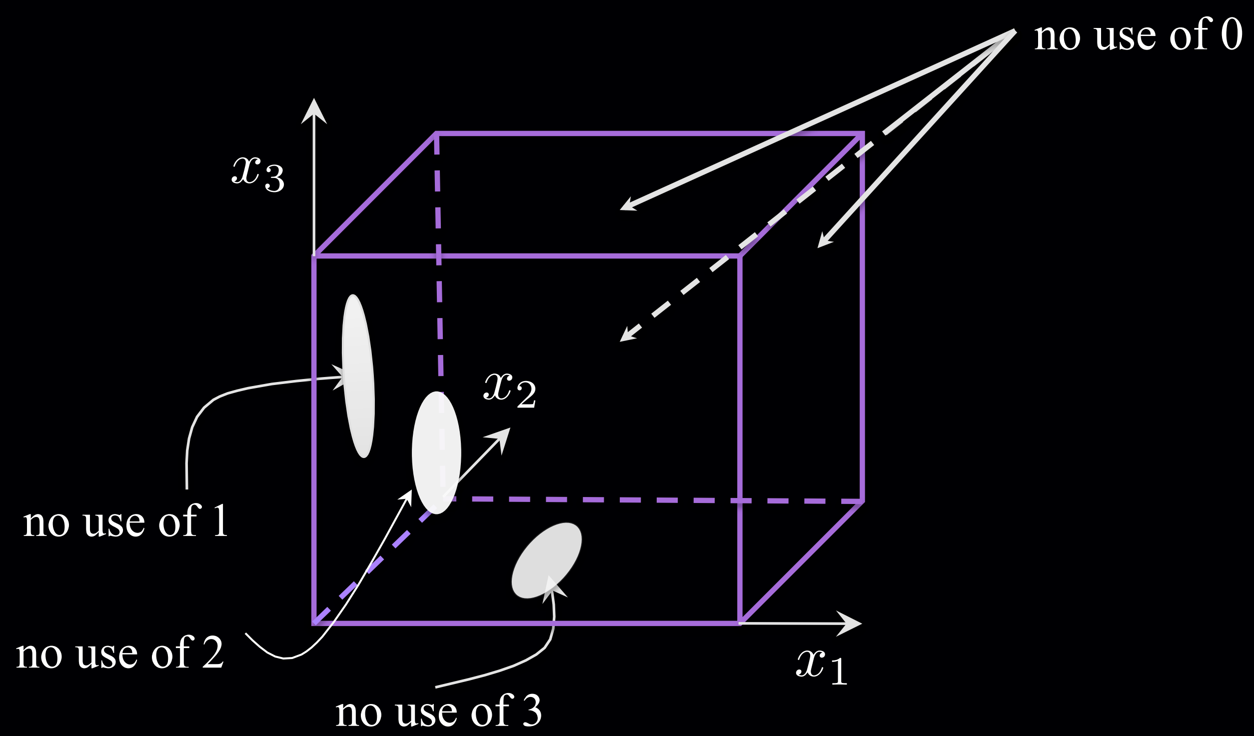
⬇

n-dimensional Sperner

↳ *n* colors: 0, 1, …, *n*

↳ (P_n): For all $i \in \{1,\dots, n\}$, none of the vertices on the face $x_i = 0$ of the hypercube uses color *i*; moreover, color 0 is not used by any vertex on a face $x_i = 1$, for some $i \in \{1,\dots, n\}$.

Legal Coloring (3-d)



C. Statement of Sperner's Lemma

Sperner's Lemma

Theorem [Sperner 1928]:

Suppose that the vertices of the canonical simplicization of the hypercube $[0,1]^n$ are colored with colors 0,1, …, n so that the following property is satisfied by the coloring on the boundary.

(P_n): For all $i \in \{1,\ldots, n\}$, none of the vertices on the face $x_i = 0$ uses color i; moreover, color 0 is not used by any vertex on a face $x_i = 1$, for some $i \in \{1,\ldots, n\}$.

Then there exists a panchromatic simplex in the simplicization. In fact, there is an odd number of those.

pan: from ancient Greek πᾶν = all, every

chromatic: from ancient Greek χρῶμα.= color

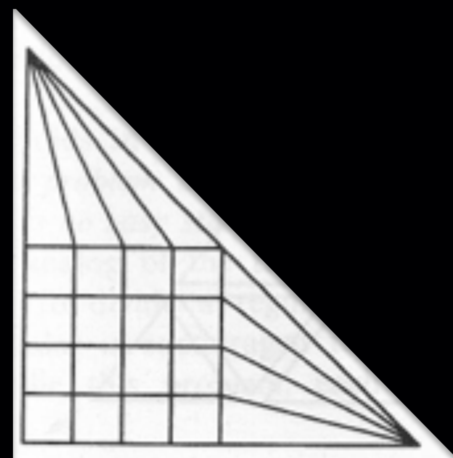
Remarks:

1. We need not restrict ourselves to the canonical simplicization of the hypercube shown above (that is, divide the hypercube into cubelets, and divide each cubelet into simplices in the canonical way shown above). The conclusion of the theorem is true for any partition of the cube into *n*-simplices, as long as the coloring on the boundary satisfies the property stated above.

   The reason we state Sperner's lemma in terms of the canonical triangulation is in an effort to provide an algorithmically-friendly version of the computational problem related to Sperner, in which the triangulation and its simplices are easy to define, the neighbors of a simplex can be computed efficiently etc. We follow-up on this in the next lecture. Moreover, our setup allows us to make all the steps in the proof of Sperner's lemma "constructive" (except for the length of the walk, see below).

