Lecture 11

Applications of Bounding Mixing w/ Multicommodity Flows

Recall:

- $$\alpha = \inf_{\substack{\varphi \text{ non-constant} \\ \varphi: \Omega \to \mathbb{R}}} \frac{\mathcal{E}_\pi(\varphi,\varphi)}{\mathrm{Var}_\pi \varphi}$$ (Poincaré Constant)

  local variance. global variance

- $\alpha = 1 - \lambda_2$ for lazy, ergodic, reversible MC's (where λ_2 is the second largest eigenvalue)

- **Thm 1 (last time)**: for any lazy, ergodic P and any starting $x \in \Omega$:
$$\tau_x(\varepsilon) \le \frac{1}{\alpha}\left(2 \ln \varepsilon^{-1} + \ln \pi(x)^{-1}\right).$$

- **Thm 2 (last time)**: For any ergodic P and any multicommodity flow f for P
$$\alpha \ge \frac{1}{\rho(f) \cdot \ell(f)},$$
where $\rho(f)$ is the cost of f & $\ell(f)$ the length of longest flow-carrying path

  $\max_e \frac{f(e)}{C(e)}$

- **Corollary**: For lazy, ergodic MC P, ~~and~~ any flow f for P, and any starting $x \in \Omega$:
$$\tau_x(\varepsilon) \le \rho(f)\,\ell(f)\left(2 \ln \varepsilon^{-1} + \ln \pi(x)^{-1}\right).$$

Cheat Sheet: Multicommodity Flow Problem for MC P

- → Demands: $D(x,y) = \pi(x) \cdot \pi(y)$ for different commodities
- → Edge Capacities: $C(u,v) = \pi(u) \cdot P(u,v)$
- → flow $f: \bigcup_{xy} P_{xy} \to \mathbb{R}_+^{\cup\{0\}}$ for P

  simple paths from x to y

  $\sum_{p \in P_{xy}} f(p) = D(x,y)$

Converses?

- to thm 1: $\tau_{mix} \geq \text{constant} \times \frac{1-\alpha}{\alpha}$

- to thm 2: $\exists$ flow f s.t. $\alpha \leq \text{constant} \times \frac{\log|\Omega|}{\rho(f)}$

  (follows from Leighton & Rao's $\log n$ approximation to the sparsest cut [LR '88])

- to corollary: $\exists$ flow f with $\tau_{mix} \geq \text{constant} \times \rho(f)$

  (this flow is exactly that "generated" by the probability mass flow in the markov chain) if the MC is run for $2 \cdot \tau_{mix}$ steps.

Examples:

(A) Lazy Random Walk on the Hypercube:

- $\Omega = \{0,1\}^n$, #vertices $N = 2^n$
- $\pi(x) = \frac{1}{N}$, $C(e) = \frac{1}{N} \cdot \frac{1}{2n} = \frac{1}{2Nn}$, $\forall e$

  $D(x,y) = \frac{1}{N} \cdot \frac{1}{N} = \frac{1}{N^2}$, $\forall x,y$

- Want flow minimizing $\rho(f) \cdot \ell(f)$: We will show this is achieved by

  (*) $\boxed{\text{Spreading flow between } x,y \text{ uniformly over all shortest paths between } x,y.}$

  - (*) achieves $f(e) = f(e')$, for all e, e' (by symmetry)

  - observe total flow on my network cannot be smaller than

  $$\sum_{x,y} \frac{1}{N^2} \cdot \{\text{length of SP between } x,y\} = \frac{n}{2},$$ for any multicommodity flow satisfying the demands

  - If all edges share it equally:

  $$f(e) = \frac{n/2}{|E|} = \frac{n/2}{n \cdot N} = \frac{1}{2N}.$$

(*) achieves: $f(e) = \frac{1}{2N}$

hence, under (*): $\rho(f) = \dfrac{\frac{1}{2N}}{\frac{1}{2Nn}} = n$ (which by the above discussion is best possible $\rho(f)$ for any multicommodity flow.

$\ell(f) = n$

hence corollary $\Rightarrow$ $\tau_x(\varepsilon) \le \rho(f)\,\ell(f)\,(2\ln\varepsilon^{-1} + \ln\pi(x)^{-1})$

$\le n^2(2\ln\varepsilon^{-1} + n)$

$\Rightarrow \tau_{mix} = O(n^3)$.

Not tight, as $\tau_{mix} \sim \frac{n}{2}\ln n$. (we got this up to a constant factor using coupling)

Where did we lose?

- Short answer: everywhere
- Notice that the spectral gap (Poincaré constant) for our Markov chain is $1-\lambda_2 = \alpha = \frac{1}{2n}$ (e.g. consider dimension cuts. +1 on one side -1 on the other)
- Even using exact value for α,
  - theorem 1 gives $\tau_{mix} = O(n^2)$ which is still off!

