6.896 Topics in Algorithmic Game Theory	February 17, 2010
Lecture 5	
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NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

We have seen that there always exists a Nash equilibrium in two-player zero-sum games. In this lecture, we prove Nash's theorem on the existence of Nash equilibrium in any n-player finite game. The proof requires Brouwer's fixed point theorem. Although this is a topological result, we give a proof for two dimensions using combinatorial methods via Sperner's lemma.

1 Preliminaries

We start by giving a few precise definitions.

Definition 1. *A n-player finite game $\langle [n], (S_p)_{p\in[n]}, (u_p)_{p\in[n]}\rangle$ is formally defined as follows.*

- *We have a set of n players $[n] = \{1, 2, \ldots, n\}$.*
- *Each player $p \in [n]$ has a finite set of strategies or actions S_p.*
- *The set $S = \prod_{p\in[n]} S_p$ is the set of strategy profiles.*
- *Each player can choose a distribution of strategies $\underset{\sim}{x}_p \in \Delta^{S_p} = \{x_p \in \mathbb{R}_+^{S_p} \mid \sum_{s_p\in S_p} x_p(s_p) = 1\}$.*
- *Each player has a utility function $u_p : S \to \mathbb{R}$.*

Here, given a mixed strategy profile $\underset{\sim}{x} \in \Delta = \prod_{p\in[n]} \Delta^{S_p}$, the payoff of player p is given by the expected utility

$$u_p(\underset{\sim}{x}) = \mathbb{E}_{s\sim \underset{\sim}{x}}\left[u_p(s)\right].$$

Given a strategy profile $s \in S$, we also adopt the notation s_p to denote player p's strategy and s_{-p} to denote everyone else's strategies. Similarly, given $\underset{\sim}{x} \in \Delta$, we let $\underset{\sim}{x}_p$ denote the distribution of strategies for player p and let $\underset{\sim}{x}_{-p}$ denote the distribution of strategies for everyone else.

Next, we give a few definitions for types of Nash equilibria.

Definition 2. *A distribution of strategies $\underset{\sim}{x} \in \Delta$ is a Nash equilibrium iff $\forall p \in [n], s_p \in S_p$, we have*

$$u_p(\underset{\sim}{x}) \geq u_p(s_p; \underset{\sim}{x}_{-p}).$$

In other words, $\underset{\sim}{x}$ is a Nash equilibrium iff no player can strictly increase his or her payoff by using pure strategies. Note that checking against all pure strategies is sufficient because expected payoff is linear in the payoff of pure strategies. Also, note that if $\underset{\sim}{x}$ is a Nash equilibrium, then for all p, the support of $\underset{\sim}{x}_p$ is a subset of the maximizers, or

$$\mathrm{supp}(\underset{\sim}{x}_p) \subset \mathrm{argmax}_{s_p\in S_p} u_p(s_p; \underset{\sim}{x}_{-p}).$$

The set of pure strategies is a subset of mixed strategies. Therefore, an equivalent definition is the following.

Definition 3. *A distribution of strategies $\underset{\sim}{x} \in \Delta$ is a Nash equilibrium iff $\forall p \in [n], s_p, s'_p \in S_p$ such that $\underset{\sim}{x}_p(s_p) > 0$, we have*

$$u_p(s_p; \underset{\sim}{x}_{-p}) \geq u_p(s'_p; \underset{\sim}{x}_{-p}).$$

Here are two definitions for approximate Nash equilibria.

Definition 4. *A distribution of strategies* $\underset{\sim}{x} \in \Delta$ *is a* ϵ*-approximate Nash equilibrium iff* $\forall p \in [n], s_p \in S_p$, *we have*

$$u_p(\underset{\sim}{x}) \geq u_p(s_p; \underset{\sim}{x}_{-p}) - \epsilon.$$

Definition 5. *A distribution of strategies* $\underset{\sim}{x} \in \Delta$ *is a* ϵ*-well-supported Nash equilibrium iff* $\forall p \in [n], s_p, s'_p \in S_p$ *such that* $\underset{\sim}{x}_p(s_p) > 0$, *we have*

$$u_p(s_p; \underset{\sim}{x}_{-p}) \geq u_p(s'_p; \underset{\sim}{x}_{-p}) - \epsilon.$$

Note that these two definitions are not equivalent. Indeed, the latter implies the former. A ϵ-well-supported Nash equilibrium needs to be within ϵ of the optimal for every pure strategy, whereas a ϵ-approximate Nash equilibrium only needs to be within ϵ of the optimal for the mixed strategy.