2. Sperner's Lemma was originally stated for a coloring of a triangulation of the *n*-simplex, (rather than the cube shown above). In that setting, we color the vertices of any triangulation of the *n*-simplex---a convex combination of points $v_0, v_1, \ldots, v_n$ in general position---with *n* colors, 0,1,…,*n*, so that the facet not containing vertex v_i does not use color *i*. Then Sperner's lemma states that there exists a panchromatic simplex in the simplicization.

   Our coloring of the *n*-dimensional cube with *n*+1 colors is essentially mimicking the coloring of a simplex whose facets (except one) correspond to the facets of the cube around the corner 0^n, while the left-out facet corresponds to the "cap" of the hypercube around 1^n.

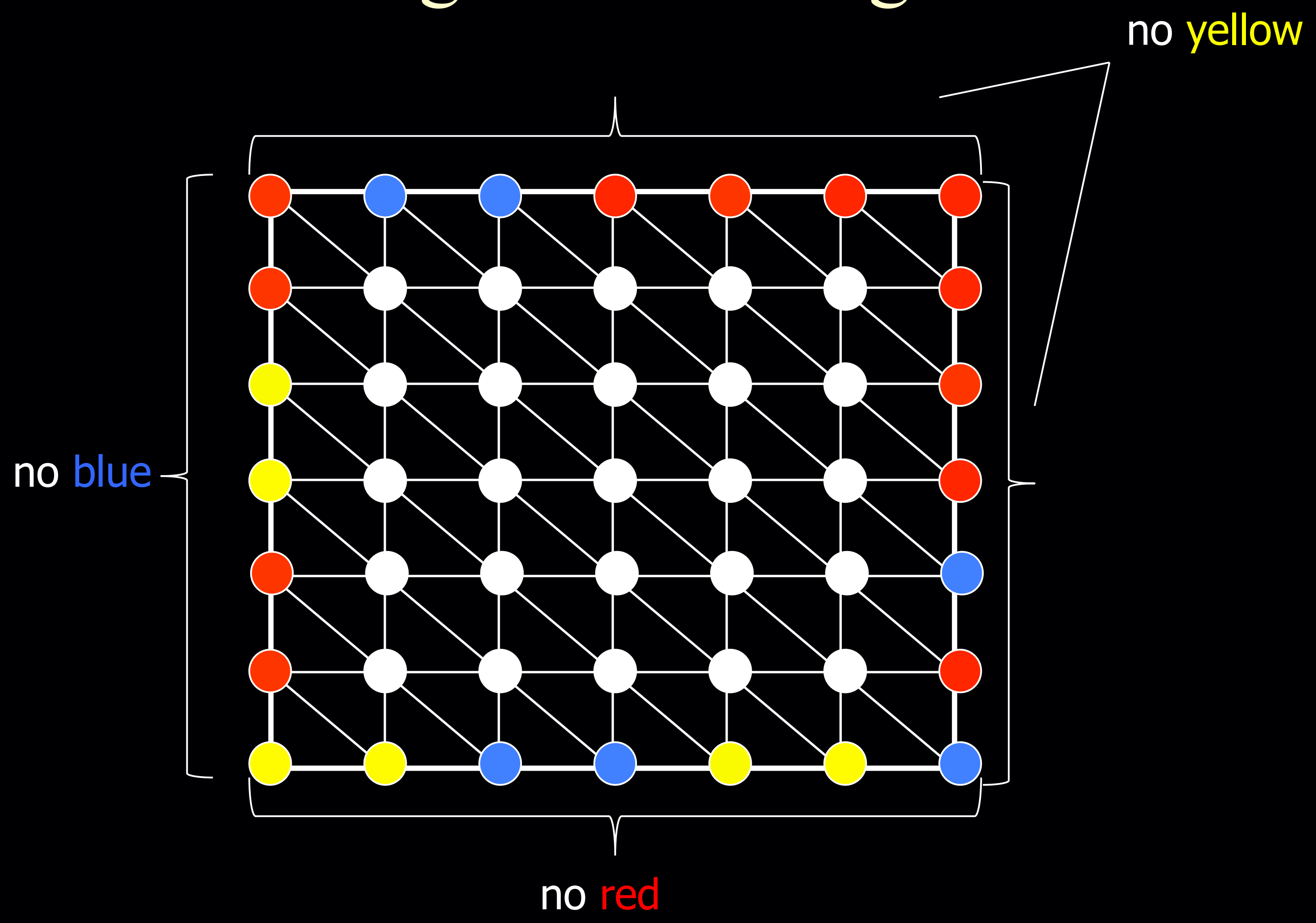


Proof of n-dimensional Sperner's Lemma

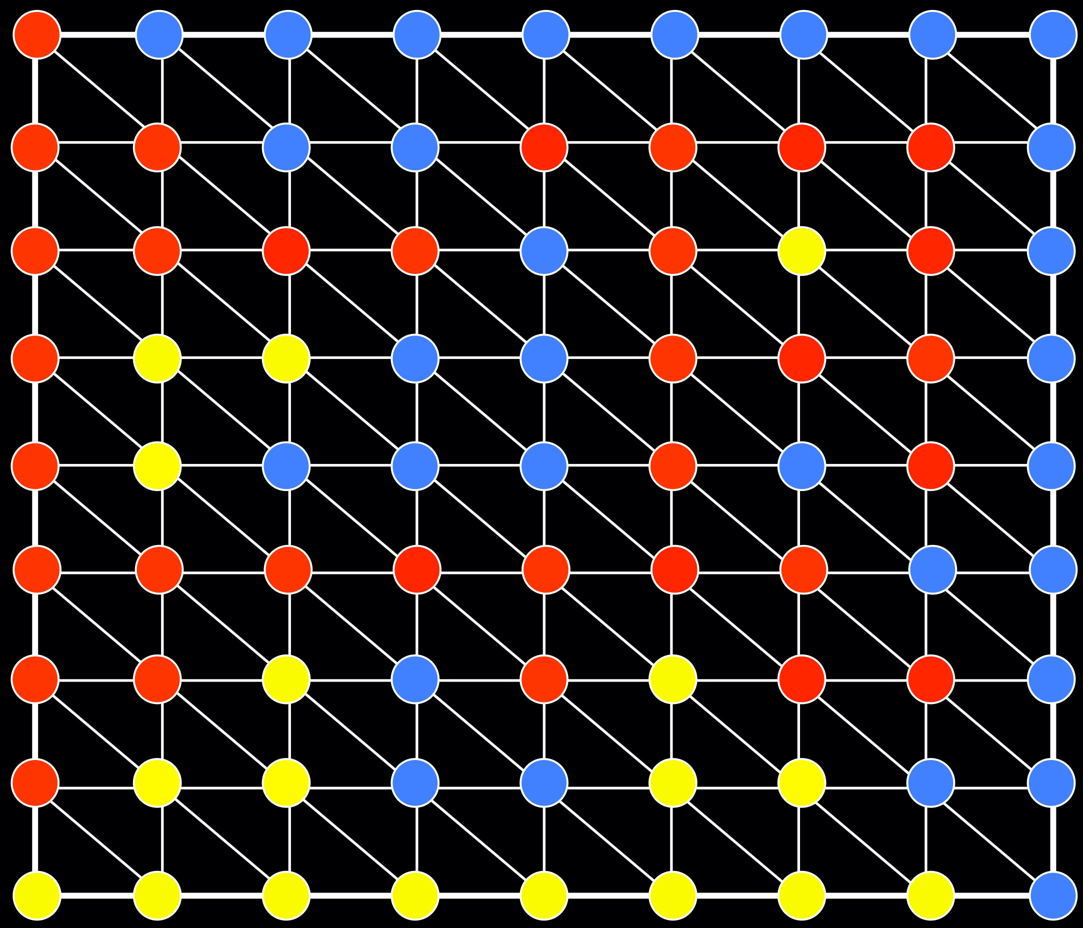
generalizing the proof of the 2-d case

1. Envelope Construction

Original Coloring

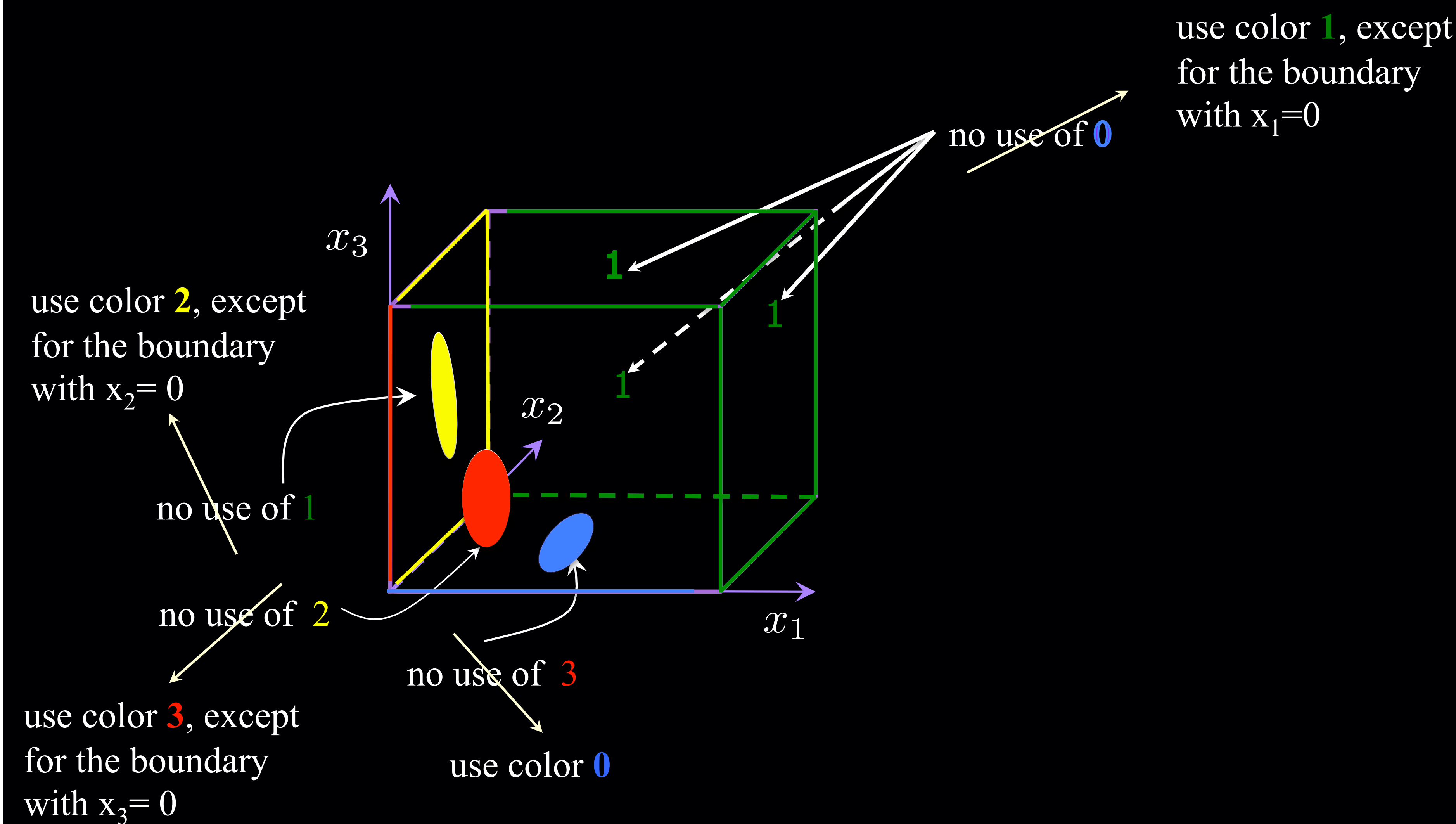


The Canonically Colored Envelope



