# Lecture 17

- Phase Transitions in the Ising Model

- **Ising Model on $\sqrt{n} \times \sqrt{n}$ lattice**

Gibbs Distribution:

$$\pi(\sigma) = \frac{1}{Z} \exp\left(\sum_{i \sim j} \beta \sigma_i \sigma_j\right) \qquad \begin{array}{cc} & \sqrt{n} \\ \sqrt{n} & \boxed{\begin{matrix} + & - & - & + \\ + & + & - & - \\ + & + & - & + \\ - & - & + & - \end{matrix}} \end{array}$$

$$\propto \exp(\beta\,(\#\text{agreeing neighbors} - \#\text{disagreeing neighbors}))$$

$Z$: partition function

$\beta$: inverse temperature

$\beta$: low (high temperature) $\pi$ approaches uniform distn'
no macroscopic order

high (low temperature) $\pi$ assigns higher weight
to organized configurations

- **Phase Transition:** $\beta_c = \frac{1}{2} \ln(1+\sqrt{2})$

$\exists\ \beta_{crit}$ where model suddenly switches from being organized to being disorganized.

More formally, consider box $[-\sqrt{n}, +\sqrt{n}]^2$ for progressively larger $n$'s

study correlation of spin at $(0,0)$ and a function of spins at the boundary (that tries to guess spin at 0); then:

- $\beta < \beta_c$, correlation $\xrightarrow{n \to +\infty} 0$ (for any function)

- $\beta > \beta_c$, $\exists$ function: correlation $\geq c > 0$, $\forall n$

- Heat Bath chain (Glauber Dynamics):
  - start at arbitrary $\sigma$;
  - at each step, choose $i$ u.a.r.

    sample $\sigma_i$ from the conditional distn' on $i$ given the spins $\sigma_{N(i)}$ of its neighbors

    [that is, if $m_i^+(\sigma)$, $m_i^-(\sigma)$ are respectively neighbors of $i$ with $+,-$ sign in $\sigma$, then the new spin at $i$ is $+$ w/ prob

$$\frac{\exp(\beta(m_i^+ - m_i^-))}{\exp(\beta(m_i^+ - m_i^-)) + \exp(\beta(m_i^- - m_i^+))}$$

MC is reversible wrt. $\pi$.

- Theorem [MO] ([Martinelli-Olivieri '94]): The mixing time of Glauber dynamics for the Ising Model on a $\sqrt{n} \times \sqrt{n}$ box of the 2-dimensional lattice is:

$$\begin{cases} O(n \log n) & , \text{if } \beta < \beta_c; \\ e^{\Omega(\sqrt{n})} & , \text{if } \beta > \beta_c; \end{cases}$$

where $\beta_{crit}$ is the critical inverse temperature ($\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$)

Remark: This theorem provides a connection (a very precise one) between spatial and temporal mixing of the Ising model.

organization in space

mixing time of Glauber Dynamics

- In today's lecture, & next lecture we show fast fixing for sufficiently low $\beta$ and slow mixing for sufficiently high $\beta$

- **Proof [fast mixing for $\beta < \frac{1}{2} \ln 5/3$]:**

  - proof by path coupling
  - pre-metric:
    - edge between two configurations if they differ at one site
    - weight of edge 1
  - induced metric: hamming distance between two configurations
  - path coupling: define coupling step $(X_t, Y_t) \longrightarrow (X_{t+1}, Y_{t+1})$ for $X_t, Y_t$ that differ at $i_0$

    $\hookrightarrow$ pick same $i$ and update it "optimally" to maximize the probability of agreement of $X_t, Y_t$; i.e. if $p_X$ respectively $p_Y$ are the probabilities that $i$ becomes a '+' then set $i$ to a '+' in both chains w/ probability $\min(p_X, p_Y)$ to a '−' in both chains w/ prob. $\min(1-p_X, 1-p_Y)$, etc.

  - Analysis:

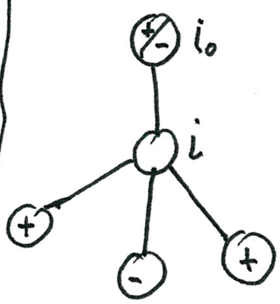
    good moves: $i = i_0$ - then $\Delta$distance $= -1$ w/prob 1
    - this case happens w/ prob $1/n$

    bad moves: $i \in N(i_0)$ - then

- then $\Delta$distance $= +1$ w/ prob $\overset{\circledast}{\leq} \dfrac{\exp(2\beta)-\exp(-2\beta)}{2+\exp(2\beta)+\exp(-2\beta)}$
- happens w/ prob $\leq \dfrac{4}{n}$

**proof of $\circledast$:** let $\alpha^+$ : # of + neighbors of $i$, excluding $i_0$

$\alpha^-$ : # of − neighbors of $i$ excluding $i_0$




$$\left|\Pr\left[X_{t+1}(i)=+\right]-\Pr\left[Y_{t+1}(i)=+\right]\right| = \frac{\exp(2\beta)-\exp(-2\beta)}{\exp(2\beta\alpha)+\exp(-2\beta)+\exp(2\beta)+\exp(-2\beta\alpha)}$$

maximized by letting $\alpha=0$

Hence:

$$\mathbb{E}\left[d(X_{t+1},Y_{t+1})\,|\,X_t,Y_t\right] \leq d(X_t,Y_t)\left(1-\overbrace{\left[\frac{1}{n}-\frac{4}{n}\cdot\frac{\exp(2\beta)-\exp(-2\beta)}{2+\exp(2\beta)+\exp(-2\beta)}\right]}^{\lambda}\right)$$