2 Nash's Theorem

The following theorem was established by John Nash in 1951 [2].

Theorem 1 (Nash)**.** *Every game* $\langle [n], (S_p)_{p \in [n]}, (u_p)_{p \in [n]} \rangle$ *has a Nash equilibrium.*

Before we delve into the proof of this theorem, we need Brouwer's fixed point theorem. This will be proved (in the two-dimensional case) in the next few sections.

Theorem 2 (Brouwer)**.** *Let* D *be a convex, compact subset of the Euclidean space. If* $f : D \to D$ *is continuous, then there exists* $x \in D$ *such that* $f(x) = x$.

The idea behind Nash's proof is to construct a function $f : \Delta \to \Delta$ that satisfies the conditions of Brouwer's fixed point theorem such that the fixed point $\underset{\sim}{x}$ is a Nash equilibrium. To do so, we introduce the idea of a gain function.

Definition 6. *Suppose* $\underset{\sim}{x} \in \Delta$ *is given. For a player* p *and strategy* $s_p \in S_p$, *we define the gain as*

$$\mathrm{Gain}_{p;s_p}(\underset{\sim}{x}) = \max\{u_p(s_p; \underset{\sim}{x}_{-p}) - u_p(\underset{\sim}{x}), 0\}.$$

In other words, the gain is equal to the increase in payoff for player p using strategy s_p if it is positive and zero otherwise.

Proof of Theorem 1: We define a function $f : \Delta \to \Delta$ as follows. Given $\underset{\sim}{x} \in \Delta$, we set $f(\underset{\sim}{x}) = y$, where for $p \in [n]$ and $s_p \in S_p$, we have

$$y_p(s_p) := \frac{\underset{\sim}{x}_p(s_p) + \mathrm{Gain}_{p;s_p}(\underset{\sim}{x})}{1 + \sum_{s'_p \in S_p} \mathrm{Gain}_{p;s'_p}(\underset{\sim}{x})}.$$

Here, the denominator ensures that $\sum_{s_p \in S_p} y_p(s_p) = 1$. We note that f is continuous. Moreover, Δ is a product of simplices, so is convex. It is easy to see that Δ is both closed and bounded, so is compact as well. Therefore, we can apply Brouwer's fixed point theorem to get a fixed point $\underset{\sim}{x}$. Given that $f(\underset{\sim}{x}) = \underset{\sim}{x}$, we claim that it is a Nash equilibrium. To do so, it suffices to prove that

$$\mathrm{Gain}_{p;s_p}(\underset{\sim}{x}) = 0 \quad \forall p \in [n], s_p \in S_p.$$

We proceed by contradiction. Assume that there is some player who can improve, i.e.

$$\exists p \in [n], s_p \in S_p \text{ s.t. } \mathrm{Gain}_{p;s_p}(\underset{\sim}{x}) > 0. \qquad (*)$$

Then, it will be enough to show the following.

Claim 1. *The above (*) implies that* $\exists s'_p \in \text{supp}(\underset{\sim}{x}_p)$ *such that*

$$u_p(s'_p; \underset{\sim}{x}_{-p}) - u_p(\underset{\sim}{x}) < 0.$$

The claim follows the fact that a linear combination of the terms $u_p(s'_p; \underset{\sim}{x}_{-p}) - u_p(\underset{\sim}{x})$, with weights corresponding to the weights of the pure strategies in $\underset{\sim}{x}_p$, gives zero. Moreover, there is at least one positive term (with positive gain) so there is at least one negative term. The corresponding strategy s'_p must belong to $\text{supp}(\underset{\sim}{x}_p)$.