For convenience introduce an outer boundary, that does not create new tri-chromatic triangles.

Envelope construction in 3-d



Envelope construction in Many Dimensions

Introduce an extra layer off of the boundary of the hypercube and color the vertices of this extra layer legally, but according to the very canonical/greedy way defined below:

where 0 is disallowed, color with 1, except for boundary with x_1=0;

where 1 is disallowed, color with 2, except for boundary with x_2=0;

$\vdots$

where i is disallowed, color with i+1, except for boundary with x_{i+1}=0;

$\vdots$

where n is disallowed, color with 0.

Claim: No new panchromatic tetrahedra were introduced during the envelope construction.

2. Definition of the Walk

Walk

Like we did in the 2-d case, we show that a panchromatic simplex exists by defining a walk that jumps from simplex to simplex of our simplicization, starting at some fixed simplex (independent of the coloring) and guaranteed to conclude at a panchromatic one.

- The simplices in our walk (except for the final one) will contain all the colors in the set {2, 3, …, n, 0}, but will be missing color 1. Call such simplices *colorful*.

- In particular, every such simplex will have exactly one color repeated twice. So it will contain exactly two *facets* with colors 2, 3,…, n, 0. Call these facets *colorful*.

- Our walk will be transitioning from simplex to simplex, by pivoting through a colorful facet.

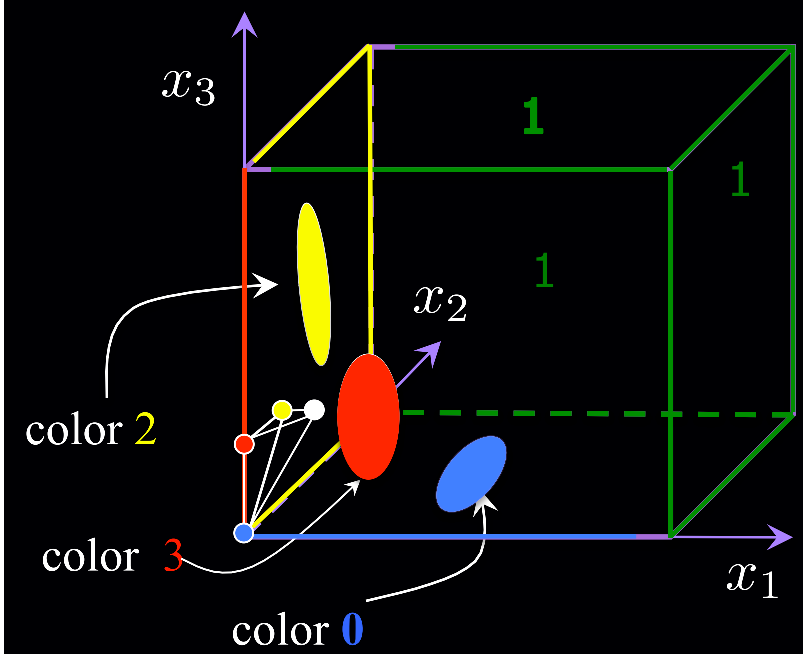
- When entering a new simplex through a colorful facet, there are two cases:

  -- either the other vertex has color 1, in which case a panchromatic simplex is found!

  -- or the other vertex has some color in {2, 3, …, n, 0}, in which case a new colorful facet is found and traversed, etc.

3. Starting Simplex

Starting Simplex



The starting simplex belongs to the cubelet adjacent to the 0^n vertex of the hypercube, and corresponds to the permutation

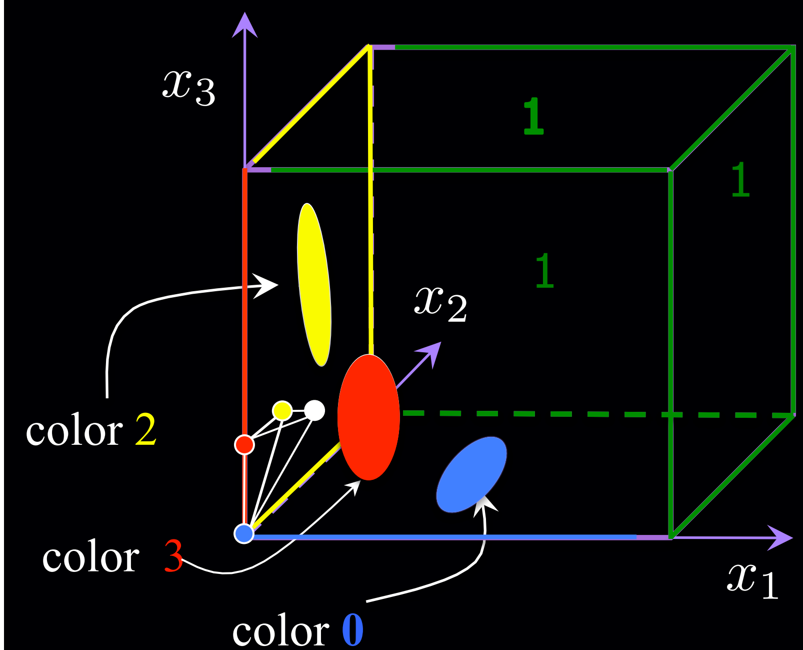
$$\pi = (1, \ldots, n-1, n)$$

This simplex has a colorful facet, lying on the face x_1=0 of the hypercube.