    The extra factor of $\frac{n}{\log n}$ comes from $\ln\pi(x)^{-1}$, which arises because we approximated the mixing using only the 2nd eigenvalue (recall calculation in end of last lecture)
- Moreover, the best bound on α that Thm 2 can possibly give is $\alpha \geq \frac{1}{\rho(f)\ell(f)} = \frac{1}{n^2}$, which is off by a factor of n from the true value of $\frac{1}{2n}$.

Example 2: RW on a Line.

- $\Omega = \{1, 2, \ldots, N\}$; self loop probability 1/2 at every state except for states 1, N where self-loop probability is 3/4.

- $\pi(x) = 1/N$

- $D(x,y) = \frac{1}{N^2}, \quad \forall\, x, y.$

- $C(e) = \frac{1}{4N}$, for all (non self-loop) edge e.

- multicommodity flow: route each commodity through unique path available.

then $f((i,i+1)) = i \cdot (N-i) \cdot \frac{1}{N^2} \leq \frac{1}{4}, \quad \forall i$

similarly $f((i+1,i)) \leq \frac{1}{4}, \quad \forall i$

$$\left.\begin{array}{l} \rightsquigarrow \rho(f) = \max_e \frac{f(e)}{C(e)} = \frac{1/4}{1/4N} = N \\ \ell(f) = N \end{array}\right\} \rightsquigarrow \alpha \geq \frac{1}{\rho(f)\ell(f)} \geq \frac{1}{N^2}$$

(asymptotically tight)

$\Rightarrow \tau_{mix} = O(N^2 \log N)$ (off by a factor of $O(\log N)$; lost because of the approximation with the 2nd eigenvalue).

- ## Example 3: Lazy RW on $K_{2,N}$.

- suppose $\Omega = \left\{ \begin{array}{ll} s & 1 \\ t & 2 \\ & \vdots \\ & N \end{array} \right\}$, self-loop probability 1/2 everywhere.

$P(t,i) = P(s,i) = \frac{1}{2N}, \quad \forall i$

$P(i,s) = P(i,t) = \frac{1}{4}, \quad \forall i$

- Stationary Distn: $\pi(s) = \pi(t) = \frac{1}{4}$

$\pi(i) = \frac{1}{2N}, \quad \forall i$

Back of the envelope Calculation

→ Usually (i.e. non-reversible / non-symmetric chains) it is tougher to find good multicommodity flow

→ need some technology to find flows in a generic setup.

→ **Desiderata:**

- Suppose $|\Omega| = N$ (usually exponential in natural size n)
- suppose $\pi(x) = \frac{1}{N}$
- and $P(u,v) \geq \frac{1}{\text{poly}(n)}$ (basically degree is not huge)

$\Rightarrow$ edge capacities: $C(u,v) = \pi(u)\, P(u,v) \approx \frac{1}{N \cdot \text{poly}(n)}$

- OUR GOAL: Find f s.t.

$$\frac{f(e)}{C(e)} \leq \text{poly}(n) \quad , \quad \ell(f) \leq \text{poly}(n)$$ to get poly(n) mixing time.

- Hence we need $\boxed{f(e) \leq \frac{\text{poly}(n)}{N}, \ \forall e}$ (*)

- But notice total number of edges $|E| \leq N \times \text{poly}(n)$ and total flow on paths $\geq \sum_{x,y} \frac{1}{N^2} \approx 1$

$\Rightarrow$ $\exists$ edge that should carry flow at least $\frac{1}{N \times \text{poly}(n)}$

i.e. $\boxed{f(e) \geq \frac{1}{N \cdot \text{poly}(n)}, \text{ for some } e.}$ (**)

- Comparing $(*)$, $(**)$ $\Rightarrow f$ should be optimal to within a poly(n) factor

- <u>Hypothesis:</u> . flow $x \to y$ goes along a single path $\gamma_{x,y}$

  - $\text{paths}(e) = |\{\gamma_{xy} \ni e\}|$

  - $f(e) = |\text{paths}(e)| \times \frac{1}{N^2}$ and $(*)$ $\Big\} \Rightarrow |\text{paths}(e)| \leq N \times \text{poly}(n)$.

$$\boxed{\text{So need to set-up flows s.t. the number of paths along each edge is } \leq N \times \text{poly}(n).}$$

- and if we don't know N in advance?

  $\Rightarrow$ construct injective map: $\eta_e : \text{path}(e) \hookrightarrow \Omega$.

  such map implies $|\text{paths}(e)| \leq |\Omega|$.

  more on that in lecture 12