This implies that $\text{Gain}_{p;s'_p}(\underset{\sim}{x}) = 0$, so

$$y_p(s'_p) = \frac{\underset{\sim}{x}_p(s'_p) + \text{Gain}_{p;s'_p}(\underset{\sim}{x})}{1 + \sum_{s''_p \in S_p} \text{Gain}_{p;s''_p}(\underset{\sim}{x})} < \underset{\sim}{x}_p(s'_p),$$

since the numerator is $\underset{\sim}{x}_p(s'_p)$, while the denominator is greater than 1 (there is at least one nonzero gain). Therefore, $\underset{\sim}{x}$ is not a fixed point, contradiction.

It follows that $\underset{\sim}{x}$ is a Nash equilibrium, as desired. □

3 Sperner's Lemma

Consider a grid composed of square cells and a triangulation of the grid where the diagonals are drawn from the upper left corner to the lower right corner of the individual cells. We color the nodes one of three colors: red, blue, or yellow.

Definition 7. *We call a coloring legal if there are no blue nodes on the left boundary, no red nodes on the bottom boundary, and no yellow nodes on both the top and right boundaries.*

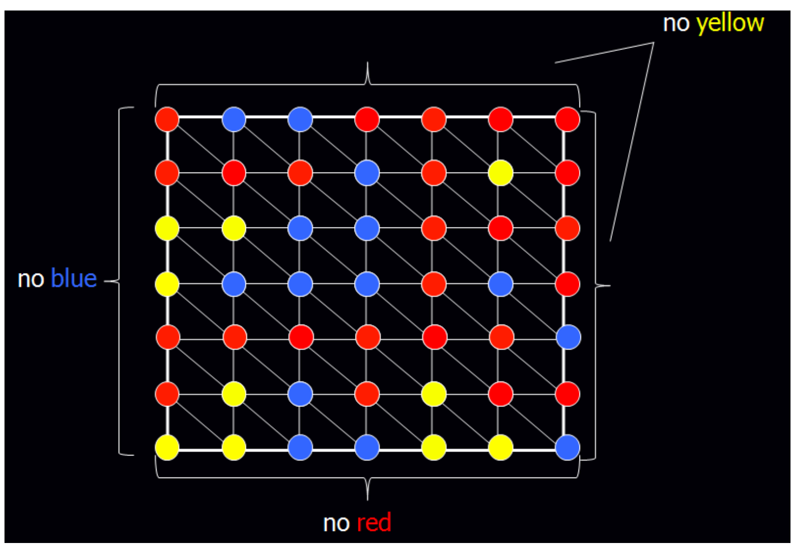

Figure 1: An example of a legal coloring

Definition 8. *A tri-chromatic triangle is a triangle in which all three nodes are different colors.*

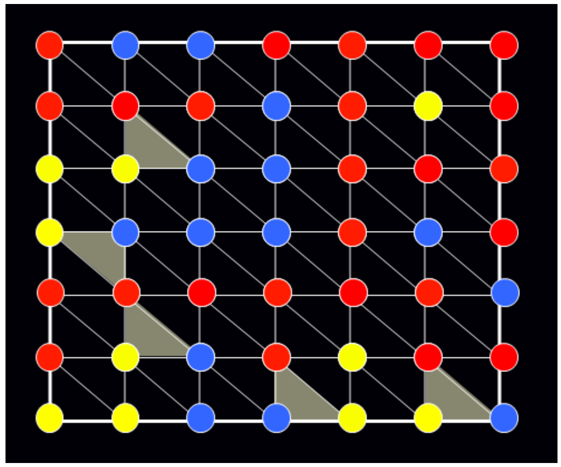

Figure 2: Tri-chromatic triangles in the above coloring

We are now ready for Sperner's lemma.

Lemma 1 (Sperner)**.** *Given a legal coloring of a grid, there exists a tri-chromatic triangle. In fact, there are an odd number of tri-chromatic triangles.*

For convenience, we introduce an outer boundary with red on the left (including the top left corner), yellow on the bottom (including the bottom right corner), and blue on the remaining sides. The following claim is easily verified.