4. Finishing the Proof

The Proof of Sperner's Lemma

a. Start at the starting simplex; this has all the colors in {2,3,…,n, 0} but not color 1 and hence it is a colorful simplex. One of its colorful facets lies in $x_1=0$, while the other is shared with some neighboring simplex.



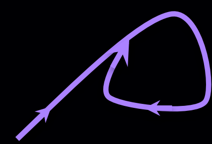
b. Enter into that simplex through the shared colorful facet. If the other vertex of that simplex has color 1 the walk is over, and the existence of a panchromatic simplex has been established. If the other vertex is not colored 1, the simplex has another colorful facet.

c. Cross that facet. Whenever you enter into a colorful simplex through a colorful facet, find the other colorful facet and cross it.

what are the possible evolutions of this walk?

Proof of Sperner

(i) Walk cannot loop into itself in a rho-shape, since that would require a simplex with three colorful facets.



(ii) Walk cannot exit the hypercube, since the only colorful facet on the boundary belongs to the starting simplex, and by (i) the walk cannot arrive to that simplex from the inside of the hypercube (this would require a third colorful facet for the starting simplex or a violation to (i) somewhere else on the path).

(iii) Walk cannot get into a cycle by coming into the starting simplex (since it would have to come in from outside of the hypercube)

The single remaining possibility is that the walk keeps evolving a path orbit, encountering a new simplex at every step while being restricted inside the hypercube. Since there is a finite number of simplices, walk must stop, and the only way this can happen is by encountering color 1 when entering into a simplex through a colorful facet.

$\Rightarrow$ a panchromatic simplex exists

Odd number of panchromatic simplices?

After original walk has settled, we can start a walk from some other simplex that is not part of the original walk.

- If the simplex has no colorful facet, stop immediately isolated node

-If the simplex is colorful, start two simultaneous walks by crossing the two colorful facets of the simplex; for each walk: if S is a colorful simplex encountered, exit the simplex from the facet not used to come in; there are two cases:

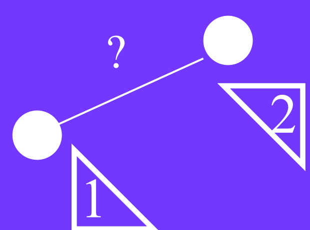
either the two walks meet cycle

or the walks stop at a different panchromatic simplex each path

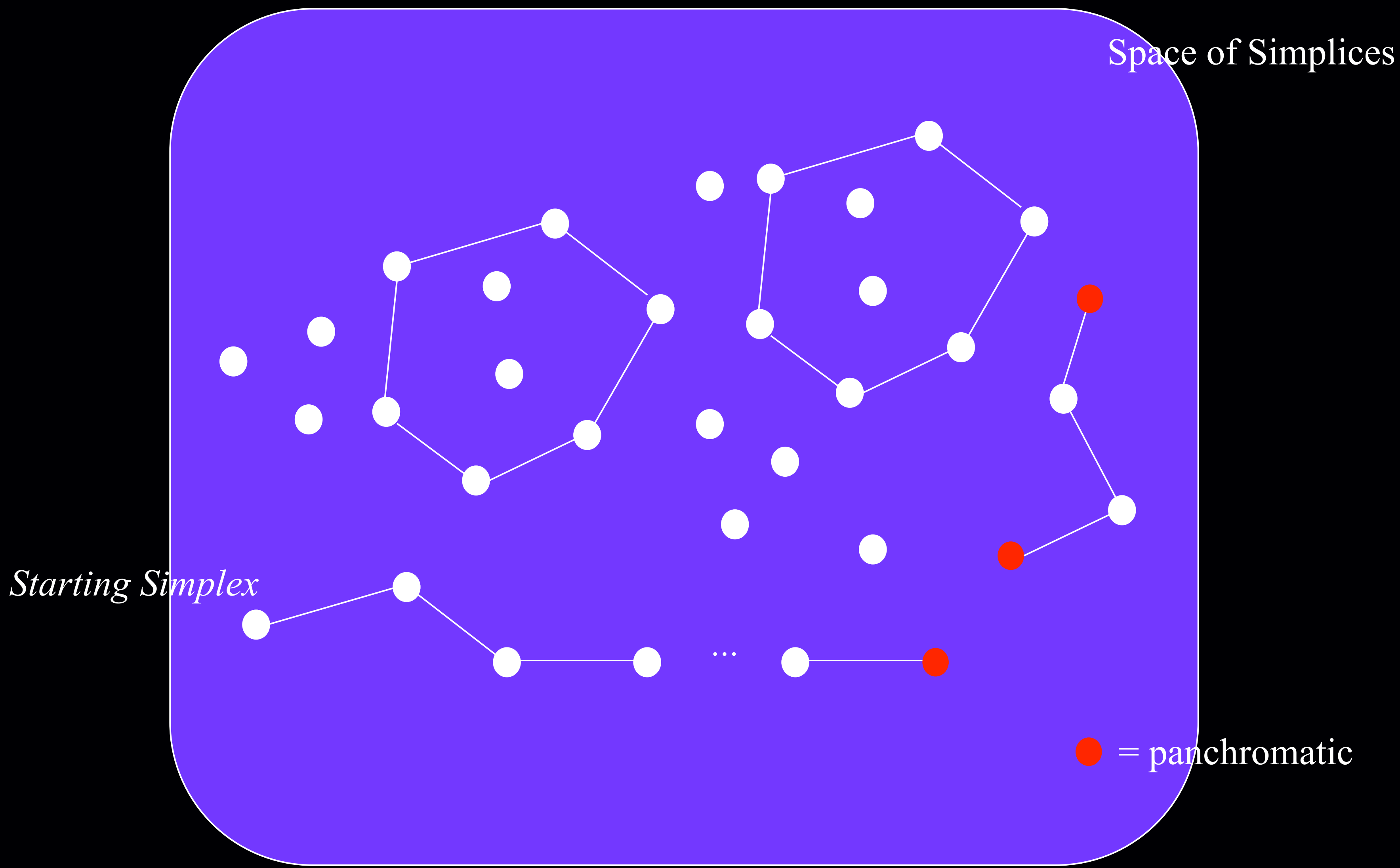
Abstractly…

Space of Simplices

Two simplices are ***Neighbors*** *iff they share a colorful facet*



Proofs constructs a graph with degree ≤ 2



5. Directing the walk

Towards a more constructive argument

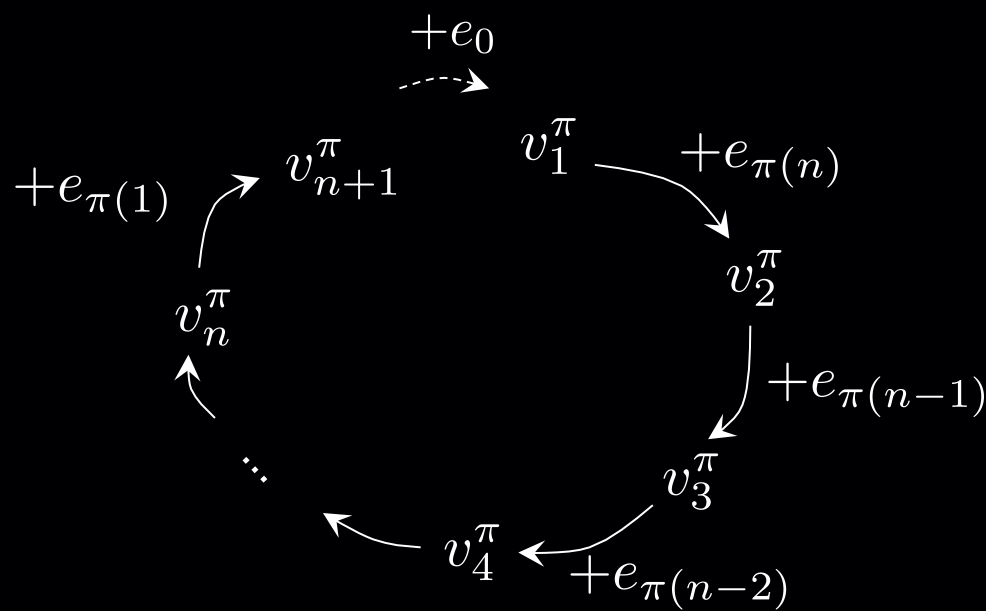
The above argument defines an undirected graph, whose vertex set is the set of simplices in the simplicization of the hypercube and which comprises of paths, cycles and isolated vertices.

We will see in the next couple of lectures that in order to understand the precise computational complexity of Sperner's problem, we need to define a ***directed*** *graph with the above structure (i.e. comprising of directed paths, directed cycles, and isolated nodes).*

We devise next a convention/efficient method for checking which of the two colorful facets of a colorful simplex corresponds to an incoming edge, and which facet corresponds to an outgoing edge.

Direction of the walk

Recall that we can cycle around the corners of $\mathcal{T}_\pi$ as follows:

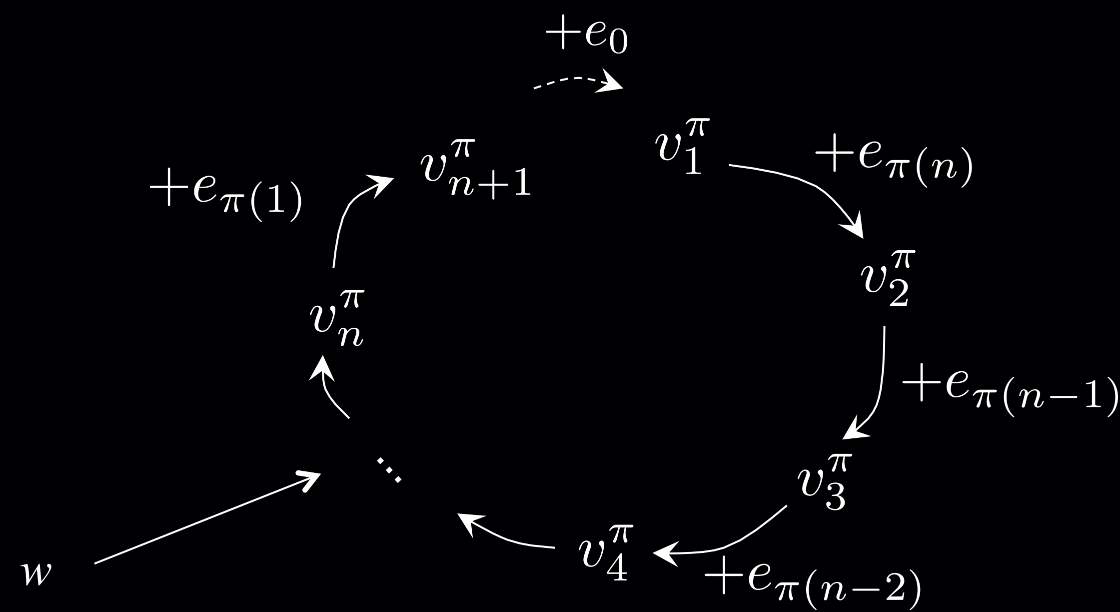


where the Hamming weight is increasing from v_1^π to v_{n+1}^π.