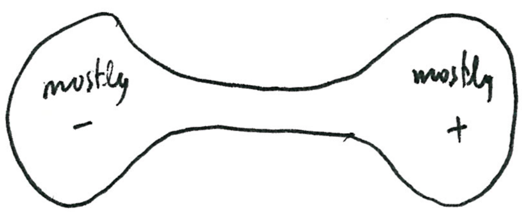
the above is true for every edge in pre-metric, and by the path coupling thm also true for every pairs of states if we extend the above coupling appropriately

if $\lambda > 0 \left(\Leftrightarrow 1 > 4\cdot\dfrac{e^{2\beta}-e^{-2\beta}}{2+e^{2\beta}+e^{-2\beta}} \Leftrightarrow \beta < \frac{1}{2}\ln 5/3\right)$

the mixing time is $T_{mix} = O(n\log n)$ $\boxtimes$

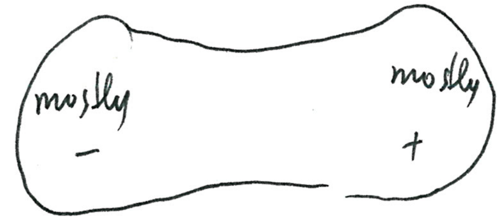
- Next time we'll show slow mixing for $\beta > \frac{1}{2}\ln 3$. We'll use the following intuition:

low temperature




configurations that are mostly + or mostly - get a lot of probability under $\pi$, but intermediate configurations get low probability mass creating a bottleneck

high temperature




the bottleneck at intermediate configurations ceases to exist.

- To show how bottlenecks affect mixing, we show the following lemma:

**Theorem:** For any Markov chain, and any $S \subseteq \Omega$ w/ $\pi(S) \leq \frac{1}{2}$,

$$\tau_{mix} \geq \frac{1}{4\Phi(S)},$$

where $\Phi(S) = \dfrac{\sum_{x \in S, y \notin S} \pi(x) P(x,y)}{\pi(S)}$ is the conductance of $S$.

**Proof:** - consider the initial distn'

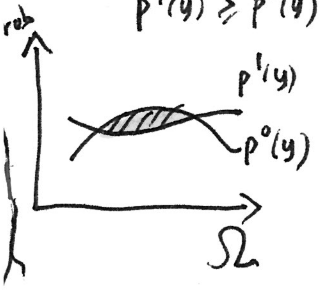
$$p^0(x) = \begin{cases} \frac{\pi(x)}{\pi(S)}, & \text{if } x \in S \\ 0, & \text{if } x \notin S \end{cases}$$

- let's see what happens after one step of the Markov chain:

$$\|p^1 - p^0\|_{TV} = \frac{1}{2} \sum_y \left| p^1(y) - p^0(y) \right|$$

$$= \frac{1}{2} \sum_y \left| \sum_x p^0(x) P(x,y) - p^0(y) \right|$$

- summing overall states $y$ where $p^1(y) \geq p^0(y)$




can be shown that these states are precisely the states in $\bar{S}$

$$= \sum_{y \in \bar{S}} \left| \sum_x p^0(x) P(x,y) - \underbrace{p^0(y)}_{0} \right|$$

$$= \sum_{y \in \bar{S}} \sum_x p^0(x) P(x,y)$$

$$= \sum_{y \in \bar{S}} \sum_{x \in S} p^0(x) P(x,y) = \Phi(S).$$

- <u>claim</u>: $\|p^{t+1} - p^t\|_{TV} \leq \|p^t - p^{t-1}\|_{TV}$

<u>proof</u>: $$\|p^{t+1} - p^t\|_{TV} = \frac{1}{2} \sum_x \left| p^{t+1}(x) - p^t(x) \right|$$

$$= \frac{1}{2} \sum_x \left| \sum_y \left( p^t(y) P(y,x) - p^{t-1}(y) P(y,x) \right) \right|$$

$$\leq \frac{1}{2} \sum_x \sum_y P(y,x) \left| p^t(y) - p^{t-1}(y) \right|$$

$$= \frac{1}{2} \sum_y \left| p^t(y) - p^{t-1}(y) \right| = \|p^t - p^{t-1}\|_{TV} \quad \square$$

- using claim and triangle inequality:

$$\|p^t - p^0\|_{TV} \leq \|p^t - p^{t-1}\|_{TV} + \|p^{t-1} - p^{t-2}\|_{TV} + \dots + \|p^1 - p^0\|_{TV} \leq t \cdot \|p^1 - p^0\|_{TV}$$

$$\leq t \cdot \Phi(S)$$

- using triangle inequality again:

$$\|p^t - \pi\|_{TV} \geq \|p^0 - \pi\|_{TV} - \|p^t - p^0\|_{TV}$$
$$\geq \frac{1}{2} - t\,\Phi(S)$$
$$\geq \frac{1}{4} \quad \left(\text{as long as } t \leq \frac{1}{4\Phi(S)}\right).$$

- So $\quad T_{mix} \geq \dfrac{1}{4\Phi(S)} \qquad \boxtimes$