Claim 2. *The boundary does not introduce any tri-chromatic triangles.*

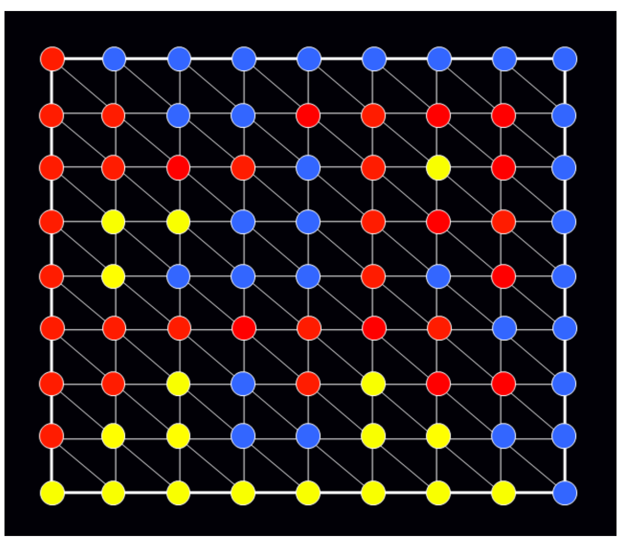

Figure 3: The grid with our additional outer boundary

Next, we define a directed walk starting from the bottom-left triangle. If there exists an edge with a red node and a yellow node (a "red-yellow door"), then cross the door with the red node on the left. Note that there is only one red-yellow door in the outer boundary, but with red on the right. If we start from the bottom-left triangle, it cannot return to itself, so must end at a tri-chromatic triangle. Moreover, note that every triangle has at most two red-yellow doors, so we cannot get any loops. Therefore, the following claim is true.

Claim 3. *The walk cannot exit the square, nor can it loop around itself in a rho-shape. Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle.*

Starting from other triangles with two red-yellow doors, we do the same going forward or backward. Considering going forward from such a triangle. We will either loop back to itself, or hit a tri-chromatic triangle. In the latter case, we can go backward and encounter another tri-chromatic triangle. The two tri-chromatic triangles cannot be the same, as there is only one red-yellow door. This allows us to complete the proof of Sperner's lemma.

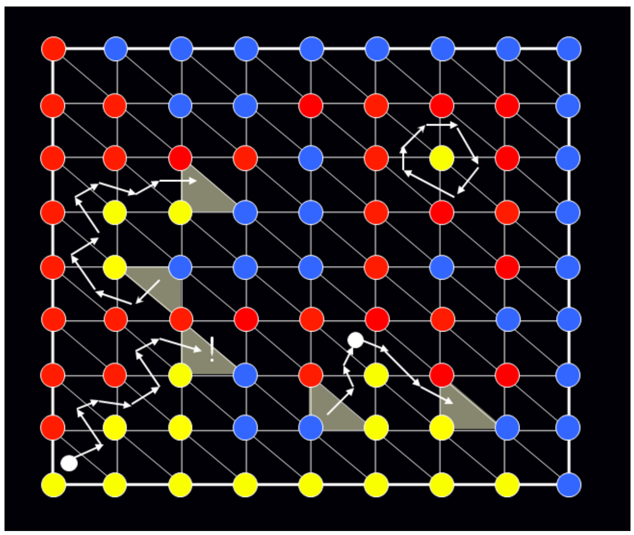

Figure 4: Illustration of the directed walks

Proof of Lemma 1: Starting from the bottom-left triangle, the walk ends at a tri-chromatic triangle. For any other triangle that has two red-yellow doors, we start both the forward and backward walk. If they meet, we end in a loop and without any more tri-chromatic triangles. Otherwise, both walks end at a tri-chromatic triangle. This means we have an odd number of tri-chromatic triangles, as claimed. □

4 Proof of Brouwer's Fixed Point Theorem

Now, we show that Sperner's lemma implies Brouwer's fixed point theorem for $D = [0,1]^2$. We will work over the ℓ_∞ norm. Suppose we have a continuous function $f : D \to D$. Because $D = [0,1]^2$ is compact, by the Heine-Cantor theorem, we get that f is uniformly continuous as well.