Direction of the walk

Given a colorful facet f of some simplex, we need to decide whether the facet corresponds to inward or outward direction. To do this we define two permutations, τ_f and σ_f as follows.

Let *w* be the vertex not on the colorful facet. *w* falls somewhere in the cycle of the simplex.

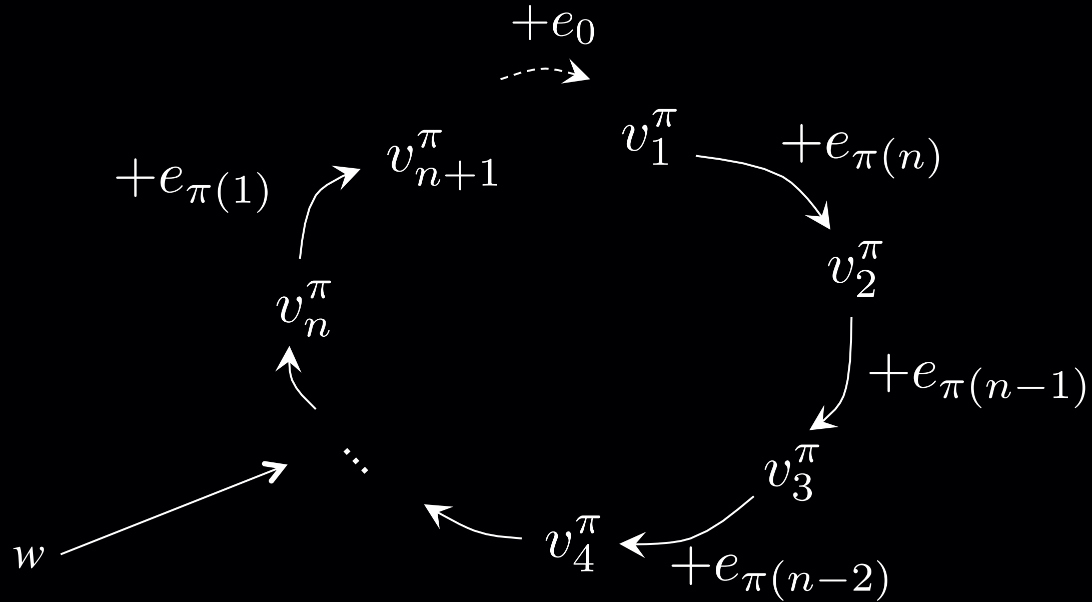


If $w = v^{\pi}_k$, let τ_f be the following permutation of 0, 1,…,n:

$$\pi_{n-k+1}, \pi_{n-k}, \pi_{n-k-1}, \ldots, 0, \pi_n, \ldots, \pi_{n-k+2}$$

Direction of the walk

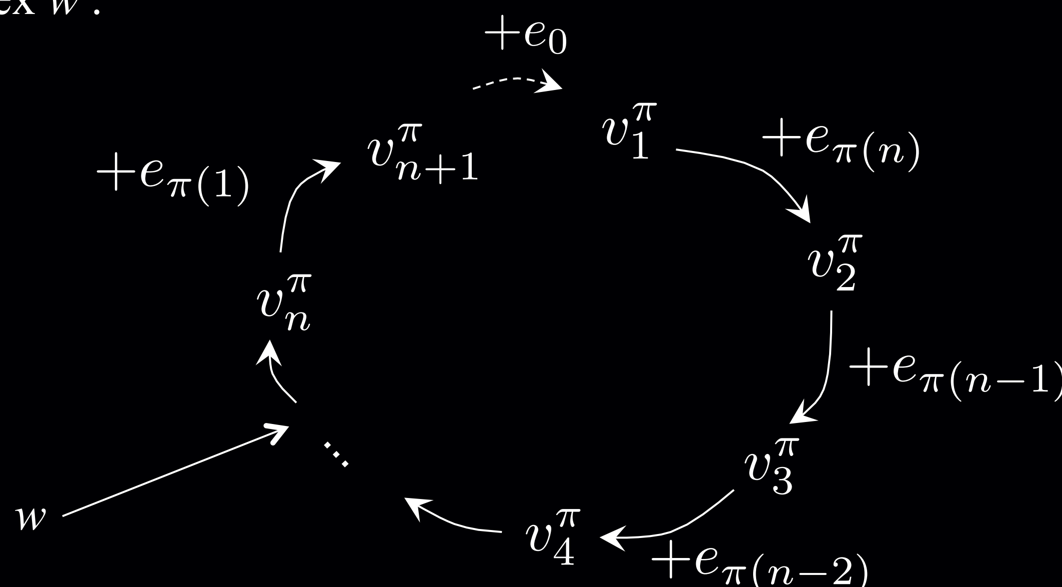
In other words, start at w and travel around the cycle to get back to w. Then τ_f is the permutation of indices that you encounter on the arrows as subscripts of e.



Direction of the walk

Permutation $\sigma_f : \{2, 3, \ldots, n, 0\} \rightarrow \{2, 3, \ldots, n, 0\}$

the order, in which the colors {2, 3,…, *n*, 0} appear in the cycle, starting from the vertex *w* .



Given τ_f and σ_f define the sign of the facet f to be:

$$\mathrm{sign}(f) = \mathrm{sign}(\sigma_f) \cdot \mathrm{sign}(\tau_f)$$

$$(-1)^{\#\text{inversions}}$$

→ parity of the number of pairwise inversions in the permutation

Interesting Properties of $\text{sign}(f)$

Suppose that f is colorful, and shared by a pair of simplices *S* and *S'*.

Claim: If *S* and *S'* belong to the same cubelet and share a faced *f, then*
$\text{sign}_S(f) \cdot \text{sign}_{S'}(f) < 0$, i.e. simplices *S* and *S'* assign different signs to their shared colorful facet *f.*

Proof: If *S* and *S'* belong to the same cubelet and share a facet *f*, then it must be that their permutations π, π' are identical, except for a transposition of one adjacent pair of indices

$$\pi_1, \pi_2, \ldots, \pi_i, \pi_{i+1}, \ldots, \pi_n$$

Hence if *w, w'* is the missing vertex from f in *S* and *S'* respectively, *w* is located in the cycle of π, π' respectively between indices *i* and *i +1* , while all the other shared vertices appear in the same order.

Hence, the color permutation σ_f is the same in *S, S'*, while the permutation τ_f has the pair of indices *i*, *i*+1 , transposed and hence has opposite sign in *S, S'*.

Interesting Properties of $\mathrm{sign}(f)$

Claim: If *S* and *S'* belong to the adjacent cubelets and share a facet *f*, then

$$\mathrm{sign}_S(f) \cdot \mathrm{sign}_{S'}(f) < 0.$$

Proof:

If S and S' belong to adjacent cubelets then f lies on a facet x_i=1 of S and x_i=0 of S'. The vertex not in f in S is 0…00, while the vertex not in f in S' is 1…11. Moreover, to obtain the vertices of f in S', replace coordinate i in the vertices of f in S with 0. In other words permutations π, π' are identical, except that *i* is moved from the last position of π to the first position of π'.

It follows that the color permutation σ_f is the same in *S, S'*, while there is exactly one transposition in going from τ_f in S to τ_f in S'.

Interesting Properties of $\text{sign}(f)$

Claim: Let *S be a colorful simplex and f, f' its two colorful facets. Then*

$$\text{sign}_S(f) \cdot \text{sign}_S(f') < 0.$$

Proof: Let *w*, *w'* be the vertices of *S* missing from *f* and *f'* respectively. W.l.o.g *w* appears before *w'* on the cycle, and they are separated by *k* arcs.

It is then easy to see that the permutations τ_f and $\tau_{f'}$ differ by a cyclic shift of *k* positions.

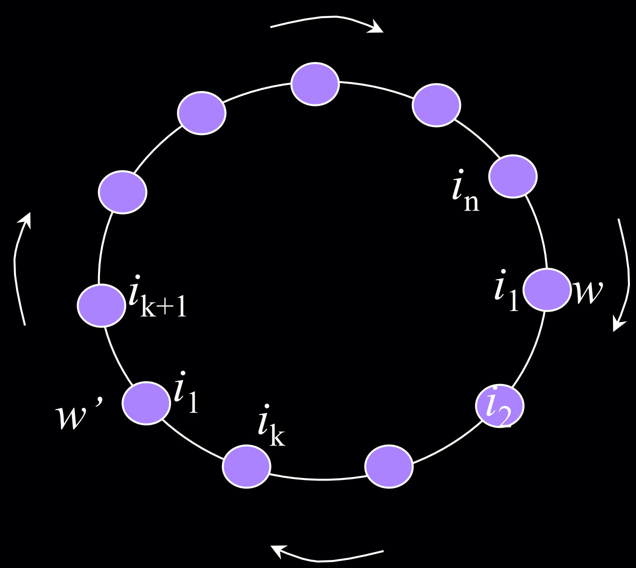
if *n* is even: $\text{sign}(\tau_{f'}) = \text{sign}(\tau_f)$

if *n* is odd: $\text{sign}(\tau_{f'}) = \text{sign}(\tau_f)(-1)^k$

We proceed to the comparison of permutations σ_f and $\sigma_{f'}$:

Interesting Properties of $\text{sign}(f)$

Proof (cont.): Let the colors be as follows



$$\sigma_f = i_2 i_3 \ldots i_k i_1 i_{k+1} \ldots i_n$$

$$\sigma_{f'} = i_{k+1} \ldots i_n i_1 i_2 \ldots i_k$$

To obtain $\sigma_{f'}$ from σ_f move color i_1 to the beginning of the permutation, then shift cyclically left k positions.

if n is even only pay for moving i_1: $\text{sign}(\sigma_{f'}) = \text{sign}(\sigma_f)(-1)^{k-1}$

if n is odd pay for moving i_1 and shift: $\text{sign}(\sigma_{f'}) = \text{sign}(\sigma_f)(-1)^{k-1}(-1)^k$

Interesting Properties of $\operatorname{sign}(f)$

Proof (cont.):

Hence:

if *n* is even: $$\operatorname{sign}(f) \cdot \operatorname{sign}(f') = (-1)^{k-1}$$

if *n* is odd: $$\operatorname{sign}(f) \cdot \operatorname{sign}(f') = (-1)^{k}(-1)^{k}(-1)^{k-1}$$

∎