Given $\epsilon > 0$, there is some $\delta(\epsilon) > 0$ such that for all $z, w \in D$,

$$d(z,w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon.$$

We triangulate D so that the diameter of the cells is at most $\delta(\epsilon)$. Next, color the nodes x according to the direction of $f(x) - x$, as shown below. In other words, given a vector (x, y), we color it yellow if

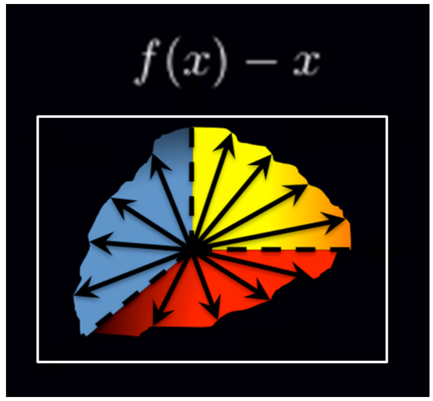

Figure 5: Determining the color of a node based on the direction of $f(x) - x$

$x, y \geq 0$, red if $x \geq y,\ y \leq 0$, and blue if $x \leq y,\ y \geq 0$. We tie-break at the boundaries to ensure that coloring is legal.

By Sperner's lemma, there is a tri-chromatic triangle. Suppose z^Y, z^R, z^B are the yellow, red, blue nodes of the triangle, respectively.

Claim 4. *We have* $|f(z^Y) - z^Y|_\infty < \epsilon + \delta$.

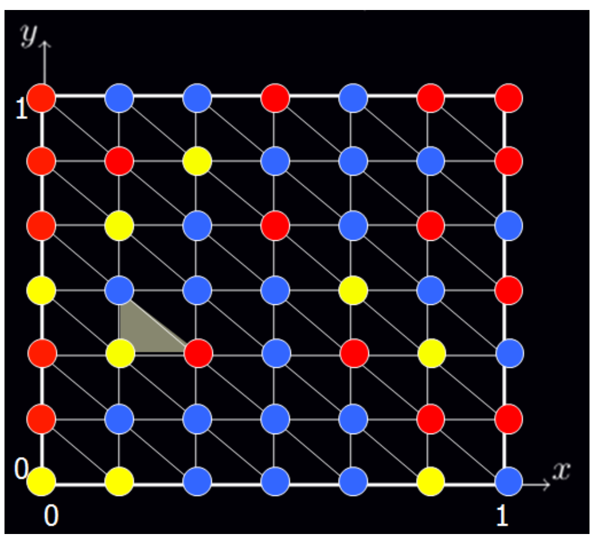

Figure 6: A triangulation of D with a tri-chromatic triangle

Proof of Claim 4: By definition of the coloring, note that

$$(f(z^Y) - z^Y)_x \cdot (f(z^B) - z^B)_x \leq 0.$$

Hence,

$$\begin{aligned}
|(f(z^Y) - z^Y)_x| &\leq |(f(z^Y) - z^Y)_x - (f(z^B) - z^B)_x| \\
&\leq |(f(z^Y) - f(z^B))_x| + |(z^Y - z^B)_x| \\
&\leq d(f(z^Y), f(z^B)) + d(z^Y, z^B) \\
&\leq \epsilon + \delta.
\end{aligned}$$

Similarly, we can show that

$$|(f(z^Y) - z^Y)_y| \leq \epsilon + \delta.$$

□

Note that choosing $\delta = \min(\delta(\epsilon), \epsilon)$ implies that

$$|f(z^Y) - z^Y|_\infty \leq 2\epsilon.$$

We are now ready to prove Brouwer's fixed point theorem for $D = [0, 1]^2$.

Proof of Theorem 2: Let $\epsilon_i = 2^{-i}$ be a sequence for $i = 1, 2, \ldots$. For each ϵ_i, define a triangulation with diameter $\delta_i = \min(\delta(\epsilon_i), \epsilon_i)$. Note that these triangulations get finer. For the triangulation corresponding to ϵ_i, let z_i^Y be the yellow node of a tri-chromatic triangle, which exists by Sperner's lemma. By compactness, this sequence has a convergent subsequence w_i with limit equal to w^*. Now, we claim that w^* is a fixed point.

Define a function $g(x) = d(f(x), x)$, which is continuous since both f and $d(\cdot, \cdot)$ are continuous. Therefore, it follows that

$$g(w_i) \;\to\; g(w^*) \;\text{ as }\; i \to \infty.$$

By the previous claim, we have $0 \leq g(w_i) \leq 2\epsilon_i = 2^{-i+1}$, so $g(w_i) \to 0$. It follows that $g(w^*) = 0$, which implies $f(w^*) = w^*$, as desired. □

5 Visualizing Nash's Construction

Although we have all the relevant theorems, the purpose of this section is to develop some intuition for Brouwer's fixed point theorem and Nash's construction. Below are a few examples of Brouwer's fixed point theorem in action, where D is a two-dimensional disk. Note that all the conditions for Brouwer's fixed point theorem are necessary to ensure a fixed point.

Finally, we give an example of Nash's construction. Consider the simple penalty shot game below. The only Nash equilibrium is mixing uniformly.

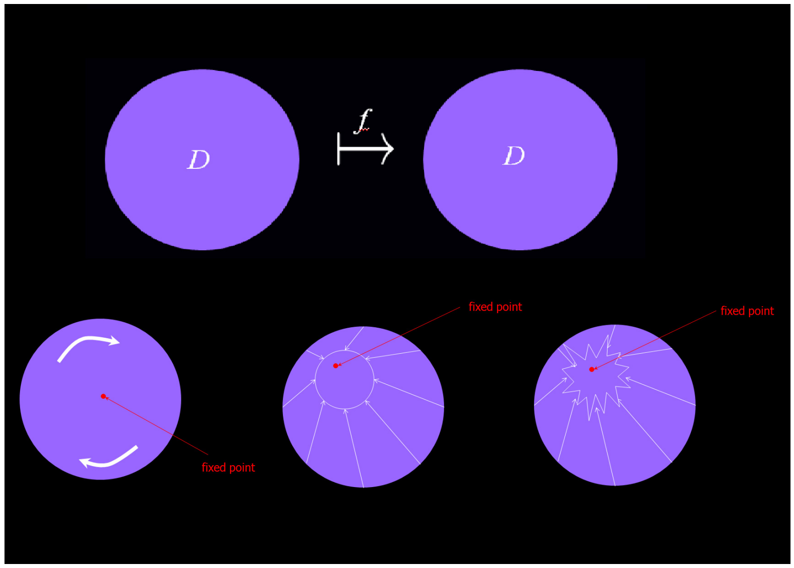

Figure 7: Examples of fixed points for three different functions

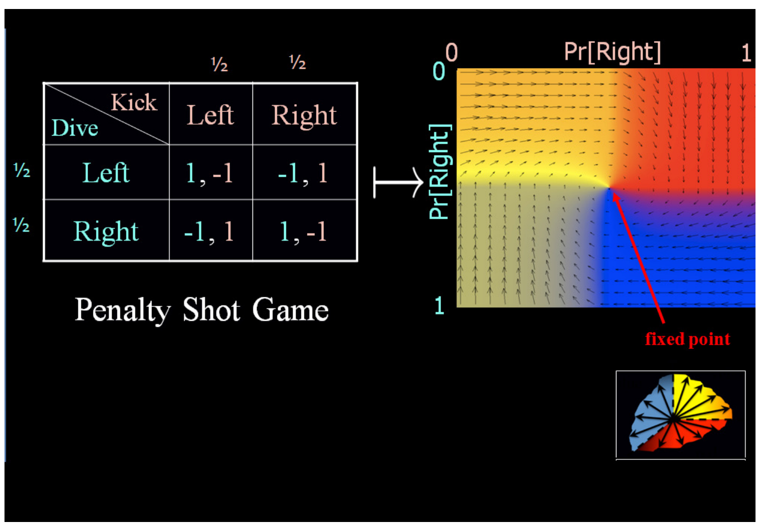

Figure 8: Nash's construction for the penalty shot game

References

[1] S. Kakutani. A generalisation of Brouwer's fixed point theorem. *Duke Math. J.*, 7:457–459, 1941.

[2] J. Nash. Noncooperative Games. *Annals of Mathematics*, 54:289–295, 1951.